Journal of
Numerical Cognition

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Decoding Fact Fluency and Strategy Flexibility in Solving One-Step Algebra Problems: An Individual Differences Analysis

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Journal of Numerical Cognition, 2022, Vol. 8(2), 281–294, https://doi.org/10.5964/jnc.7093

Received: 2021-07-06 • **Accepted:** 2022-06-22 • **Published (VoR):** 2022-07-28

Handling Editor: André Knops, Université de Paris, Paris, France

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Supplementary Materials: Data, Materials [see Index of Supplementary Materials]

Abstract

Algebraic thinking and strategy flexibility are essential to advanced mathematical thinking. Early algebra instruction uses 'missing-operand' problems (e.g., x − 7 = 2) solvable via two typical strategies: 1) direct retrieval of arithmetic facts (e.g., 9 − 7 = 2) and 2) performance of the inverse operation (e.g., 2 + 7 = 9). The current study investigated the strategies people choose when solving these problems, and whether some people are more flexible in their choices than others. U.S. undergraduates (n = 59) solved missing-operand problems and made speeded verifications of arithmetic sentences corresponding to the direct- and inverse-matched facts. To 'decode' their strategy as direct or inverse, each participant's response times (RTs) for missing-operand problems were regressed on their RTs for the corresponding direct and inverse facts. Our findings replicated the problem size effect for the arithmetic verification task and extended this effect to missing-operand (i.e., one-step) algebra problems, suggesting that the two tasks draw on common representations and processes in the addition (but not subtraction) context. We found individual differences in strategy choice and flexibility such that participants varied both in terms of fluency for retrieving the direct fact and sensitivity to the potential benefit of switching to the inverse fact, which was validated by self-report. We did not find a predicted relation between strategy flexibility and standardized mathematical achievement. These findings inform our understanding of the cognitive processes involved in strategy flexibility in algebra and establish an RT-decoding paradigm for future examination of individual differences in students' learning of early algebra concepts.

Keywords

algebra, arithmetic, problem size effect, flexibility, individual differences, strategies

Success in algebra depends in large part on *fact fluency* (Fuchs et al., 2016) and *strategy flexibility* in problem-solving (Koedinger et al., 2008; Newton et al., 2010; Star & Newton, 2009; Star & Rittle-Johnson, 2008). Recent educational reforms, such as the Common Core State Standards for Mathematics initiative in the United States, have emphasized the value of being able to flexibly choose between multiple strategies and conceptually understand how they are mathematically related (National Governors Association, 2010). Solving simple algebra problems may be said to extend complex arithmetic thinking about inverse operations (Campbell, 2008; Peters et al., 2010; Torbeyns et al., 2018) to become a foundation for algebra, enabling more complex algebraic problem solving (Koedinger et al., 1997). Thus, investigating performance on problems that are commonly used in early algebra instruction provides a window into the shift from arithmetic thinking to the strategy flexibility characteristic of algebraic problem solving.

Early algebra instruction formalizes ‘open number sentences’ from arithmetic (e.g., [ ] − 7 = 2; Carpenter et al., 1988; Groen & Poll, 1973; Lamb et al., 2016) into one-step algebraic equations in one variable (e.g., $x - 7 = 2$; Herscovics & Linchevski, 1994; Pillay et al., 1998). Students in early algebra can solve these ‘missing-operand problems’ via at least two strategies. The *direct* strategy involves direct retrieval of the corresponding arithmetic fact (e.g., ***9*** − 7 = 2). The *inverse* strategy involves applying the inverse operation and then retrieving the inverse fact (e.g., 2 + 7 = ***9***). Importantly, moving from number-specific direct retrieval to number-general ‘inverse thinking’ is a cornerstone of algebraic problem solving, where the inverse strategy is formalized as ‘isolating’ the unknown (e.g., by adding 7 to both sides of $x - 7 = 2$; Pillay et al., 1998). We selected problems like $x - 7 = 2$ because they are conceptually linked to arithmetic problems, and their solution may involve elements of both arithmetic and algebraic problem solving.

The direct strategy involves bottom-up pattern-matching from the algebraic equation to its arithmetic fact, while the inverse strategy requires an additional top-down symbol-manipulation step before retrieval (i.e., transformation or re-representation; Anderson, 2005; Qin et al., 2003). Despite this additional step, the inverse strategy may sometimes be more efficient depending on features of the problem at hand (e.g., the size of the operands) and a person’s relative access to the direct and inverse arithmetic facts (i.e., *fluency*), which may vary both by problem features and the individual. Herscovics and Linchevski (1994) found that a class of seventh graders overwhelmingly used inverse strategies for missing-operand problems when direct fact retrieval was inaccessible. Thus, students who are *flexible* problem solvers may switch between direct and inverse strategies depending on their fluency for the direct fact and whether switching to the inverse strategy would benefit or cost their efficiency.

Essential to a deep understanding of mathematics is the ability to conceptually understand and choose appropriately between multiple strategies (Schneider et al., 2011). However, research suggests that strategy flexibility varies considerably between individuals. Children and adults both vary in flexibility in solving open number sentences for integer arithmetic (Bofferding & Richardson, 2013; Lamb et al., 2016). Algebra students also vary in their strategy flexibility (Star & Rittle-Johnson, 2008), and mere knowledge of multiple strategies is not sufficient for flexible use (Newton et al., 2010). Even experts, who tend to be more flexible, do not always choose the most efficient strategy (Star & Newton, 2009).

Much of the research on strategy flexibility in algebra has primarily used more complex problems involving multiple steps and multiple variables. However, success in solving simple algebra problems is foundational for and predictive of learning to solve more complex problems (Booth et al., 2014), and many students struggle to transition from arithmetic to algebra (Herscovics & Linchevski, 1994; Kieran, 1992; Pillay et al., 1998). Thus, investigating strategy flexibility in solving simple algebra problems can inform our understanding of strategy flexibility in algebra and its developmental trajectory.

The current study tested a new approach to ‘decoding’ strategy flexibility in simple algebra problems (i.e., one-step missing-operand problems) using participants’ response times. It also investigated individual differences in use of the direct vs. inverse strategy for solving missing-operand problems, as well as the potential association between strategy flexibility and mathematical achievement. As an antecedent step, it investigated temporal profiles of the retrieval of arithmetic facts during the solving of missing-operand problems. Accordingly, our hypotheses derived from the arithmetic problem solving literature.

Prior Research

Previous research has revealed reasoners’ cognitive processes in arithmetic (Ashcraft, 1992), which may undergird their understanding of algebra, especially missing-operand problems. One method used to solve arithmetic problems is retrieval, in which the relevant arithmetic fact is directly retrieved from memory to answer the question (Ashcraft & Guillaume, 2009; LeFevre et al., 1996). Research indicates that people are faster to evaluate arithmetic expressions or to verify arithmetic sentences involving ‘smaller’ operands (e.g., to evaluate 2 + 3 or verify 2 + 3 = 5) than ‘larger’ operands (e.g., to evaluate 8 + 9 or verify 8 + 9 = 17). This ‘problem size effect’ (Parkman & Groen, 1971) has been demonstrated robustly and throughout mathematical development (Ashcraft & Guillaume, 2009). The problem size effect is a key characteristic of the representations involved in arithmetic problem solving.

Reliance on direct retrieval to solve arithmetic problems is a common strategy even for young children (Ashcraft & Fierman, 1982; Siegler & Shrager, 1984). However, it is not the only strategy. People also use non-retrieval strategies, including more effortful transformational strategies. There are various transformational strategies used by young children learning arithmetic (Carpenter et al., 1981; Hiebert, 1982), but transformation remains common among adults, especially for larger problems. For example, as addition problem sums exceed 10, young adults become increasingly likely to self-report that they decomposed one operand to create a more familiar arithmetic fact (e.g., 7 + 4 = 7 + (3 + 1) = (7 + 3) + 1 = 10 + 1 = 11; LeFevre et al., 1996). Non-retrieval strategies may be especially prominent for subtraction, for which reasoners increasingly shift from retrieval to non-retrieval for minuends 11 and greater, sometimes using an inverse strategy for subtraction expressions (e.g., 13 − 7 = _) and recruiting the corresponding addition facts (e.g., 6 + 7 = 13) (Seyler et al., 2003). This subtraction-by-addition strategy is more common when the operands are large (Campbell, 2008) and when the subtrahend is considerably smaller than the difference (Peters et al., 2010; Torbeyns et al., 2018). It is even demonstrated among children, sometimes flexibly (Hickendorff, 2020; Torbeyns et al., 2018). However, it is less clear whether inverse transformations play a role in missing-operand problems, especially for single-digit operands (i.e., below 10), and how this may relate to individual differences in arithmetic fluency.

Earlier research has shown that young children struggle with open number sentences (e.g., [ ] − 3 = 5; Groen & Poll, 1973), especially when the unknown is the first operand or when represented as word problems (Briars & Larkin, 1984; Hiebert, 1982). Children's performance on such word problems improves after instruction mapping their informal strategies onto the mathematical symbols of open number sentences (Carpenter et al., 1988). Across these and other studies, children vary in their strategy choice, which can relate to achievement (Siegler, 1988) and working memory capacity (Seyler et al., 2003).

Finally, research has demonstrated that when individuals switch between strategies across problems, they may incur a 'switch cost' (Lemaire & Lecacheur, 2010). Despite this cost, most participants in the above studies were willing to use different strategies depending on problem characteristics. This suggests that people may also perceive a strategy 'switch benefit' for some problems which outweighs the switch cost enough to compel them to switch strategies for those problems. If so, sensitivity to this strategy switch benefit may be a key characteristic of flexibility in algebraic problem solving and mathematical achievement more generally. We examine this 'switch benefit' in missing-operand algebra problems and whether individuals vary in sensitivity to it.

The Current Study

These findings provide an arithmetic basis on which to investigate strategy flexibility in simple algebraic problem solving. Given the importance of simple algebra problems and the foundational nature of arithmetic to algebra, we sought to assess whether individual participants' arithmetic and algebraic problem-solving patterns are related for single-step missing-operand problems, and if so, whether an individual's unique patterns could be used to 'decode' their strategies for missing-operand problems. The current study does this through addressing five research questions.

The first and second research questions concern whether arithmetic and algebraic problem solving utilize common mental representations and processes. After ensuring replication of the problem size effect on the arithmetic verification task, the first research question extends the problem size effect to one-step algebra problems: Is there a problem size effect on the missing-operand task? The second research question connects arithmetic and algebraic problem solving through an individual differences analysis: Is an individual's problem size effect on the arithmetic verification task positively correlated with their problem size effect on the missing-operand task? Given the centrality of the problem size effect to the representations involved in arithmetic thinking, extending the problem size effect to one-step algebra problems and finding correlations between the effect in arithmetic and algebra tasks would suggest that participants may repurpose arithmetic facts for algebraic problem solving.

The third research question addresses the arithmetic-algebra relation at the problem level to measure individuals' flexibility in algebraic problem solving. Recall that there are two primary ways to solve missing-operand problems: by retrieving the direct fact or by transforming and retrieving the inverse fact. We predict that people will differ in strategy use as a function of their fluency for the direct fact and their ability to flexibly switch to the inverse retrieval when it is beneficial. To assess individual variation in flexibility, we formalize a *switch benefit* (alternatively, a *stay cost*) for missing-operand problems as the potential gain in switching from the direct fact − given its *dis*fluency − to the inverse fact (or alternatively, the potential penalty of staying with the direct fact given its *dis*fluency). We operationalize this at the participant level: On missing-operand problems (e.g., $x - 7 = 2$), is a given individual more likely to switch to the inverse strategy if their time to retrieve the direct arithmetic fact ($9 - 7 = 2$) is *longer* than their time to retrieve the inverse arithmetic fact ($2 + 7 = 9$) — that is, if direct retrieval of the corresponding arithmetic fact is relatively *dis*fluent for that individual on that problem? An individual's switch benefit for a problem may predict which strategy the individual uses to solve the problem. We propose a participant-level regression model in the analysis to address this question.

The fourth research question concerns the validity of this switch benefit operationalization. We assessed this question by comparing individuals' 'switch benefit' estimates from the regression model to their explicit strategy self-reports collected at the end of the study.

The fifth research question concerns the relation between strategy selection, flexibility, and mathematical achievement. If flexibility is important for mathematical thinking more generally, then is it the case that sensitivity to switch benefit when solving missing-operand problems is associated with higher mathematical achievement more generally (as measured by standardized ACT and SAT scores)? We predicted that sensitivity to switch benefit would be positively associated with mathematical achievement.

To address these research questions, we created an experimental paradigm to 'decode' participants' strategy choice flexibility from their response times (RTs). In this paradigm, participants make speeded judgements in the missing-operand task of interest (e.g., $x - 7 = 2$) and also in an arithmetic verification task where they verify the corresponding direct and inverse arithmetic facts (e.g., $\underline{\boldsymbol{9}} - 7 = 2$ and $2 + 7 = \underline{\boldsymbol{9}}$). We addressed the first and second research questions by analyzing participant RTs on these tasks as a function of problem size and also operation (addition or subtraction). We addressed the third research question by using regression to estimate how well each participant's missing-operand RTs are predicted by direct fact fluency and/or the benefit of switching to an inverse strategy. This switch benefit is operationalized for each participant using their own RTs on the corresponding arithmetic fact verifications. We addressed the fourth research question by correlating participants' switch benefit terms from the regression model with their self-reported use of the inverse strategy. We addressed the fifth research question by using regression to evaluate whether an individual's sensitivity to switch benefit was associated with their own standardized mathematical achievement score.

Method

Participants and Design

We recruited an initial sample of 62 of an intended 80 undergraduate students at a university in the Midwest U.S., before the onset of COVID-19 brought an end to face-to-face data collection. Because the experimental paradigm is new, we could not rely on the literature to estimate the number of participants and trials. We were instead guided by a pilot study in our lab of 64 participants experiencing 224 trials across the two tasks, which detected notable individual variation between-participants. Although we were unable to increase the sample size of the current study for external reasons beyond our control, we almost doubled the number of trials across the two tasks, to 432. (This is the number that they could complete in approximately 40 minutes before beginning to experience fatigue.) Given that most of our analyses are *within*-participants, we believe our increased number of trials provided adequate power. Two participants were excluded for having also participated in the pilot, and one participant was excluded because their accuracy was considerably lower than 90% on the tasks, leaving 59 participants.

Participant ages ranged from 18 to 23 years old (M = 20.04, SD = 1.28); 42 identified as women and 17 as men; and 45 identified their race as white, 12 as Asian, 1 as Black, and 1 as multiracial. Recruitment and study procedures were approved by the local IRB. Each participant received $12 USD compensation for the study, which lasted approximately 40 minutes.

Materials

The experiment was implemented using PsychoPy (Peirce et al., 2019). Participants were instructed in both tasks to respond to each trial as quickly and accurately as possible. Response time and accuracy were recorded. The experiment script and all stimuli files are provided as Supplementary Materials.

Missing-Operand Task

Participants were given one-step algebra problems and asked to identify the unknown (e.g., $x + 5 = 8$; $x - 7 = 2$) by pressing the corresponding number key (1 to 9) on the computer keyboard. There were 88 unique trials, each appearing twice, once in each of two blocks. The within-block trial order was randomly shuffled for each participant. The trials were constructed from all possible combinations of single-digit operands for addition and subtraction problems, excluding tie problems (i.e., equal operands), non-positive results (e.g., $x - 4 = -2$ was excluded), and problems with sizes less than 6 (e.g., $x + 3 = 5$ was excluded). This yielded 64 addition problems and 24 subtraction problems.

Arithmetic Verification Task

Participants were shown arithmetic sentences, which they had to judge as 'true' or 'false' by pressing the corresponding key as quickly and accurately as possible. The stimuli consisted of 129 true trials such as 9 − 7 = 2 and 127 false trials such as 9 − 7 = 4, which appeared once in one of two blocks.[1] Trial order was randomly shuffled for each participant with a break inserted halfway. The true trials were derived from the 88 missing-operand problems that participants also solved, e.g., for $x - 7 = 2$, both the direct arithmetic fact 9 − 7 = 2 and the inverse arithmetic fact 2 + 7 = 9 appeared as true arithmetic verification sentences. The problem size of each verification trial was defined by the sum of the two operands. The subtraction arithmetic verification trials had a wider range of problem size (16 in the example; 7 − 26 overall) than the addition arithmetic verification trials (9 in the example above; 6 − 17 overall). This was a consequence of the mathematical nature of the problems and also the experimental constraint that all answers to the missing-operand problems had to be a single digit to equate for motor output time. Following standard practice (Parkman & Groen, 1971), the false trials had the same left-hand side as true trials but a result that differed from the correct result by 2, either increased or decreased by 2 systematically across trials.[2] This preserved the parity relationship to the operands, e.g., the sum of two even numbers must be even. For data analysis, we used the RT of the true trials only; the false trials were included as controls.

ACT and SAT Scores

Participants' scores on the math section of the ACT or SAT tests were obtained with consent from university records. The ACT and SAT are comprehensive standardized tests used in U.S. university admissions and are used as estimates of mathematical achievement. The mathematics subtests of these exams include topics such as early algebra (linear equations and systems-of-equations), polynomial equations, probability and statistics, geometry, and trigonometry.

Procedure

For each task, participants first completed eight practice trials with feedback. The experimental trials that followed were divided into two blocks with a break halfway. The order of the two tasks was counterbalanced across participants. After finishing both tasks, participants completed a strategy questionnaire about how they solved the missing-operand problems and what percentage of the time they utilized each strategy type (i.e., direct vs. inverse). Lastly, they completed a demographics form.

1) The intended design was 128 true and 128 false trials, but due to an error in the stimuli file discovered in analysis, the true trial 2 + 9 = 11 was presented a second time instead of its corresponding false trial, 2 + 9 = 13. For data analysis, we used the RT of the intended true trial.

2) There were two necessary deviations. For false subtraction trials with a correct difference of 1 or 2 (e.g., 4 − 3), the results were always increased by 2 to prevent the introduction of negative results. For false addition trials with 2 as an operand (e.g., 2 + 7), the results were always increased by 2 (e.g., to 11) to prevent the sum from being equal to the second operand, in which case the statement could trivially be seen to be false by using surface features alone (e.g., 2 + 7 = 7).

Results

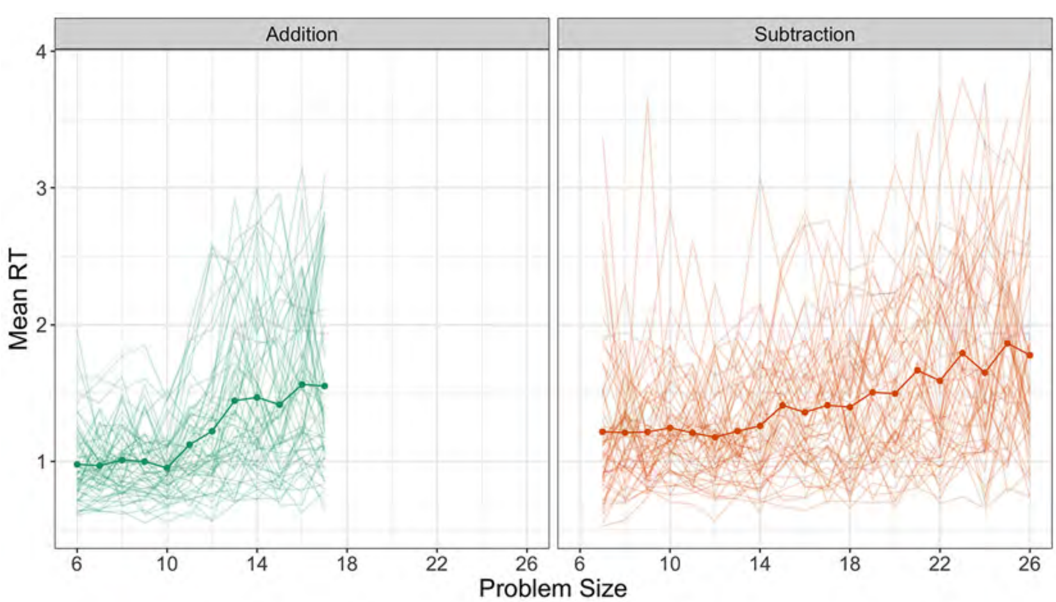
The analysis script, data files, codebooks, and survey are provided as Supplementary Materials. Additionally, Bayes Factors were computed in JASP (JASP Team, 2021) following Faulkenberry et al. (2020). All analyses include only response time (RT) data for trials in which the participant answered correctly and quickly – in less than 5 seconds for missing-operand problems and less than 4 seconds for arithmetic verification problems, as determined from a pilot experiment. All statistical tests were conducted two-tailed with $\alpha = .05$.

Problem Size Effects

Before answering our first research question, we assessed whether we replicated the problem size effect for the true trials in the arithmetic verification task. Each participant's mean RT for each problem size and operation type (addition and subtraction) are visualized in Figure 1, along with the overall means. For each participant and each operation type (addition and subtraction), we computed the Kendall's rank correlation (τ) between their RT and the problem size, operationalized as the sum of the operands.[3] For the addition problems, all participants had descriptively positive correlation coefficients, with individual τs ranging from .05 to .55 (*Median* = .29). The individual τ coefficients were significantly different from zero for 48 of the 59 participants. A one-sample *t*-test found that overall, participants' correlations for addition problems were different from zero, $t(58) = 18.8$, $p < .001$. For the subtraction problems, 57 of the 59 participants had descriptively positive correlation coefficients, with individual τs ranging from -.005 to .42 (*Median* = .19), of which 32 were significantly different from zero. Participants' correlations for subtraction problems were different from zero, $t(58) = 14.5$, $p < .001$. Thus, the problem size effect was replicated across both operations at the group level and also at the individual level for the majority of participants.

Figure 1

Problem Size Effects for Arithmetic Verification Task



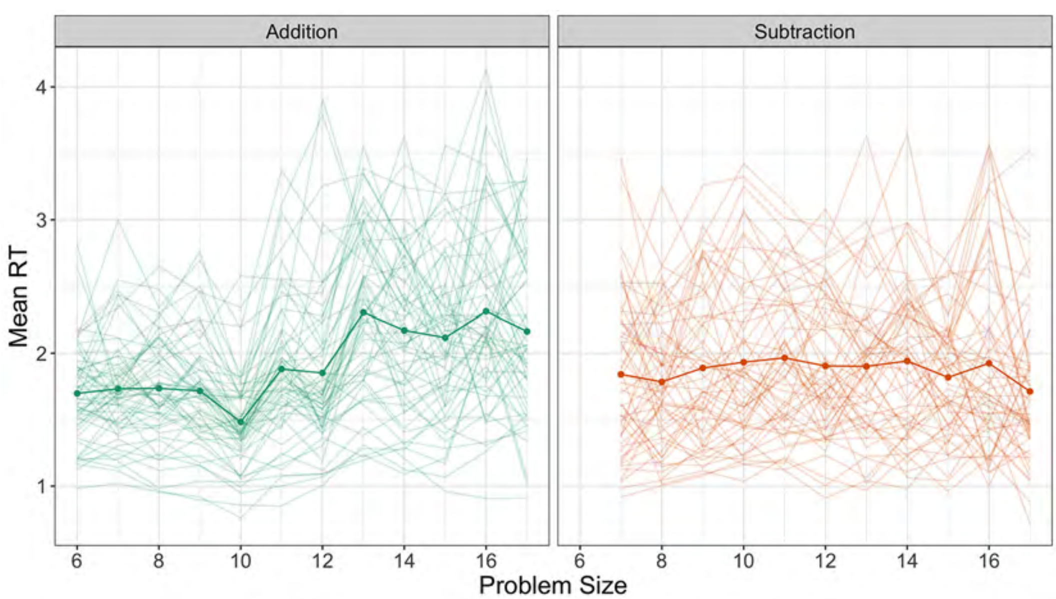
Note. Each line represents an individual participant's mean RT (*y*-axis) for each problem size (*x*-axis) for each operation (panels) in the arithmetic verification task. The thicker line indicates the mean across all participants.

3) For this and subsequent correlations, Kendall's τ is used instead of Pearson's *r* given the skew in the RT data.

The first research question was whether the problem size effect extended to the missing-operand problems (Figure 2). We operationalized the problem size as the sum of the problem's operands, one unknown and the other known. We computed the Kendall's rank correlation (τ) between RT and problem size for each participant and operation. For the addition problems, most participants had descriptively positive correlation coefficients (individual τs ranging from -.11 to .36, median of .20), and the individual τ coefficients were statistically different from zero for 36 of the 59 participants. A one-sample *t*-test found evidence that overall, participants' correlations for addition missing-operand problems were different from zero, $t(58) = 11.6$, $p < .001$. A Bayesian one-sample *t*-test (Faulkenberry et al., 2020) indicated that these correlations were more likely under a problem size effect model than the null model ($BF_{10} = 5.5 \times 10^{13}$). For the subtraction problems, participants were almost evenly split between positive and negative correlation coefficients (individual τs ranging from -.19 to .25, median of -.004), and only 2 correlations differed statistically from zero. Participants' correlations for subtraction problems were not significantly different from zero, $t(58) = 0.5$, $p = .61$. The Bayes factor was $BF_{01} = 6.211$, indicating the observed correlations are 6 times more likely under a null model. Thus, the problem size effect was extended from arithmetic problems to missing-operand problems, but only for the addition context. This provides initial evidence for some overlap in the mental representations and processes recruited for arithmetic, specifically addition, and simple algebra. Notably, we observe a dip in the mean RT for addition facts with a problem size of 10 (Figure 2, left). This may reflect practice effects given the role of 10 sums in 'transformation' strategies (LeFevre et al., 1996).

Figure 2

Problem Size Effects for Missing-Operand Task

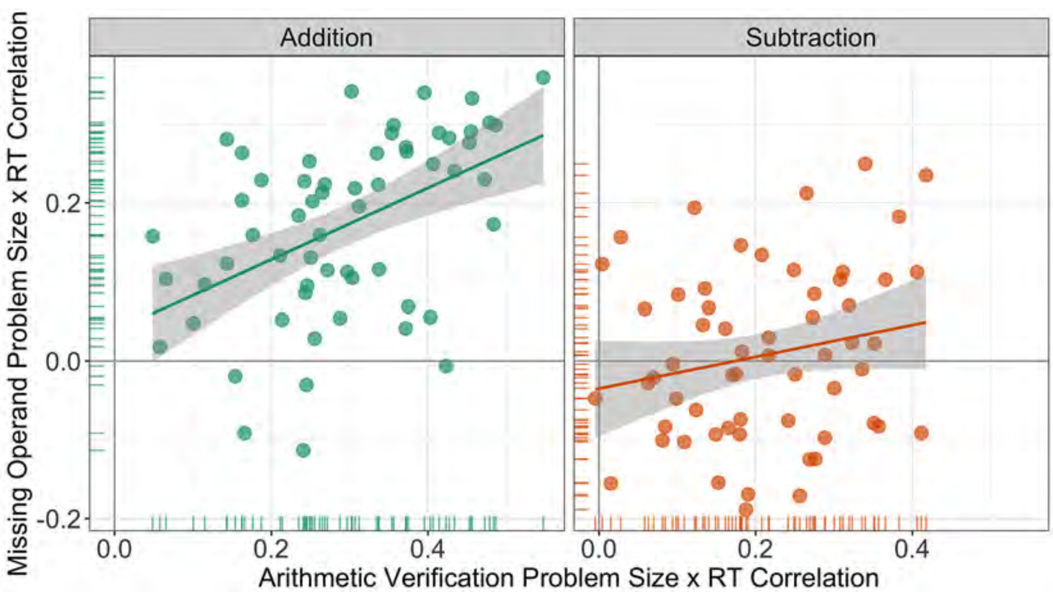


Note. Each line represents an individual participant's mean RT (*y*-axis) for each problem size (*x*-axis) for each operation (panels) in the missing-operand task. The thicker line indicates the mean across all participants.

The second research question concerned whether overlapping mental representations and processes are recruited for arithmetic and algebra. We addressed this question using an individual differences approach. Specifically, individuals varied in the strength of their problem size effect in both arithmetic and simple algebra contexts (Figures 1 and 2). We assessed whether these differences covaried across the tasks by performing a simple linear regression predicting participants' problem size effect for missing-operand problems from their problem size effect for arithmetic problems. For each problem class (i.e., missing-operand and arithmetic verification), the problem size effect was operationalized as the Kendall's rank correlation between the problem size and the RTs (Figure 3). We conducted these analyses separately for the addition and subtraction contexts. For the addition context, there was an association between the missing-operand and arithmetic problem size effects, b = .45, 95% CI [.23, .67]), $t(57)$ = 4.13, p < .001, R^2 = .23. A Bayesian linear regression indicated this association was more likely under the model where problem size effects are correlated across task than uncorrelated (BF_{10} = 191.1). This is consistent with the recruitment of the same mental representations and processes for the two problem classes. For the subtraction context, there was no such association, b = .20, 95% CI [-.05, .46], $t(57)$ = 1.58, p = .11, R^2 = .04. Here, the Bayes factor was BF_{01} = 1.34, indicating the observed correlations are slightly more likely under a null model, i.e., weak support for the absence of this correlation.

Figure 3

Between-Task Correlations of Problem Size Effect



Note. Scatterplot of each participant's Kendall's correlation coefficient between problem size and RT for arithmetic verification task (x-axis) and the missing-operand task (y-axis), separated by operation (panes). The marginal distributions are shown as rug plots. Point estimate (line) and confidence envelope (shaded region) from simple linear regressions were computed separately for each operation.

Flexible Strategy Use

The third research question asks whether participants are fluent and/or flexible across problems in their strategy selection during the missing-operand task. Can we 'decode' whether participants are sensitive to the benefit of switching to the inverse strategy when solving missing-operand problems? For this analysis, multiple linear regressions were run for each participant individually, using all cases where the missing-operand problem and the direct and inverse arithmetic facts were all three answered correctly and within the time cutoffs.

We operationalized the 'switch benefit' for missing-operand problems as the ratio of an individual's speed in retrieving the direct vs. inverse arithmetic facts during the arithmetic verification task:

$$\text{SwitchBenefit} = \frac{\text{DirectRT}}{\text{InverseRT}}$$

This switch benefit ratio reflects the degree to which switching from direct retrieval to the inverse strategy would *reduce* an individual's RT (or alternatively, the degree to which staying with direct would *cost* their RT). Consider the missing-operand problem $x - 7 = 2$. If participant A verified the direct match $9 - 7 = 2$ in 1500 ms and the inverse match $2 + 7 = 9$ in 1000 ms, then this ratio would be 1500 / 1000 = 1.5. That is, A would reduce their RT if they switched to the inverse route when solving $x - 7 = 2$. If participant B verified these same arithmetic facts in 1200 ms and 1800 ms, respectively, B's switch benefit ratio would be 1200 / 1800 = .67, indicating that they would *not* reduce their RT if they switched; instead, *B* should stick with direct retrieval. To measure whether participants were sensitive to their individual switch benefit, we included it as a term in a regression predicting their missing-operand RT. Note that this variable is a *reverse* indicator: The larger the switch benefit ratio, the greater the benefit of switching from the direct strategy to the inverse strategy. Thus, if a participant is sensitive to this ratio and switches adaptively to the inverse strategy, then this term will be *negatively* predictive of their missing-operand RT (by reducing RT).

Thus, we regressed each participant's RTs for missing-operand problems on their switch benefit term, and also on their RT for the directly matched arithmetic fact. The direct match RT operationalizes the participant's retrieval fluency. We also included block number (1 or 2) as a predictor to control for practice effects. The full equation is:

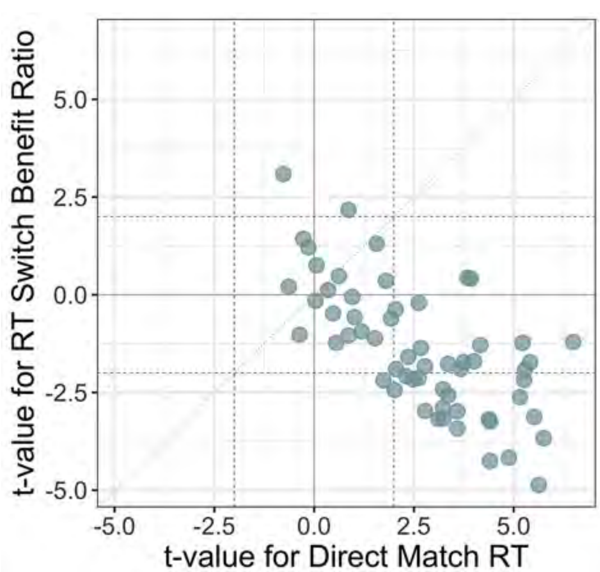
$$\text{MissingOperandRT} \sim \frac{\text{DirectRT}}{\text{InverseRT}} + \text{DirectRT} + \text{BlockNum}$$

Given the full regression model, the switch benefit ratio predictor thus measures strategy flexibility *over and above* fact fluency.

Figure 4 visualizes the results of the individual-level regressions. Each point shows the *t*-value for the direct RT predictor (*x*-axis) and the *t*-value for the switch benefit ratio predictor (*y*-axis) for a participant. The figure shows an overall trend (τ = -.49, p < .001) as well as a range of individual differences. When participants' missing-operand RTs are more *positively* predicted by their direct match RT (more positive on Figure 4's *x*-axis), they tend to also be more *negatively* predicted by their switch benefit (more negative on *y*-axis), as indicated by the overall negative trend between the two *t*-values. Those participants with positive *t*-values for direct RT may be inferred to have high arithmetic fact fluency, and those with negative *t*-values for switch benefit may be inferred to have both (1) sensitivity to the ratio of each fact's fluency (recall that this variable is reverse-coded) and (2) the strategy flexibility to switch to the inverse strategy when beneficial. This suggests an overall positive relationship between fact fluency and strategy flexibility. Additionally, there are clear individual differences, with a couple of participants positively predicted by switch benefit and not direct RT alone (top-middle of Figure 4), several by direct RT alone and not switch benefit (middle-right), some by both (lower-right), and some by neither (middle).

Figure 4

Regression Scatterplot



Note. Scatterplot of individual participants' *t*-values for predicting missing-operand problem RTs from the direct arithmetic match RT (*x*-axis) and the switch benefit (ratio of direct over inverse arithmetic match RTs; *y*-axis). Horizontal and vertical dashed lines indicate cutoffs for statistical significance, and the diagonal line represents identity between *t*-values.

The regression analyses indirectly estimated the switch benefit for each participant from their RTs. The fourth research question sought converging evidence for these analyses. Do the resulting estimates correspond to participants' self-reports about their strategy usage? We evaluated this by correlating the *t*-values for the switch benefit ratio to the percentage of the time they self-reported using the inverse strategy on the survey at the end of the experiment. As predicted, the two variables were negatively, though modestly, correlated, τ = -.25, p = .007. Thus, the more sensitive participants were to their switch benefit (the more negative the *t*-value associated with the switch benefit ratio variable), the higher their self-reported use of the inverse strategy. This correlation provides external validation for the strategy flexibility of participants that we indirectly measured (i.e., decoded) in our regression models.

The fifth research question asked whether strategy flexibility on missing-operand problems was associated with higher mathematical achievement more generally as measured by college entrance exam scores. We obtained standardized test scores for 58 of the 59 participants, 50 for the ACT college entrance exam and 8 for the SAT; the latter were converted to equivalent ACT scores using a concordance table.[4] We fit a regression model predicting their ACT-Math (or equivalent) scores from the *t*-value for their switch benefit ratio predictor and the *t*-value for their direct RT predictor. In contrast to our predictions, we did not find a significant negative relationship between switch benefit and ACT-Math scores, b = -0.65, $t(55)$ = -1.53, p = .13. However, direct match fluency was a significant and negative predictor of ACT-Math, b = -1.09, $t(55)$ = -2.92, p = .005. We did not make any explicit hypotheses regarding this latter relation and do not have a clear interpretation of it. (These two results remained the same after removing an outlier with an ACT-Math score more extreme than 3 SDs below the mean, p = .26 and p = .027, respectively.) A Bayesian linear regression with both terms revealed the model with the highest odds increase from prior to posterior included only the direct match fluency predictor (BF_M = 2.5), followed by the model with both predictors (BF_M = 1.9). These results provide moderate evidence that direct match fluency is associated with ACT-Math scores (posterior inclusion BF = 5.2), but not switch benefit (inclusion BF = 0.73).

Discussion

The current study investigated the strategies that undergraduate students use when solving simple algebra (missing-operand) problems. We developed an RT-decoding paradigm to infer whether a person uses bottom-up pattern-matching to retrieve a *direct* arithmetic fact or, depending on the problem, whether they switch from retrieving the direct fact to retrieving the *inverse* arithmetic fact (i.e., applying top-down algebraic rules). This paradigm revealed individual differences in undergraduates' strategy choice. Namely, the correlation between the *t*-values for direct match and switch benefit terms suggests that many participants who are highly *fluent* for direct facts tend to also be highly *flexible* in choosing a solution strategy based on the benefit of switching. Our results extend prior research (Campbell, 2008; Peters et al., 2010) by providing evidence for individual variation, particularly that individuals vary in their sensitivity to the benefit of switching to an inverse strategy, and that they do so for smaller operands than have previously been studied in arithmetic (Hickendorff, 2020; Seyler et al., 2003; Torbeyns et al., 2018).

More specifically, we examined five research questions. First, we extended the problem size effect from arithmetic verification to missing-operand problems, but only for addition problems; participants did not consistently show this effect for subtraction problems. Second, we conducted an individual differences analysis of whether participants' problem size effect for missing-operand problems predicted their problem size effect for arithmetic problems. We found a significant relationship for addition, suggesting that missing-operand problems in addition recruit similar mental representations and processes to arithmetic verification. There was no such relationship for subtraction. Third, we 'decoded' individual differences in missing-operand RTs through individual-level regression models, predicting missing-operand performance by an individual's RTs for both the direct arithmetic fact and the 'switch benefit' ratio of direct vs. inverse facts. We found an overall trend where higher direct fact fluency was associated with higher sensitivity to

4) We used the 2018 ACT/SAT Concordance Tables published by ACT, Inc. We judged 2018 to be the year that most participants in our sample likely took the ACT, i.e., the Spring of their junior year and the Fall of their senior year in high school. The tables are available for download: https://www.act.org/content/dam/act/unsecured/documents/ACT-SAT-Concordance-Tables.pdf

the switch benefit, suggesting higher strategy flexibility. Fourth, we validated the RT-based switch benefit measures by participants' self-reported use of the inverse strategy. Fifth, we failed to find a predicted relationship between our 'decoded' strategy flexibility variable (i.e., switch benefit) and standardized mathematics scores.

The predicted extension of the problem size effect to missing-operand problems (our first research question) and the correlation between problem size effects across arithmetic problems and missing-operand problems (our second research question) were only found for addition problems. The mathematical cognition literature offers multiple possible explanations for this asymmetry. It may be that subtraction facts are less fluent for direct recall compared to addition facts, perhaps due to less practice (Ashcraft & Guillaume, 2009). Alternatively, reasoners may solve subtraction problems by transformation to addition (e.g., Peters et al., 2010), although previous studies demonstrating adults' inverse strategy used considerably larger subtraction facts (e.g., 81 − 37 = _), with Seyler et al. (2003) finding it with minuends as low as 11. By comparison, all of our minuends and subtrahends were single-digit numbers. Future research might attempt to extend the current findings to missing-operand problems for another pair of operations, multiplication and division, which are also inversely related.

Of primary theoretical importance are the individual-level regression models that implicitly 'decode' participants' strategies during the missing-operand task. Prior work has shown individual differences in students' (i.e., children's) strategy flexibility (Lamb et al., 2016; Star & Rittle-Johnson, 2008). Here, we show that even undergraduate students appear to vary in strategy across missing-operand problems with single-digit operands, and moreover this is conditioned on their own fact fluency. Many participants appeared sensitive to a switch benefit for performing an inverse operation over direct retrieval when their individual fluency was higher for the inverse fact. That is, participants varied not just in their flexibility but in how *efficient* their flexibility may be.

These 'decoding' results were validated by explicit self-report. While self-report of strategies can be informative (LeFevre et al., 1996), there are also limitations on its reliability: Self-report can be less veridical for automatic processes, and task descriptions can induce demand characteristics (Kirk & Ashcraft 2001). While this may have played a role in the present study's self-report data, any unreliability of self-report would only attenuate the correlation estimate, meaning the true correlation would be higher. Thus, the fact that participant self-report for the inverse strategy correlates with decoded sensitivity to the trial-level switch benefit provides convergent evidence for the individual differences found in the current study.

Future studies should investigate performance characteristics of the direct and inverse strategies for missing-operand problems to further assess the validity of this study's paradigm. One way to do so is by varying the instructions given to participants about which strategy to use when solving problems or whether they can choose (i.e. choice/no-choice); another is by varying problem characteristics that are more or less conducive to certain strategies. Such work has already improved our understanding of strategy choice in arithmetic problems (Eaves et al., 2019; Hickendorff, 2020; Siegler & Lemaire, 1997; Torbeyns et al., 2018).

This study built on prior work emphasizing the importance of strategy flexibility to student success in algebra (Star & Rittle-Johnson, 2008) by investigating strategy flexibility in missing-operand problems, which involve elements of both arithmetic and algebra. We found that arithmetic representations play a crucial role in algebra problem solving strategy. RT-based regression models decoded individual differences in undergraduates' strategy choice and brought to light individual differences in fact fluency and flexibility based on participants' own fact retrievability. Our RT-based decoding paradigm is promising for future work examining when and how such differences emerge in younger students learning early algebra, as it allows for the potential identification of the cognitive processes that enable reasoners to flexibly select strategies for solving algebra problems.

Funding: The authors have no funding to report.

Acknowledgments: The authors thank Kori Terkelsen, Em Hayward, and Corissa Rohloff for their assistance running participants.

Competing Interests: The authors have declared that no competing interests exist.

Data Availability: For this article, a data set is freely available (Bye, Harsch, & Varma, 2021).

Supplementary Materials

The Supplementary Materials contain the following items (for access see Index of Supplementary Materials below):

- Experiment script and stimuli
- Data files
- Codebooks
- Analysis script
- Survey

Index of Supplementary Materials

Bye, J. K., Harsch, R. M., & Varma, S. (2021). *Supplementary materials to "Decoding fact fluency and strategy flexibility in solving one-step algebra problems: An individual differences analysis"* [Research data, codebook, code, and materials]. OSF. https://osf.io/gxypt

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Journal of Numerical Cognition (JNC) is an official journal of the Mathematical Cognition and Learning Society (MCLS).

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