MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2022, 18(8), em2141
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/12245

Argumentative orchestration in the mathematical modelling cycle in the classroom

Horacio Solar [1*], Andrés Ortiz [2], Victoria Arriagada [1], Jordi Deulofeu [3]

[1] Faculty of Education, Pontificia Universidad Católica de Chile, Santiago, CHILE
[2] Faculty of Education, Universidad Católica de la Santísima Concepción, Concepción, CHILE
[3] Department of Didactics of Mathematics and Science, Universitat Autònoma de Barcelona, Barcelona, SPAIN

Received 30 March 2022 ▪ Accepted 5 July 2022

Abstract
Given the importance of modelling in mathematics classrooms, and despite the extensive body of research on teacher support for promoting the mathematical modelling cycle in the classroom, authors have overlooked how teacher support for argumentation can contribute to this cycle. This study is aimed at characterizing teacher support for argumentation in the mathematical modelling cycle in the classroom. We analyzed 10 class episodes taken from the cases of two teachers, Soledad and Ángeles. The episodes were analyzed considering teacher support for argumentative orchestration (communicative strategies and pattern recognition). In the two cases studied, we found that argumentative orchestration exhibited different types of overall presence and recurrence throughout the stages of the mathematical modelling cycle, with communicative strategies being more present across the board and more recurrent in the mathematical modelling cycle than pattern recognition strategies.

Keywords: mathematical modelling cycle, modelling process, collective argumentation, argumentative orchestration, case study

INTRODUCTION

Researchers agree that modelling must play a relevant role in mathematical education, for instance, through the modelling of real-world problems in the mathematics classroom (Stillman et al., 2020). However, several empirical studies describe the difficulties encountered by students when attempting to complete modelling tasks in mathematics lessons (Blum, 2011). Therefore, teachers play a key role in helping students to navigate modelling processes (Brown, 2017) through the use of strategic guidance interventions (Blum & Borromeo-Ferri, 2009). There is an extensive body of literature on teacher strategies for promoting modelling processes in the mathematics classroom. Such strategies include collaborative learning communities (Mueller et al., 2014), measures for promoting modelling processes in students (Schukajlow et al., 2015; Tropper et al., 2015), or metacognitive strategies (Vorhölter, 2019), revealing that these types of activities characterize modelling processes in the classroom.

However, the above teacher strategies fail to consider an aspect that may be highly relevant: the role of discussion among students in modelling processes in the mathematics classroom. The importance of discussion can be observed in several aspects; for instance, given the open-ended nature of modelling tasks, it is common for productive discussions to emerge (Cai et al., 2014; Manouchehri et al., 2020). Teamwork skills (Maaß et al., 2019) are another relevant element, as promoting small-group discussions is a frequent teacher strategy in the modelling process. These discussions tend to include conflicts among the members of the group, who eventually reach a consensus solution (Tekin-Dede, 2019). In large group discussions, when the teacher selects and sequences students' responses, the opportunity emerges to contrast the multiple models generated by the students and examine their validity or pertinence (Smith & Stein, 2011).

Argumentation is a good opportunity for encouraging discussion in mathematics classrooms. Argumentation is aimed at convincing both oneself and others of the validity of a line of reasoning (Ayalon &

✉ hsolar@uc.cl **(*Correspondence)** ✉ aortiz@ucsc.cl ✉ vparriagada@uc.cl ✉ jordi.deulofeu@uab.cat

Contribution to the literature

- A professional development program was implemented seeking to help teachers to promote argumentation and modelling in mathematics classroom.
- In this research we characterize teacher's support for argumentation (participation opportunities, dealing with errors, deliberate questions eliciting students' thinking and recognizing students' thinking patterns) through the mathematical modelling cycle.
- The argumentative orchestration exhibited different types of overall presence and recurrence throughout the stages of the mathematical modelling cycle, with communicative strategies being more present across the board and more recurrent in the mathematical modelling cycle than pattern recognition strategies.

Hershkowitz, 2018; Krummheuer, 1995). Even though the literature on modelling is extensive, only a handful of studies connect argumentation processes with the mathematical modelling cycle (Tekin-Dede, 2019; Guc & Kuleyin, 2021), revealing how arguments are constructed in the modelling cycle that students go through. Given the important role that argumentation can play in facilitating students' navigation of modelling processes, it is relevant to examine teacher support for argumentation in the mathematical modelling cycle in the classroom.

Research Question

Several empirical studies have shown the importance of the teacher's encouragement of argumentation in the mathematics classroom (Conner et al., 2014; Yackel, 2002). However, the few studies linking argumentation with modelling do not examine teacher support for argumentation in the mathematical modelling cycle in the classroom; therefore, this constitutes a topic to be explored. In this context, our research question is:

How can we characterize teacher support for argumentation in the mathematical modelling cycle in the classroom?

THEORETICAL FRAMEWORK

The Mathematical Modelling Cycle

In our proposal, *modelling* is defined as a correspondence that makes it possible to generate an interaction between a real problem and the mathematical world, considering that the problems of reality are complex and that mathematical models are simplified representations of reality observed through mathematical methods (Blum & Niss, 1991). Thus, authors have proposed cycles for using and creating a mathematical model that comprise five modelling processes: simplifying, mathematizing, working within mathematics, interpreting, and validating (Blum & Borromeo-Ferri, 2009; Maaß, 2006). Blum and Leiß (2007) argue that, to select the type of cycle to be used in modelling tasks, it is necessary to consider the purposes of the study or the requirements of the mathematical activity conducted by students, where some cycles are more geared toward individual problem-solving. In this study, we utilized the modelling cycle proposed by Maaß (2006), which is an adaptation of that described by Blum (1996). The cycle clearly shows the interaction between reality and mathematics throughout the modelling process, establishing a relationship between them.

According to Maaß (2006), in order to model a real problem, it is necessary to move between reality and mathematics, capturing students' activity in each of the transitional phases of the modelling cycle, because this process is non-linear due to students' freedom to move forward or backward (Blum & Leiß, 2007; Borromeo, 2009). The mathematical modelling process begins with a real-world problem that is *simplified* to generate a real model, fragmenting the problem or presuming that certain variables are irrelevant. At this stage, the participants discuss the problem, identifying initial conditions, restrictions, and variables, and employ representations to produce the real model. The aim of this process is to find similarities with a known aspect of the mathematical world. The *mathematizing* of the real model results in a mathematical model generated through a translation process that involves symbols, representations, and mathematical expressions. In the *working within mathematics* stage, students employ mathematical methods such as properties, theorems, and algorithms, make calculations, and verify results using known or unknown mathematical procedures. *Interpreting* is linked to explaining and discussing solutions. To do this, it is necessary to consider the context where the data were extracted and identify the optimal solution to the problem, within a reasonable range that meets the initial conditions. The *validating* stage comes next, where the participants must justify the validity of the model using numeric approximations or estimations, determining the model's error margins and its strengths and limitations (Aravena, 2016; Blum & Borromeo-Ferri, 2009; Blum & Leiß, 2007; Maaß, 2006).

Teacher Strategies for Promoting Modelling

Several studies have reported a variety of strategies whereby teachers can help students to navigate modelling processes. One potential strategy is to promote small-group collaborative work to complete modelling tasks (Mueller et al., 2014), where the participants present solutions to the task and the strategies used to the whole class, while the teacher supports them through permanent monitoring (de Oliveira & Barbosa, 2013). Even though collaborative work monitored by the teacher has been shown to be more effective in modelling tasks than a teacher-centered instructional strategy (Schukajlow et al., 2012), it has been reported to be only partly satisfactory for dealing with modelling tasks. Therefore, researchers have sought to identify more specific tasks for covering the five processes of modelling: simplifying, mathematizing, work within mathematics, interpreting, and validating (Blum & Borromeo-Ferri, 2009; Maaß; 2006). In general, these strategies have a large metacognitive component, because they are aimed at ensuring that students become aware of modelling processes. Therefore, another possible strategy aligned with this feature is the solution plan (Blum, 2011), which comprises four steps: understanding the task, establishing the model, using mathematics, and explaining the results (Schukajlow et al., 2012, 2015). Other authors have divided the fourth step into two parts (interpreting and evaluating), resulting in a five-step plan (Beckschulte, 2020). These studies show differences in the impact of the solution plan depending on the stage of the modelling process, with results improving toward the end of the plan as a result of interpreting and evaluation (Schukajlow et al., 2015). More specifically, the five-step solution plan has a significant impact in the interpreting stage, but the rest of the stages of the mathematical modelling cycle exhibit no results, suggesting that more qualitative research on the use of the solution plan is needed (Beckschulte, 2020).

Few studies have been conducted on teachers' role in the multiple stages of the mathematical modelling cycle (Tropper et al., 2015). Czocher (2018) argues that validating actions, which can occur at any point of the mathematical modelling cycle, guarantee that the model will yield a reasonably accurate prediction, noting that such actions take place in the modelling process and are not performed at the end just to verify it. Therefore, validating must be given more attention within the modelling process. These dissimilarities in the relevance of modelling actions support the notion that the teacher's role may differ depending on the moment of the modelling process that the student is conducting.

The encouragement of students' metacognition is another of the strategies reported in the literature. Several studies describe the relationships between metacognition and modelling processes (Blum & Schukajlow, 2018; Schukajlow et al., 2012). Vorhölter (2019) proposes three types of metacognitive strategies: strategies to move forward in the process, strategies to regulate the solution process, and strategies to evaluate the modelling process as a whole. The author indicates that the strategies for moving forward are the most commonly used by students, while regulation and evaluation strategies must be improved in order to foster students' metacognitive modelling strategies. In this context, teachers must be aware of these metacognitive strategies and the difficulties associated with their application (Vorhölter, 2019).

The teacher strategies described in this section, despite being devised to encourage students to move through modelling processes, do not consider teacher support for argumentation in the mathematical modelling cycle.

Teacher Support for Argumentation

Argumentation in the classroom has been defined as a communicative situation in which contrasting ideas are expressed in order to convince other people of their validity (Ayalon & Hershkowitz, 2018; Krummheuer, 1995). The structure advanced by Toulmin (2003) is one of the most prevalent models for analyzing argumentation in the mathematics classroom. Toulmin's (2003) structure comprises six elements: the "claim", which is the assertion that the speaker wants to prove to his/her interlocutors; the "grounds", which is the evidence presented to initiate the argumentation process; the "warrant", which makes it possible to infer the claim based on the grounds; the "rebuttal", which establishes the conditions in which the warrant or the claim are not valid; the "modal qualifier", which qualifies the claim considering how certain it is; and the "backing", which adds legitimacy to the warrant. Argumentation can be understood as the process whereby these components are assembled (Knipping, 2008). Toulmin's (2003) structure is useful for analyzing argumentation in the mathematics classroom given the fundamental role played by rebuttals in convincing others (Solar & Deulofeu, 2016).

Several authors have focused on collective argumentation, which occurs when two or more people interact to reach and conclusion and engage in argumentation (Knipping, 2008; Yackel, 2002). Collective argumentation requires teacher support because specific actions can strengthen different steps in students' argumentative processes (Conner et al., 2014). Teachers play a key role in the establishment of norms and standards for mathematical argumentation in the classroom (Ayalon & Hershkowitz, 2018). Specifically, collective argumentation requires teacher support because specific actions can strengthen different steps in students' argumentative processes (Yackel, 2002). This support can be provided through teacher actions or questions (Conner et al., 2014) or specialized methods to promote argumentation such as argumentative orchestration (Solar et al., 2021). This support for collective argumentation comprises a variety of resources and strategies: first, mathematical tasks open to discussion that require a variety of resolution strategies or allow for various positions in order to promote debate among students; second, communicative strategies such as the encouragement of participation, dealing with errors, and deliberate questions (Solar & Deulofeu, 2016); third, strategies for recognizing students' thinking, which include the teacher's elicitation and acknowledgment of thinking patterns in their students (Ball et al., 2009).

These three strategies–tasks open to discussion, communicative strategies, and strategies aimed at recognizing students' thinking–are part of the resources at teachers' disposal to support argumentation. Therefore, we will study how these strategies can generate the necessary conditions for conducting modelling processes in the mathematics classroom.

METHODOLOGY

We adopted a qualitative, exploratory approach and a multiple case study design (Yin, 2014). The study is part of a larger project aimed at characterizing students' learning outcomes when modelling and argumentation competences are jointly promoted in mathematics classrooms. The present study seeks to describe the teacher support argumentation process in each of the stages of the mathematical modelling cycle in the classroom.

Contexts and Participants

Since it is uncommon for lessons to feature argumentation in modelling tasks, we implemented a professional development program aimed at helping mathematics teachers to promote argumentation and modelling competences in the classroom. This program was implemented in two Chilean cities (Santiago and Concepción) and benefited 22 teachers (all women), 13 from the Metropolitan Region and nine from the Biobío Region. These teachers work in primary education (with students aged 6-11 years) or the first two years of secondary education (with students aged 12-14 years). The 22 teachers were invited to participate due to their experience in the development and analysis of argumentation in the mathematics classroom, which means that the selection process was purposive (Creswell, 2011). This approach was necessary because the professional development program required prior argumentation knowledge to be articulated with modelling when designing and implementing the program's classroom activities. The teachers in the group worked in private schools, State-subsidized private schools, and municipal schools.

The professional development program consisted in 15 sessions lasting three hours each and was implemented between August and December 2018 in accordance with the teacher experience improvement model (Solar et al., 2016). Based on this model, the participants were shown video recordings of other teachers' experiences and were then requested to implement a teaching proposal. After the completion of the training process, 10 teachers from each region were selected as case studies to follow-up their lessons in 2019. During this year, they designed a task composed of 3-4 classes to promote students' modelling and argumentation skills. These teachers were selected due to exhibiting a constant attendance level and working at different school levels during 2019. The latter criterion was adopted to observe the development of argumentation and modelling at multiple ages.

All the sequences–implemented through a task for promoting modelling and argumentation–were collaboratively conducted in small groups. Each of the 10 teachers implemented three-four classes lasting 45-70 minutes each, which were video-recorded using three cameras. One camera tracked the teacher's movements, while the other two focused on two fixed small groups selected by the teacher (group 1 and group 2). The conversations of these two groups were recorded (including their gestures) and their written production collected to serve as data for the study. All participants (teachers and students) gave their consent to be recorded in the classroom.

We then selected five of the 10 teachers due to their successful implementation of a sequence of lessons that displayed the cycle of mathematical modelling and argumentation promotion. This article presents the results of 2 two of these five cases, which were selected because the students reached the last stage in the mathematical modelling cycle: validating (Maaß, 2006).

Data Analysis Strategy

The class recordings were coded using Atlas ti, a software package for qualitative data analysis.

We reviewed all the videos and coded the moments of the lesson where we observed teacher and/or student actions present in our code books, which we will term "episodes". They are described below.

We first coded the episodes during, which the teacher interacted with the students promoting argumentation, which lasted between 30 and 200 seconds. This phase was conducted by four coders who analyzed the teachers' actions according to a rubric of five levels of argumentation promotion according to Toulmin's (2003) structure: the teacher does not encourage students to provide any justifications for their answers and positions (level 1); the teacher encourages students to justify their answers and positions (level 2); in addition to the actions of level 2, the teacher also encourages students to refute their answers and positions (level 3); in addition to the actions of levels 2 and 3, the teacher also encourages students to discuss their answers and positions (level 4); in addition to the actions of levels 2, 3, and 4, the teacher also encourages students to support the discussion of their answers and positions (level 5). We selected episodes from levels 3 to 5, in which the teacher promoted argumentation through justifications and

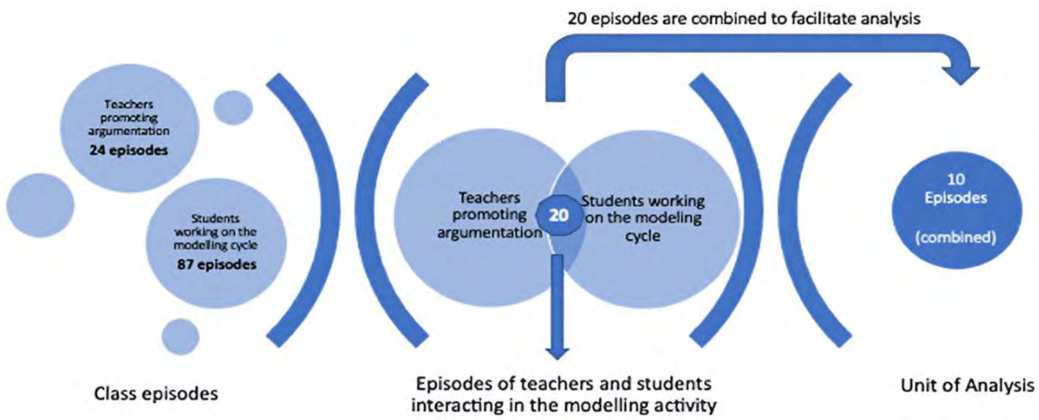

Figure 1. Selection of the unit of analysis (summarized)

refutations. A total of 24 episodes exhibited these characteristics.

Second, we coded the episodes that featured the students working on the mathematical modelling cycle, which lasted between five and 242 seconds. This phase was conducted by two coders who analyzed the students' actions using the modelling framework developed by Maaß (2006), which contains the multiple stages of the cycle (simplifying, mathematizing, working within mathematics, interpreting, and validating). This framework was complemented with codes generated by our team to refer to student actions in each stage. Then, these episodes were reviewed by the two coders and a researcher from the team. They selected only the most representative episodes for each code; therefore, they were able to choose more than one episode, one, or none. A total of 87 episodes exhibited these characteristics.

Third, two coders identified episodes that showed the two actions mentioned previously at the same time; that is, the teacher promoting argumentation and eliciting a response from the student regarding the modelling activity. The two cases yielded 20 episodes of this type, that is, 83% of the argumentation episodes generated student responses with respect to the modelling cycle. Furthermore, continuous episodes were combined to facilitate the analysis of teacher-student interaction, resulting in 10 episodes that comprise the unit of analysis in this study (**Figure 1**).

Lastly, we identified the teacher support actions present in each of these 10 resulting episodes. For the purposes of this study, when we use the term "teacher strategies", we specifically refer to two of the three dimensions of argumentative orchestration (Solar et al., 2021): *communicative strategies* (Lee, 2006; Solar & Deulofeu, 2016): participation opportunities, dealing with errors, and deliberate questions; and *recognizing students' thinking* (Ball et al., 2009): eliciting students' thinking and recognizing their thinking patterns. We excluded *mathematical tasks open to discussion* because they lie outside the scope of the present study. **Table 1** shows the teacher strategies and support actions (Solar et al., 2021).

Table 1. Argumentative orchestration (Solar et al., 2021)

Argumentative orchestration	Teacher strategies argumentative orchestration	Teacher support
Communicative strategies (Lee, 2006, Solar & Deulofeu, 2016)	Participation opportunities	O1: Not validating responses (or procedures before discussion among peers). O2: Flexibility for students to join the discussion. O3: Asking questions that encourage all students to explain and describe procedures and ideas.
	Dealing with errors	E1: Focusing on making students converse (with explanations or correct and incorrect answers). E2: Not mentioning errors (before collective discussion among peers). E3: Not reviewing errors in advance, but only after students have noticed them.
	Deliberate questions	Q1: Asking questions that encourage students to explain and describe answers and procedures rather than yes-no questions. Q2: Asking students follow-up questions based on their own answers. Q3: Asking questions that do not shift focus too quickly; trying to ask questions that encourage development in students' ideas.
Recognizing students' thinking (Ball et al., 2009)	Eliciting students' thinking	E1: Encouraging mathematical communication (through multiple forms of expression: oral or written language, drawings, and/or models). E2: Formulating questions that foster comparisons and discussion (about solutions or alternative procedures).
	Identifying thinking patterns	P1: Recognizing and organizing ideas (to clarify a concept). P2: Identifying the causes of frequent errors.

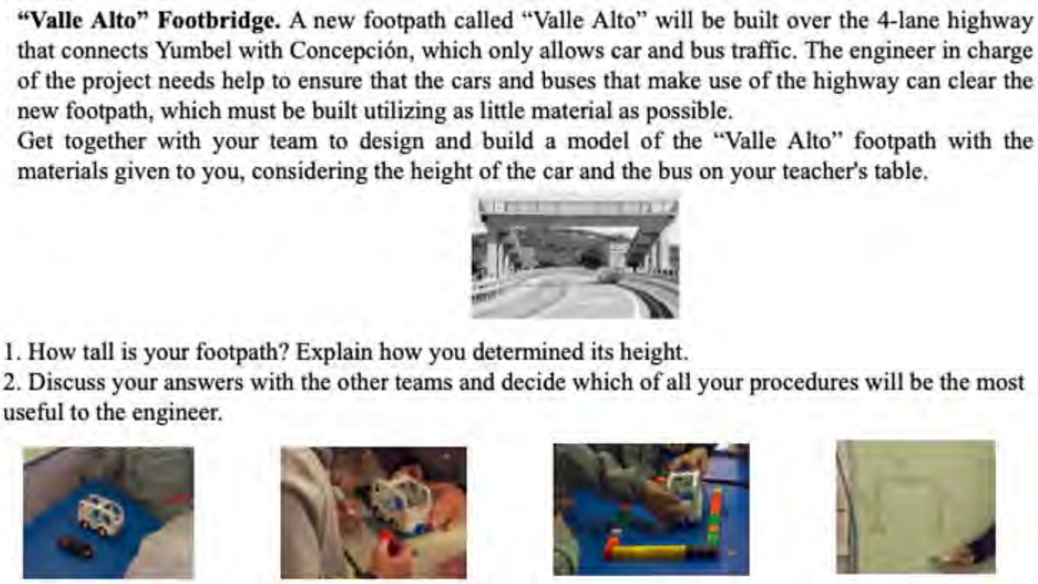

Figure 2. Activity implemented by Soledad for her 3rd grade class

To boost the internal validity of the study (Creswell, 2011), this analysis was conducted by the research team using recursive processes of discussion and re-coding. This made it possible to ensure the reliability and validity of our findings (Bogdan & Biklen, 2007; Petty et al., 2012).

RESULTS

Results will be presented by case (Soledad and Ángeles). In each of them, we will present the mathematical task conducted by the students to contextualize the teacher actions associated with the sub-dimensions of argumentative orchestration (see **Table 1**). We present Soledad'scase and Ángeles' case, where we note the teacher support actions present in each of the stages of the mathematical modelling cycle: simplifying (Sim), mathematizing (Mat), working within mathematics (WwM), I interpreting (Int), and validating (Val). Transcripts of selected episodes are also included to illustrate said actions.

Soledad's Case

Soledad presented her activity–which she designed with the support of the research team–to 3rd grade students (7-8 years old). The activity is presented exactly as it was posed to her students. The children were instructed to make a model footbridge with building blocks, using as reference points two toy vehicles that the teacher provided (a car and a bus). At the start of the sequence (class 1), the groups of students were allowed to look at the vehicles for one minute without touching them. In class 2, the groups were allowed to touch and measure the vehicles and then make the model with whatever technique they chose (class 3). The teacher agreed to let the students touch both vehicles and use them to compare them to the structure that they had already designed. In class 4, the teacher led a discussion on the sum of the sides of the footbridge. This is shown in **Figure 2**, which includes images of the process.

Soledad's teacher support actions in the mathematical modelling cycle

Table 2 shows Soledad's argumentative orchestration actions during each stage of the mathematical modelling cycle.

Table 2 shows that some teacher support actions such as participation opportunities, dealing with errors, and deliberate questions were promoted in multiple stages of the mathematical modelling cycle. However, other strategies such as eliciting students' thinking were specific to certain stages of the cycle, while actions classed as recognizing thinking patterns were not found in this case. Regarding the stages of the mathematical modelling cycle, teacher support actions were only found in the simplifying, working within mathematics and interpreting stages, during which the teacher played a more active role. Her involvement in the teacher support argumentation was especially active in the simplifying stage. For this reason, we present two episodes that illustrate the whole range of support actions performed by the teacher in the modelling task.

At the start of the activity, the teacher gave all the students a few minutes to handle a toy car and bus that they would be able to use as models to build the footpath using base ten blocks. She also allowed the students to measure the vehicles, using either a ruler or a pencil in any direction that they wished (e.g. width, height). During the first part of class 2, during the simplifying stage, the teacher approached one of the groups and they discussed the problem to understand it. Specifically, the teacher asked whether the measurements that they had

Table 2. Quantification of Soledad's teacher support actions during the mathematical modelling cycle

Teacher strategies argumentative orchestration	TS	Sim	Mat	WwM	Int	Val
Participation opportunities	O1	2			1	0
	O2	2			1	1
	O3	1			0	0
Dealing with errors	E1	0			1	1
	E2	1			1	1
	E3	1			1	1
Deliberate questions	Q1	2			0	1
	Q2	1			1	0
	Q3	2			1	1
Eliciting students' thinking	T1	2			0	0
	T2	0			0	1
Identifying thinking patterns	P1	0			0	0
	P2	0			0	0

Note. Sim: Simplifying; Mat: Mathematizing; WwM: Working within mathematics; Int: Interpreting; Val: Validating; & TS: Teacher support

taken at the start of the activity had been useful to them. The following is the transcript of episode 1.

Episode 1

> Soledad: The measurements you took, were they useful? [one student answers "yes"] Are they useful, Paula? You looked at me like you thought they didn't work… are those measurements any use?
>
> Carlo: Yes, because Berta measured upward, on a straight line [pointing upward] and also to the side.
>
> Berta: Yes, I measured sideways, backward, forward, both sides, up, and down.
>
> Soledad: And what measurements did you forget to take?
>
> Berta: The top of the bus.
>
> Soledad: And was it necessary to measure that?
>
> Carlo: Yes.
>
> Soledad: Why?
>
> Carlo: Because otherwise we wouldn't know… how tall the buses are so that…
>
> Berta: I was going to take that measurement, but I was measuring the other parts when the thing rang, I was going to measure it.
>
> Soledad: And did you measure the car?
>
> Carlo: Paula had to do that.
>
> Soledad: And did she measure the car?
>
> [the student shows the teacher a sheet of paper with measurements]
>
> Soledad: Okay, are these measures clear, Paula? It says "car" and three numbers. Okay, if the measurements aren't useful, calculate, estimate how much it was, how tall you think it was…

In episode 1, the teacher offered *participation opportunities* by not validating the students' answers. This is illustrated by the question "Was it necessary to measure that?" when Berta stated that she had failed to measure the length of the top of the bus. In this case, the teacher may have directly instructed the students to take that measurement, however, she first asked whether it was necessary to do so, to which Carlo answered "yes". The teacher also asked questions that encouraged the explanation and description of procedures; for instance, she asked "Why?" when the students stated that measuring the height of the bus ("the top") was necessary and that they had failed to do so. In addition, the teacher's discourse shows that the students were able to join the discussion flexibly. She asked *deliberate questions*: instead of just asking yes-no questions, she encouraged the students to explain their answers and procedures. For instance, she asked "Why?" with respect to the measurement that the students had failed to make, which also indicates that the teacher asked follow-up questions based on the students' answers. During the conversation, the teacher also asked questions whose focus did not change abruptly, for instance, she asked "Are those measurements any use?", "What measurements did you forget to take", and "Was it necessary to measure that?" This approach allowed the teacher to examine an issue in depth without shifting to another one too quickly. Finally, during this part of the episode, the teacher *elicited students' thinking* by promoting mathematical communication through oral language.

Therefore, it is clear that the students understood the problem because they discussed the initial conditions, for example that the width of the tracks should be enough for the bus and the car to pass, which meant that it was necessary to measure both vehicles, not just one. In addition, the footpath would need to be tall enough for the vehicles to clear it.

At the start of class 2, after the students were allowed to touch a toy car and bus that they would use as reference points to build the footpath with base ten blocks, they were also able to measure the toys in any way that they wished, using either a ruler or a pencil in any direction (e.g. width, height). Afterward, as part of the simplifying stage, the teacher approached another group of students who were building their footpath and discussing the problem to understand it. Specifically, the teacher asked which measurements they had taken when touching the car and the bus. The following is the transcript of episode 2.

Episode 2

> Soledad: Okay, what measurements did you take? [Nobody answers] Did you take any measurements, with your ruler?
>
> Marta: yes, with my ruler.
>
> Soledad: Okay, are the measurements you took useful?
>
> Marta: [moves her head from side to side, indicating disagreement]
>
> Soledad: Why not? Why do you think those measurements are useless?… Luis, are the measures you took with your ruler useful?
>
> Luis: No.
>
> Soledad: Why not?
>
> Luis: Because we need to take many.
>
> Antonia: Because we need to make it bigger.
>
> Soledad: Why?
>
> Andrés: Because four cars won't fit here.
>
> Soledad: I see.
>
> Andrés: There isn't enough room for two.
>
> Soledad: There's no room for two cars with these measurements!
>
> Andrés: Right, not even one.

Episode 2 shows the teacher *dealing with errors* through various actions. The questions "Are the measurements you took useful?" and "Why not?" reveal that the teacher not only avoided pointing out the students' mistake–since she never stated that the measurements that they had taken were inadequate–, but she also did not preemptively review the mistake until after the students had noticed it; in other words, she waited for the students to detect it. The teacher also offered *participation opportunities* by not validating either correct or incorrect procedures. This approach is illustrated by the question "Why do you think those measurements are useless?" In this case, the teacher waited until Luis, Antonia, and Andrés realized that the measurements that they had taken would lead them to build a footpath that could not be cleared by all the vehicles that use a four-lane highway. The teacher's discourse shows that the students were able join the discussion flexibly. Soledad asked *deliberate questions* instead of yes-no questions. She encouraged the students to explain their answers and procedures; for instance, she asked "Why do you think those measurements are useless?" The episode also shows that the teacher asked questions whose focus did not shift too rapidly. By asking about the students' measurements and their relevance to the problem, without changing the subject, she managed to understand the students' thinking in more depth. Finally, during this part of the episode, the teacher *elicited students' thinking* by promoting mathematical communication through oral language.

Ángeles' Case

Working with an 8th grade class (children aged 13-14 years), Ángeles presented an activity entitled Cinema Paradiso (see **Figure 2**) for groups of four members. The activity was conducted in a three-lesson sequence lasting 185 minutes. In the first session, the students discussed and solved the problem using a table without developing a mathematical model. Some groups wrote down their results on a poster that would be used later to give a presentation to the whole class. In the second class, all the groups presented their calculations and procedures on a flip chart and shared their position with the rest of the class using the whiteboard. In the third session, the remaining groups presented their answers to the whole class using a flip chart. Then, the teacher led a whole-class discussion to analyze the response provided by each group and find the one that offered the best answer to the real-life problem. Also, the whole class devised a mathematical model to cover the cases of being a member and not being one (**Figure 3**).

Ángeles' teacher support actions in the mathematical modelling cycle

Table 3 shows Ángeles' argumentative orchestration actions in the selected episodes of each stage of the mathematical modelling cycle.

Table 3 shows that, like in the previous case, some teacher support actions such as participation opportunities, deliberate questions, and eliciting students' thinking were promoted in multiple stages of

Cinema Paradiso. The "Paradiso" movie theater offers a yearly membership card that allows people to buy tickets at a reduced price.

The following table indicates how much a person would spend buying tickets as a member and as a non-member.

Unfortunately, the report got wet and some information was lost.

Nº de entradas compradas	Dinero gastado sin ser Socio	Dinero gastado Siendo socio
2	5000	9000
3	7500	11000
5	12500	
6		
20		

A family says that, by selecting the most affordable choice, they saved $29,000 last year. How many tickets did they buy?

Figure 3. Activity implemented by Ángeles for her 8th grade class

Table 3. Quantification of Ángeles' teacher support actions in the mathematical modelling cycle

Teacher strategies argumentative orchestration	TS	Sim	Mat	WwM	Int	Val
Participation opportunities	O1	2		1	0	1
	O2	2		1	0	1
	O3	2		1	0	0
Dealing with errors	E1	1		0	0	1
	E2	1		0	0	0
	E3	0		0	0	1
Deliberate questions	Q1	1		1	1	1
	Q2	2		1	1	0
	Q3	2		1	0	1
Eliciting students' thinking	T1	2		1	1	0
	T2	1		1	0	0
Identifying thinking patterns	P1	0		0	1	0
	P2	0		0	0	0

Note. Sim: Simplifying; Mat: Mathematizing; WwM: Working within mathematics; Int: Interpreting; Val: Validating; & TS: Teacher support

the mathematical modelling cycle. However, actions such as dealing with errors and recognizing thinking patterns characterized other specific stages of the cycle.

Regarding the stages of the mathematical modelling cycle, teacher support actions were found in all stages except for mathematizing. This does not mean that the mathematizing stage elicited no reactions from the students, but that there were no episodes in which the teacher participated in the students' interaction, the essential criterion for selecting the episodes defined as the unit of analysis.

This specific teacher made major argumentative support efforts in the interpreting and validating stages. For this reason, we present two episodes that illustrate the variety of support actions that she conducted in these stages of the mathematical modelling cycle.

At the beginning of the third class, as part of the interpreting stage, the students considered the real problem while interpreting the model constructed. The groups went to the board with their posters to explain what they had, the discussions that they had held, and the ideas that they had arrived at. One of the groups noted that not being a member was convenient until one purchased nine tickets, when the situation became equal for both members and non-members; after purchasing 10 tickets, the opposite occurred. Afterward, the teacher pointed out that this group's ideas revealed that it was better to remain a non-member until the 9th ticket. The following is the transcript of episode 3.

Episode 3

Cristián: [unintelligible explanation] Over time, they will become equal… from the 11th ticket onward…

Mario: In the end, there was always going to be a point, 10 tickets, when it became equal, but then it changed.

Ángeles: How much does a member and a non-member spend on 10 tickets?

Cristián: It would have to be the same.

Ángeles: Okay, according to what you're saying, we can conclude that not being a member is convenient until ticket number 9, right?

Mario: Yes [other students nod in agreement]

Ángeles: But the other groups did not say that. But you agree? Was that part of your calculations?

Mario: We reached another conclusion, that if we went much higher, we'd reach a discount of 29,000, so that was like a starting point for…

Episode 3 shows that the teacher *recognized the students' thinking patterns* after they explained that the convenience of being a member and a non-member became equal upon buying 10 tickets. At that point, the teacher said "Okay, according to what you're saying, we can conclude that not being a member is convenient until ticket number 9", which Mario and other students agreed with. This indicates that the teacher recognized and organized the students' ideas to emphasize their conclusions. During this episode, the teacher also asked *deliberate questions*, encouraging the students to explain their answers and procedures instead of only asking yes-no questions. This is illustrated by her decision to ask them to go to the board to describe their procedures. Then, she asked a follow-up question based on their answers: "Okay, according to what you're saying, we can conclude that not being a member is convenient until ticket number 9, right?" Lastly, the teacher briefly *elicited the students' thinking* by promoting mathematical communication orally and with the support of posters placed on the whiteboard.

In the middle of the third session, the students discussed in their groups the intermediate models constructed and the answers to the problem, both their own and those of other groups. In other words, they discussed models and results as part of the validating stage. The groups had already presented their results, which were focused on completing the table with missing data (intermediate model of arithmetic regularity), and had also managed to produce the algebraic model for buying as a member (2500*x); however, they did not present a model that generalized purchasing behavior. In addition, the students' posters did not show the answer to the question of the number of tickets that must be bought to save $29,000, which also involves identifying the most convenient choice. The following is the transcript of episode 4.

Episode 4

Ángeles: According to what we're saying, we can find some restrictions. Cristián and Mario's group found the point of equality, where did they become the same?

Students: At 10.

Ángeles: At 10 tickets. So, what is the most convenient choice? Paula and Karen, if I spend the same money to buy 10 tickets being a member or a non-member, what's the best choice? Being a member or a non-member? When is it more convenient not to be a member?

Students: Up to 10.

Ángeles: The cost becomes equal at 10, it's the same for both.

Armando: Yes, if someone wants to be a member it's because he goes to the movies a lot. I don't think someone who goes to the cinema once a month would want to be a member.

Ángeles: Okay, so when is it not convenient to be a member?

Armando: When you don't go to the movies much, when you go once in a while.

Ángeles: How many times? [a student says 10 times per year].

Ángeles: It becomes the same at 10 tickets. When does it get convenient? [students are heard discussing]

Ángeles: But look, at one point it's not convenient to be a member, because the cost becomes equal when you buy 10 tickets, it costs 25,000, right? [two students answer "up to 9" and the teacher writes on the board "not being a member until you buy 9"]

Armando: Miss, but if one wants to be a member and wants to get a good discount, that person goes to the movies a lot.

Ángeles: Okay. This person goes to the cinema a lot, but how many tickets does she need to buy? [a student says "11"]

Mario [at the board]: Miss, I can go once a month, but if I take 20 people with me it would also be convenient.

Ángeles: You could spend a lot.

Mario [to Armando]: Even if you don't go to the movies often, it's still better to be a member if you go with many people.

Ángeles: But if I want to be a member, how many tickets do I have to buy to make it worthwhile?

Armando [to Mario]: But it's not convenient for you to become a member. If you go with 20 people once.

Mario: But you'd be paying less.

Ángeles: But I'm buying 20 tickets. [Ariel tries to refute Martín's point but does not complete his idea]

Ángeles: But at what point do I start saving money?

Students: From 11 tickets onward.

Ángeles: From 11 tickets onward, right? So, that's when it becomes convenient to be a member.

In episode 4, the teacher, aware that the students were mistaken, generated *participation opportunities* by not validating any correct or incorrect answers or procedures before the peer discussion. Furthermore, she *dealt with errors* by encouraging them to discuss correct and incorrect answers. This process was prompted by the view shared by Mario's group that both choices become equal when purchasing 10 tickets.

In the rest of the episode, Ángeles continued to offer *participation opportunities*. For instance, she said: "If I spend the same money to buy 10 tickets being a member or a non-member, what's the best choice? Being a member or a non-member? When is it more convenient not to be a member?" Some students incorrectly stated that the threshold is 10 tickets, but one said that the cost of both choices becomes equal at 10 tickets [with a hesitant tone of voice]. At this point, one student said 9, that is, Ángeles did not validate the answer before it was socialized and instead returned to the idea that originated the discussion, which reveals the flexibility with which students were able to enter the discussion. For instance, when Armando mistakenly stated that it is convenient to be a member when one goes to the cinema often, the teacher asked again: "This person goes to the cinema a lot, but how many tickets does she need to buy?" [a student is heard saying 11]. With respect to *dealing with errors*, Ángeles did not check the students' mistake in advance, because she expected them to notice it. At this point, she generated new *participation opportunities* by allowing any student to join the discussion flexibly. For instance, Mario refuted Armando without requesting permission, saying "I can go once a month but with many people, and even if I don't go that often it's still convenient" [Mario's group validates his views]. Armando then challenged Mario's view, saying that it is not convenient if you go once with 20 people, but the teacher pointed out that you would still need to buy 20 tickets. *Deliberate questions* are also observed in this episode, for instance, questions that do not shift focus too quickly, allowing ideas to develop. This is exemplified by the following questions: "So, what is the most convenient choice? Being a member or a non-member? When is it more convenient not to be a member? How many tickets do you need to buy? At what point do I start saving money?" These questions steered the discussion in a way that allowed the students to realize that it is not convenient to become a member when one buys nine tickets, that it is irrelevant when one buys 10, and that it is better to become a member when one purchases 11. This was not clear to the students at that point; therefore, before starting to generalize the model for members, they needed to discover that buying less than a certain number of tickets made it unnecessary to be a member and that purchasing more made it convenient.

Table 4 summarizes the support actions performed by Soledad and Ángeles in each of the stages of the mathematical modelling cycle. **Table 4** makes it possible to visualize two important findings: argumentative orchestration is present in the mathematical modelling cycle and is present throughout the whole modelling process, but with some differences in gradation and intensity depending on the stage considered. The following section offers more details about these two findings.

Overall Presence of Teacher Support Actions throughout the Modelling Cycle

We detected three types of presence across the modelling cycle: full, partial, and limited. *Full overall presence* is used when an argumentative orchestration strategy is present in all the stages of the mathematical modelling cycle reported. *Partial overall presence* is used when a teacher strategy is present in some of the stages

Table 4. Teacher support actions present in the mathematical modelling cycle (both cases)

Argumentative orchestration	Teacher strategies argumentative orchestration	Sim		Mat		WwM		Int		Val	
		S	A	S	A	S	A	S	A	S	A
Communicative strategies	Participation opportunities	5	6				3	2	0	1	2
	Dealing with errors	2	2				0	3	0	3	2
	Deliberate questions	5	5				3	2	2	2	2
Recognizing students' thinking	Eliciting students' thinking	2	3				2	0	1	2	0
	Identifying thinking patterns	0	0				0	0	1	0	0

Note. Sim = Simplifying; Mat = Mathematizing; WwM = Working within mathematics; Int = Interpreting; Val = Validating. S = Soledad; A = Ángeles.
Teacher support in five-six episodes (■); Teacher support in two-three episodes (■); Teacher presents one episode (□); No teacher support observed (□); & No teacher support actions were coded in one stage of the modelling cycle (▨)

of the mathematical modelling cycle. Finally, *limited overall presence* is used when a teacher strategy is found in only one stage of the mathematical modelling cycle.

In Soledad's case, teacher support actions classed as communicative strategies–*participation opportunities, dealing with errors*, and *deliberate questions*–exhibit full overall presence, being found in all the modelling stages reported with episodes. As for the strategy recognizing students' thinking, teacher actions aimed at *eliciting students' thinking* exhibit full overall presence. In contrast, no episodes in this case displayed any support actions grouped under *recognizing thinking patterns*.

In Ángeles' case, the communicative strategies p*articipation opportunities* and *deliberate questions* exhibit full overall presence, whereas *dealing with errors* displays partial overall presence, being found only in the simplifying and validating stages. With respect to the teacher strategy recognizing students' thinking, support actions associated with *Eliciting students' thinking* display partial overall presence, since they are not observed in the validating stage. *Recognizing students' thinking* exhibits limited overall presence, being observed only in the interpreting stage.

Recurrence of Teacher Support Actions in Each Modelling Stage

Teacher support actions are differentially recurrent in the modelling stages. **Table 4** shows that teacher support actions are most recurrently found in the simplifying stage (five-six episodes), with *participation opportunities* and *deliberate questions* predominating. At a lower recurrence level (two-three episodes), we found teacher support actions in the simplifying, interpreting, and validating stages. In addition, Ángeles also performed support actions in the working within mathematics stage, which Soledad's case does not exhibit. These episodes include the following actions: *participation opportunities*, *dealing with errors*, *elicitation of students' thinking*, and *recognizing students' thinking patterns*. At a lower recurrence level (one episode), we found teacher support actions in the interpreting and validating stages: Soledad's case, in the validating stage, includes *participation opportunities*, while Ángeles' case, in the interpreting stage, displays the actions *eliciting students' thinking* and *recognizing students' thinking patterns*. Finally, in the mathematizing and working within mathematics stages for Soledad, and in mathematizing for Ángeles, we found no episodes featuring support actions due to the teachers' lack of participation.

Therefore, the teachers were found to employ a variety of argumentative orchestration support actions, with the most recurrent ones being communicative strategies (Solar et al., 2021): *participation opportunities, dealing with errors*, and *deliberate questions*. In contrast, teacher support actions associated with the recognition of thinking patterns–*eliciting students' thinking* and *recognizing students' thinking patterns*–were less recurrent, with the latter action only being found in Ángeles' case.

DISCUSSION AND CONCLUSIONS

Given the importance of modelling in mathematics classrooms, a large body of research exists which examines teacher methods for promoting modelling processes, including collaborative work (Mueller et al., 2014), four- or five-step solution plans (Beckschulte, 2020; Schukajlow et al., 2015), and metacognitive strategies (Vorhölter, 2019). However, these strategies do not consider the role of discussions among students, a context, where argumentation can be fruitfully encouraged. Even though the literature on modelling and argumentation is vast, it is still necessary for researchers to connect these topics and determine how argumentation support can influence the mathematical modelling cycle in the classroom. In this context, the present study sought to characterize teacher support for argumentation in the mathematical modelling cycle in the classroom.

From the two cases studied (Soledad and Ángeles), we selected 10 episodes that met two criteria: the students perform a relevant action belonging to the mathematical modelling cycle and the teacher participates in the students' interaction. The episodes were analyzed considering teacher support actions associated with an argumentative orchestration (Solar et al., 2021).

Our first finding was that teacher support can exhibit three types of overall presence across the modelling cycle: full, partial, and limited. Regarding communicative strategies (Lee, 2006; Solar & Deulofeu, 2016), our joint analysis of both cases revealed that the teacher support actions entitled *participation opportunities* and *deliberate questions* exhibit full overall presence, being found across all the modelling stages reported in both cases. In contrast, *dealing with errors* only displays partial overall presence, since it is not displayed in all the stages in which teacher support actions are reported. For their part, the teacher support actions grouped under the strategy recognizing students' thinking (Ball et al., 2009) are less prevalent across the modelling cycle: *eliciting students' thinking* exhibits partial overall presence, whereas *recognizing thinking patterns* displays limited overall presence, being reported in the interpreting stage of Ángeles' case only.

The overall presence of the communicative strategies *participation opportunities* and *deliberate questions* is consistent with the nature of these actions, since the teacher activates them from the start of the class–by monitoring the students–until its closing stage–by encouraging the students to find connections between their answers and procedures–(Smith & Stein, 2011). In contrast, the overall presence of *dealing with errors* can be justified because, in modelling tasks, the teacher initially monitors the students and later, from the interpreting stage onward, she can require error dealing actions when the students share their answers and models. It is worth noting that, in an argumentative orchestration, error dealing does not include the teacher's evaluation of the students' answers; rather, she makes the students themselves evaluate their peers' answers. For its part, the partial overall presence of *eliciting students' thinking* can be explained considering the nature of its support actions–promotion of mathematical communication and question formulation–, which can take place at any point of the class. The limited overall presence of the teacher support actions belonging to *recognizing students' thinking* can be explained considering that pattern recognition tends to be infrequent in teachers' practice and can be difficult to incorporate into modelling tasks, as the cases studied show.

Our second finding concerns the recurrence of teacher support actions. Regarding the stages of the modelling cycle, the highest recurrence of support actions in both cases was detected in the simplifying stage, whereas the mathematizing and working within mathematics stages display few teacher actions. In the latter stages, the teachers regain presence. These gradations in the recurrence of teacher actions can be explained considering that, in modelling tasks, teachers perform interventions at the beginning, in the monitoring period. Then, students have freedom to perform the task in the mathematizing and working within mathematics stages. Finally, teacher presence is more necessary in the interpreting and validating stages, during which the students share their answers and models. Teacher support makes it easier for students to go through modelling processes (Schukajlow et al., 2015; Tropper et al., 2015; Vorhölter, 2019) and engage in collective argumentation (Ayalon & Hershkowitz, 2018 Conner et al., 2014). Even though only a few studies convey a joint understanding of argumentation and modelling (Guc & Kuleyin, 2021; Tekin-Dede, 2019), this study has highlighted the relevance of teacher support in argumentation (Solar et al., 2021), taking into account the high recurrence of communicative strategies such as participation opportunities and deliberate questions across all the stages of the mathematical modelling cycle.

In brief, in the two cases studied, we found that argumentative orchestration exhibited different types of overall presence and recurrence throughout the stages of the mathematical modelling cycle, with communicative strategies being more present across the board and more recurrent in the mathematical modelling cycle than pattern recognition strategies.

One of the limitations of the study concerns our video recording approach, since the cameras tracked the teacher and two groups, leaving out other students who also worked on the task. Even though recording and analyzing all the interventions in all the sessions that comprise the mathematical modelling cycle may be complex and even unfeasible, this is a factor that must be considered, especially because the coding of the modelling cycle only took into account the students from these two groups. Another limitation of our study is that, in order to observe an argumentative orchestration in modelling tasks, it was necessary for the participating teachers to enroll in a professional development program. Of the 22 teachers who started the process, 10 were selected for the follow-up stage. Finally, only five teachers were analyzed because they completed the mathematical modelling cycle with argumentation promotion. In this manuscript, the findings reported are limited to two of the five cases that reached the validating stage. Therefore, future studies could employ more cases to analyze the type of overall presence and the recurrence of argumentative orchestration in each stage of the mathematical modelling cycle.

In a similar vein, research has established the importance of mathematical tasks in promoting argumentation (Solar & Deulofeu, 2016; Solar et al., 2021), however, this was not the focus of the present study. A future study could explore how the mathematical task connects with argumentative orchestration and modelling in the classroom.

The results of this study have major didactic implications for teachers. Nowadays, teachers can benefit from ample resources and guidance when designing modelling tasks aimed at the mathematics classroom, but there is much less information on how to enact said tasks. Therefore, promoting argumentative orchestration to foster modelling represents a contribution to mathematics classrooms. Furthermore, teachers who are used to generating discussion in the classroom will be familiar with many of the support actions present in argumentative orchestration. In consequence, rather than requiring practices different from those that they already know, it is important for them to be aware of the importance of actions such as generating *participation opportunities*, *asking deliberate questions*, *and eliciting students' thinking* to ensure that students take part in all the stages of the mathematical modelling cycle.

Author contributions: All authors have sufficiently contributed to the study, and agreed with the results and conclusions.

Funding: This work was supported by ANID Fondecyt 1180880.

Ethics committee approval: This work was approved by the Scientific Ethical Committee of Science, Arts and Humanities of the Pontifical Catholic University of Chile on 28 May 2018 with Protocol number: 170706036.

Declaration of interest: No conflict of interest is declared by authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Aravena, M. (2016). Modelización matemática en Chile [Mathematical modeling in Chile]. In J. Arrieta, & L. Díaz (Eds.), *Investigaciones Latinoamericanas en modelación matemática educative* [*Latin American research in educational mathematical modeling*] (pp. 195-234). Gedisa.

Ayalon, M., & Hershkowitz, R. (2018). Mathematics teachers' attention to potential classroom situations of argumentation. *The Journal of Mathematical Behavior, 49*, 163-173. https://doi.org/10.1016/j.jmathb.2017.11.010

Ball, D., Sleep, L., Boerst, T., & Bass, H. (2009). Combining the development of practice and the practice of development in teacher education. *Elementary School Journal*, *109*(5), 458-474. https://doi.org/10.1086/596996

Beckschulte, C. (2020). Mathematical modelling with a solution plan: An intervention study about the development of grade 9 students' modelling competencies. In G. Stillman, G. Kaiser, & C. Lampen (Eds.), *Mathematical modelling education and sense-making. International perspectives on the teaching and learning of mathematical modelling*. Springer. https://doi.org/10.1007/978-3-030-37673-4_12

Blum, W. (1996). Anwendungsbeziige im Mathematikunterricht - Trends und Perspektiven [Applications in mathematics teaching - trends and perspectives]**.** *Schriftenreihe Didaktik der Mathematik, 23*, 15-38.

Blum, W. (2011). Can modelling be taught and learnt? Some answers from empirical research. In G. Kaiser, W. Blum, R. Borromeo-Ferri, & G. Stillman (Eds.), *Trends in teaching and learning of mathematical modelling. International perspectives on the teaching and learning of mathematical modelling: ICTMA 14* (pp. 15-30). Springer Science & Business Media. https://doi.org/10.1007/978-94-007-0910-2_3

Blum, W., & Borromeo-Ferri, R. (2009). Mathematical modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Application, 1*(1), 45-58.

Blum, W., & Leiß, D. (2007). Investigating quality mathematics teaching: The DISUM project. In C. Bergsten & B. Grevholm (Eds.), *Developing and researching quality in mathematics teaching and learning, proceedings of MADIF5*, (pp. 3-16). SMDF.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects: State, trends and issues in mathematics instruction. *Educational Studies in Mathematics, 22*(1), 37-68. https://doi.org/10.1007/bf00302716

Blum, W., & Schukajlow, S. (2018). Selbständiges Lernen mit Modellierungsaufgaben – Untersuchung von Lernumgebungen zum Modellieren im Projekt DISUM [Independent learning with modeling tasks - investigation of learning environments for modeling in the DISUM project]. In S. Schukajlow & W. Blum (Eds.), *Evaluierte Lernumgebungen zum Modellieren* (pp. 51-72). Springer Fachmedien Wiesbaden. https://doi.org/10.1007/978-3-658-20325-2_4

Bogdan, R. C., & Biklen, S. K. (2007). *Qualitative research for education: An introduction to theories and methods*. Pearson Education.

Cai, J., Cirillo, M., Pelesko, J., Borromeo Ferri, R., Borba, M., Geiger, V., Stillman, G. A., English, L., Wake, G., & Kaiser, G. (2014). Mathematical modelling in school education: Mathematical, cognitive, curricular, instructional, and teacher education perspectives. In *Proceedings of the 38th Meeting of the International Group for the Psychology of Mathematics Education* (pp. 145-172). PME.

Conner, A. M., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401-429. https://doi.org/10.1007/s10649-014-9532-8

Creswell, J. W. (2011). *Educational research: planning, conducting, and evaluating quantitative and qualitative research*. Pearson.

Czocher, J. A. (2018). How does validating activity contribute to the modelling process? *Educational Studies in Mathematics, 99*(2), 137-159. https://doi.org/10.1007/s10649-018-9833-4

de Oliveira, A. M. P., & Barbosa, J. (2013). Mathematical modeling, mathematical content and tensions in discourses. In G. A. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), *Teaching mathematical modeling: Connecting to research and practice* (pp. 67-76). Springer. https://doi.org/10.1007/978-94-007-6540-5_5

Guc, F. A., & Kuleyin, H. (2021). Argümantasyon kalitesinin matematiksel modelleme sürecine yansıması [Reflection of argumentation quality on mathematical modeling process]. *Uludağ Üniversitesi Eğitim Fakültesi Dergisi* [*Journal of Uludag University Faculty of Education*], *34*(1), 222-262. https://doi.org/10.19171/uefad.850230

Knipping, C. (2008). A method for revealing structures of argumentations in classroom proving processes. *ZDM–International Journal on Mathematics Education*, *40*(3), 427-441. https://doi.org/10.1007/s11858-008-0095-y

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb, & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229-269). Lawrence Erlbaum.

Lee, C. (2006). *Language for learning mathematics: Assessment for learning in practice*. McGraw-Hill Education.

Maaß, K. (2006). What are modelling competencies? *ZDM, 38*(2), 113-142. https://doi.org/10.1007/BF02655885

Maaß, K., Geiger, V., Ariza, & Goss, M. (2019). The role of mathematics in interdisciplinary STEM education. *ZDM, 51*(6), 869-884. https://doi.org/10.1007/s11858-019-01100-5

Manouchehri, A., Bekdemir, M., & Yao. X. (2020). Facilitating modelling activities in a grade 5 classroom. In G. A. Stillman, G. Kaiser, & C. Lampen (Eds), *Mathematical modelling education and sense-making. International perspectives on the teaching and learning of mathematical modelling* (pp. 187-197). Springer. https://doi.org/10.1007/978-3-030-37673-4_17

Mueller, M., Yankelewitz, D., & Maher, C. (2014). Teachers promoting student mathematical reasoning. *Investigations in Mathematics Learning, 7*(2), 1-20. https://doi.org/10.1080/24727466.2014.11790339

Petty, N., Thomson, O., & Stew, G. (2012). Ready for a paradigm shift? Part 2: Introducing qualitative research methodologies and methods. *Manual Therapy, 17*(5), 378-384. https://doi.org/10.1016/j.math.2012.03.004

Schukajlow, S., Kolter, J., & Blum, W. (2015). Scaffolding mathematical modelling with a solution plan. *ZDM Mathematics Education, 47*(7), 1241-1254. https://doi.org/10.1007/s11858-015-0707-2

Schukajlow, S., Leiss, D., Pekrun, R., Blum, W., Müller, M., & Messner, R. (2012). Teaching methods for modelling problems and students' task-specific enjoyment, value, interest and self-efficacy expectations. *Educational Studies in Mathematics, 79*(2), 215-237. https://doi.org/10.1007/s10649-011-9341-2

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematics discussions*. National Council of Teachers of Mathematics.

Solar, H., & Deulofeu, J (2016). Condiciones para promover el desarrollo de la competencia de argumentación en el aula de matemáticas [Conditions to promote the development of argumentative competence in the mathematics classroom]. *Bolema - Mathematics Education Bulletin, Rio Claro, 30*(56), 1092–1112. https://doi.org/10.1590/1980-4415v30n56a13

Solar, H., Ortiz, A., & Ulloa, R. (2016). MED: Modelo de formación continua para profesores de matemática, basada en la experiencia [MED: Model of continuous training for mathematics teachers, based on experience]. *Estudios Pedagógicos, 42*(4), 281-298. https://doi.org/10.4067/S0718-07052016000500016

Solar, H., Ortiz, A., Deulofeu, J., & Ulloa, J. (2021). Teacher support for argumentation and the incorporation of contingencies in mathematics classrooms. *International Journal of Mathematical Education in Science and Technology, 52*(7), 977-1005. https://doi.org/10.1080/0020739X.2020.1733686

Stillman, G. A., Kaiser G., & Lampen, E. (2020). Sense-making in mathematical modelling and applications educational research and practice. In G. A. Stillman, G. Kaiser, C. Lampen (Eds), *Mathematical modelling education and sense-making. International perspectives on the teaching and learning of mathematical modelling*. Springer. https://doi.org/10.1007/978-3-030-37673-4_2

Tekin-Dede, A. (2019). Arguments constructed within the mathematical modelling cycle. *International Journal of Mathematical Education in Science and Technology, 50*(2), 292-314. https://doi.org/10.1080/0020739X.2018.1501825

Toulmin, S. (2003). *The uses of argument.* Cambridge University Press. https://doi.org/10.1017/CBO9780511840005

Tropper, N., Leiss, D., & Hänze, M. (2015). Teachers' temporary support and worked-out examples as elements of scaffolding in mathematical modelling. *ZDM, 47*(7), 1225-1240. https://doi.org/10.1007/978-3-030-11069-7_4

Vorhölter, K. (2019). Enhancing metacognitive group strategies for modelling. *ZDM, 51*(4), 703-716. https://doi.org/10.1007/s11858-019-01055-7

Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *The Journal of Mathematical Behavior, 21*(4), 423-440. https://doi.org/10.1016/S0732-3123(02)00143-8

Yin, R (2014). *Case study research: Design and methods.* SAGE.