Journal of Numerical Cognition

PsychOpen GOLD

Empirical Research

Check for updates

Perceptual and Number Effects on Students' Initial Solution Strategies in an Interactive Online Mathematics Game

Ji-Eun Lee [1], Caroline Byrd Hornburg [2], Jenny Yun-Chen Chan [1], Erin Ottmar [1]

[1] *Department of Social Science & Policy Studies, Worcester Polytechnic Institute, Worcester, MA, USA.* **[2]** *Department of Human Development and Family Science, Virginia Tech, Blacksburg, VA, USA.*

Journal of Numerical Cognition, 2022, Vol. 8(1), 166–182, https://doi.org/10.5964/jnc.8323

Received: 2020-10-08 • **Accepted:** 2021-05-28 • **Published (VoR):** 2022-03-31

Handling Editor: Jo-Anne LeFevre, Carleton University, Ottawa, Canada

Corresponding Author: Ji-Eun Lee, Learning Sciences & Technologies, Worcester Polytechnic Institute, 100 Institute Rd, Worcester, MA 01609, USA. E-mail: jlee13@wpi.edu

Abstract

This study investigated the effects of 1) proximal grouping of numbers, 2) problem-solving goals to make 100, and 3) prior knowledge on students' initial solution strategies in an interactive online mathematics game. In this game, students transformed an initial expression into a perceptually different but mathematically equivalent goal state. We recorded students' solution strategies and focused on the productivity of their first steps—whether their initial action led them closer to the goal. We analyzed log data within the game from 227 middle-school students solving four addition problems and four multiplication problems consisting of a total of 1,816 problem-level data points. Logistic regression modeling showed that students were more likely to use productive initial solution strategies to solve addition and multiplication problems when 1) proximity supported number grouping, 2) 100 was the problem-solving goal, and 3) students had higher prior knowledge in mathematics. Furthermore, when problem-solving goals were non-100s, students with lower prior knowledge were less likely to use productive initial solution strategies than students with higher prior knowledge. The findings of the study demonstrated that perceptual and number features influenced students' initial solution strategies, and the effect of number features on initial solution strategies varied by students' prior knowledge. Results yield important implications for designing instructional activities that support mathematics learning and problem-solving.

Keywords

mathematical strategies, perceptual grouping, individual differences, mathematical structure, algebraic knowledge

Mathematics problems can often be solved by using several different strategies. As an example, students can solve the equation $2(3 + y) = 16$ with a three-step standard strategy (i.e., distribute 2 into the parentheses, subtract 6 from both sides, and divide both sides by 2) or a two-step efficient strategy (i.e., divide both sides by 2, and subtract 3 from both sides). Although efficient and flexible problem solving is a primary goal in K-12 mathematics education in the United States (NGA Center & CCSSO, 2010), students often apply standard procedures by rote without noticing important patterns in problem structures (e.g., the multiplicative relation between 2 and 16) that afford more efficient strategies (Carpenter et al., 1980; Star & Rittle-Johnson, 2008). Prior studies have demonstrated that perceptual and conceptual problem structures (Alibali et al., 2018; Geary, Hoard, Byrd-Craven, & DeSoto, 2004; Landy & Goldstone 2010; Lemaire & Callies, 2009) and students' prior knowledge (Siegler, 1988) impact problem-solving performance. In this study, we examined the effects of problem structures (i.e., proximal grouping of numbers, problem-solving goals of making 100), and middle-school students' (age 11 to 13 years) prior knowledge on their initial solution strategies (see Figure 1) within an interactive online mathematics game that involves transforming expressions (e.g., “47 + 33 + b + 52 + 68”) into a specified goal state (e.g., “99 + b + 101” in Figure 2). We describe each of these factors in the following sections.

Figure 1

A Conceptual Model Linking Student-Level and Problem-Level Predictors to the Students’ Initial Solution Strategies

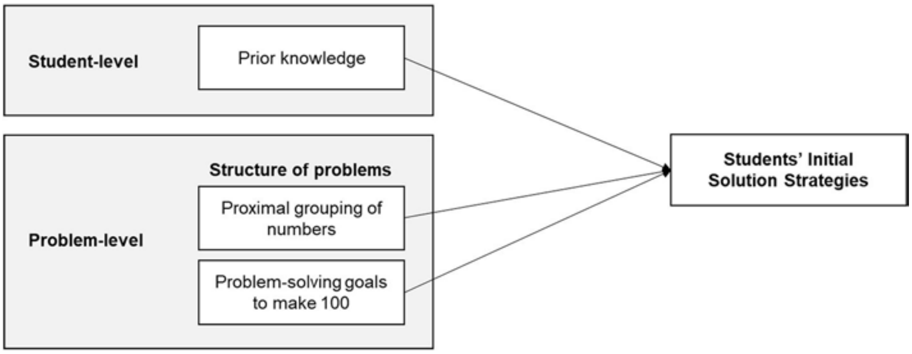
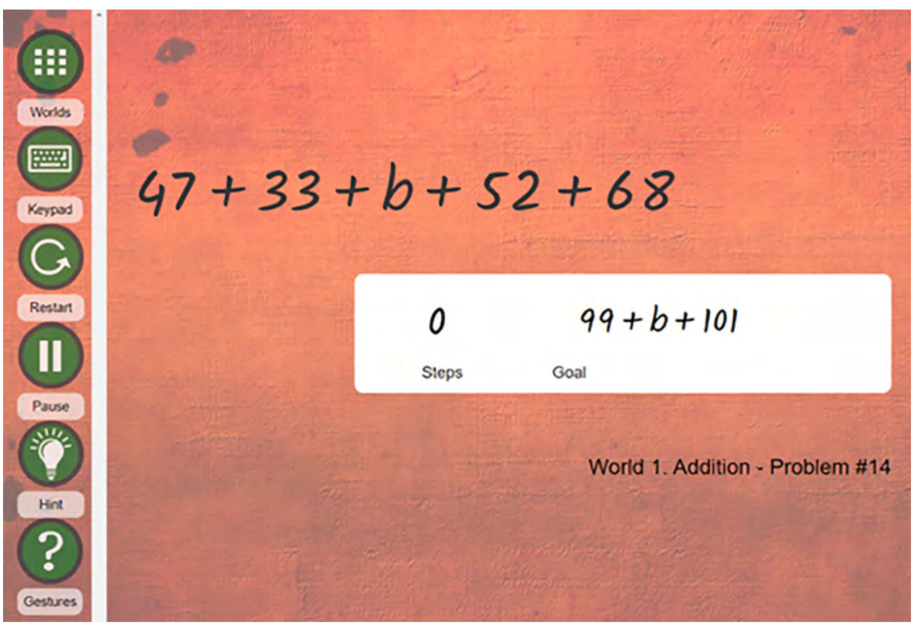


Figure 2

An Example of a Problem Consisting of a Start State (47 + 33 + b + 52 + 68) and a Goal State (99 + b + 101)



Factors That May Affect Students’ Solution Strategies

Proximal Grouping of Numbers

Substantial empirical work suggests that mathematical reasoning is grounded in perceptual and embodied processes (Alibali & Nathan, 2012; Goldstone et al., 2010; Kirshner & Awtry, 2004; Marghetis et al., 2016). For instance, undergraduate students use proximity as a perceptual cue to group symbols aligning with the order of operations, and write numbers around the multiplication sign closer than around the addition sign (e.g., 2   +   3 × 4; Landy & Goldstone, 2007). Irrespective of education levels, students tend to interpret and solve expressions incorrectly (i.e., adding before multiplying) when the symbols are spaced in a manner that is incongruent with the order of operations (e.g., 2 + 3   ×   4; Braithwaite et al., 2016; Harrison et al., 2020; Jiang et al., 2014).

Here, we test a similar concept but operationalize proximal grouping as the location of numbers within the problem rather than the physical spacing of numbers and operators within the problem. For instance, in 3 + 5 + 4 = 8 + 4, 3 and 5 are adjacent to each other, whereas 3 and 5 are not adjacent to each other in the equation 3 + 4 + 5 = 8 + 4. Drawing on the Gestalt laws of perceptual organization (Hartmann, 1935), stimuli that are proximal in terms of spatial location are more likely to be grouped together than stimuli that are non-proximal. Thus, we hypothesize that students will be more likely to make a productive first step towards the goal (e.g., 99 + b + 101) when the numbers to be added are adjacent to each other (e.g., 47 + 52 + b + 33 + 68) as opposed to farther apart (e.g., 47 + 33 + b + 52 + 68).

Problem-Solving Goals to Make 100

The mathematical system in the U.S. has an underlying base-10 structure. Base-10 structure knowledge is a key predictor of students' mathematics performance (Geary, 2006; National Council of Teachers of Mathematics, 2000; National Research Council, 2001) and is related to advanced problem-solving strategies such as decomposition (Laski et al., 2014). Several studies have shown that adults are faster and more accurate on addition problems that sum to 10 compared to those that do not sum to 10 (Aiken & Williams, 1973; Krueger & Hallford, 1984). Similarly, adults are faster and more accurate on problems in which answers are multiples of 5 as opposed to not multiples of 5 (Campbell, 1995). Adults also demonstrate superior performance on problems with 10 as a multiplicand (e.g., 16 × 10) compared with those that do not contain 10 as a multiplicand (e.g., 16 × 13; Siegler & Lemaire, 1997).

State mathematics standards tend to focus on students' facility in making 10 or 100. In the U.S., one Kindergarten (age 5 or 6 years) standard is "For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation" (NGA Center & CCSSO, 2010). Furthermore, K-12 curricula activities often focus on addition and subtraction with 10 or 100. For example, *Everyday Math* includes the vocabulary term "friendly number," which is described as "a number that is easy to work from, typically 10 or a multiple of 10." (University of Chicago School Mathematics Project, 2014). In the upper elementary grades (age 9 to 11 years), standards involve flexible addition/multiplication of 10 and 100 (NGA Center & CCSSO, 2010), and curricular activities also focus on addition combinations to reach 100 (Pearson Education, Inc, 2008).

Given this focus on 10s and multiples of 10s, we anticipate that 100 is a "friendly number" for which students can create combinations more easily as compared to other numbers. Thus, we hypothesize that students will be more likely to make productive transformations if the problem goal is to make 100 compared to problems in which the goal is to make numbers that are not 100 (e.g., 98; 102).

Prior Knowledge

To tailor strategies to a mathematical equation flexibly and adaptively, students need to have mathematical content knowledge—understanding the concepts and procedures for potential solutions (Star & Rittle-Johnson, 2008). Researchers have found the association between mathematical knowledge and strategy use across age groups and mathematics topics. For example, when solving arithmetic equations, second-grade students (age 7 to 8 years) with lower mathematics achievement tend to stick with one familiar strategy, whereas students with higher mathematics achievement use a variety of strategies and tend to find the answer more quickly and accurately (Torbeyns, Verschaffel, & Ghesquière, 2006). Similarly, middle-school students (age 11 to 14 years) with higher mathematics achievement are more likely to use a strategy that involves fewer steps when solving algebraic equations compared to students with lower mathematics achievement (Newton et al., 2020; Wang, Liu, Star, Liu, & Zhen, 2019). Furthermore, the influence of mathematical knowledge on strategy use has been experimentally tested in classroom studies that aim to improve students' algebraic knowledge through teacher instruction. After participating in an instructional intervention that teaches a variety of strategies for solving algebraic equations, sixth-grade students (age 11 to 12 years) tend to use more efficient strategies that involve fewer steps during algebraic equation solving (Star & Rittle-Johnson, 2008). Here, we hypothesize that students with higher prior algebraic knowledge will be more likely to use productive solution strategies than students with lower prior algebraic knowledge.

In addition to the main effects of problem structures and prior knowledge, students' prior knowledge may moderate the effects of problem structures (proximal grouping of numbers, problem-solving goals to make 100) on students' solution strategies. One study (Novick, 1988) examined the relation between different types of problem features (e.g., structural features, surface features) and undergraduate students' level of expertise in their transfer of mathematical knowledge. The results revealed that mathematics experts were more likely to identify structural analogs for problems and showed more positive transfer across problems, whereas novices were more likely to notice surface features that did not support positive knowledge transfer. Similarly, Sidney and Alibali (2015) noted that perceiving structural features of problems is often challenging for novice students because they tend to focus on surface attributes. However, drawing fifth- and sixth-graders' (age 10 to 12 years) attention to problem structures (e.g., similarities between whole number and fraction division) helps support their mathematics learning and transfer (Sidney & Alibali, 2017).

Here, we examine the influence of students' prior knowledge on the association between problem structures and initial solution strategies. We hypothesize that there may be a significant interaction effect between students' prior knowledge and the problem structures. Specifically, students with lower prior knowledge may benefit from the support in problem structures; they may use more productive initial solution strategies when the numbers to be added are adjacent to each other, and the goal is to make 100. Students with higher prior knowledge may use productive solution strategies regardless of whether the problem structures support these strategies.

The Present Study

The goal of this study is to understand the ways in which specific factors—the proximal grouping of numbers, problem-solving goals of making 100, and prior knowledge—affect students' initial solution strategies in an interactive online mathematics game. In this game, students were presented with a series of problems consisting of a starting expression and a mathematically equivalent but perceptually different goal state; the objective was to transform the expression into the goal state. We focus on three main research questions:

1. Do proximal grouping, problem-solving goals of making 100, and prior knowledge uniquely influence students' initial solution strategies?
2. Do these factors interact to affect students' initial solution strategies?
    a. Interaction between proximal grouping and problem-solving goals of making 100
    b. Interactions between proximal grouping and prior knowledge as well as problem-solving goals of making 100 and prior knowledge
    c. Three-way interaction among proximal grouping, problem-solving goals of making 100, and prior knowledge
3. Are these effects consistent across addition and multiplication problems?

First, we hypothesize that students may use a productive initial solution strategy when (a) the numbers to be combined are adjacent to each other, (b) the problem-solving goal is to make 100, and (c) students have higher prior knowledge. Second, students may be more likely to use a productive initial solution strategy when the numbers to be combined are adjacent to each other *and* the goal is to make 100, and students with lower prior knowledge may be more likely to benefit from these problem structures. Third, we explore whether these effects are consistent in addition and multiplication problems, and we do not have a directional hypothesis on this comparison.

Method

Participants

This study used data from a larger randomized controlled trial conducted in six middle schools located in the southeastern U.S. In the larger study, we examined the efficacy of an interactive online mathematics game compared to an online problem set (see details in Chan et al., 2022). In the current study, we used data from 227 students who received the game as their intervention and completed at least 50% of the items on the pretest (see prior knowledge measure below). This 50% cut-off approach, determined in our prior study (Chan et al., 2022), allowed us to include as much data as possible from students who completed at least enough of the pretest to yield an adequate estimate of their mathematical understanding.

In the final sample of 227 students (56% male, 44% female; age 11 to 13 years), most students (96%) were in sixth grade, and the rest (4%) were in seventh grade. In terms of instruction level, 85% of students were in advanced mathe-

PsychOpen GOLD

matics classes (i.e., accelerated mathematics programs) that were designed for students who excelled at mathematics and implemented more challenging course materials, 6% were in regular on-level classes, and 9% were in support classes that were designed for students who needed additional help in learning mathematics. Although only 4% of the sample were seventh-grade students, these students were included in analyses because they were enrolled in seventh-grade support mathematics classes that were comparable to sixth-grade on-level mathematics classes. The race/ethnicity of the final sample was 53% Asian, 36% White, 4% Hispanic, and 7% other races/ethnicities. This sample consisted of a larger percentage of Asian students in comparison with the district-wide population (52% White, 25% Asian, 15% Hispanic, 8% other races/ethnicities).

Materials

In the present study, we analyzed log data collected in an interactive online mathematics game in which students explore algebraic notations by performing mouse- or touch-based gestures. In the game, mathematical symbols are turned into virtual objects that students can touch and move according to mathematical principles.

For each problem in the game, students are presented with two mathematical expressions—a start state, which is a transformable expression, and a goal state, which is perceptually different but mathematically equivalent to the start state (see Figure 2). The objective of the game is to transform the start state (47 + 33 + b + 52 + 68 in Figure 2) into the goal state (99 + b + 101 in Figure 2) by using various permissible gesture-actions (e.g., dragging, tapping). As students manipulate the expression (e.g., dragging 52 on top of 47), the system automatically performs the computation (e.g., 52 + 47) and reveals the result of the transformation (e.g., 99 + 33 + b + 68). With these opportunities to manipulate the expressions, students can learn that mathematical transformations are dynamic rather than procedural steps. Prior research on the game has shown that it is effective in improving students' algebraic understanding and mathematics performance (Chan et al., 2022; Hulse et al., 2019).

Problem Structure

Within the game, we designed and embedded two quartets of problems—four addition problems (P10, P7, P13, P14) and four multiplication problems (P24, P32, P30, P26), that each varied on the proximal grouping of numbers in the start state and making 100 in the goal state (see Table 1).

Table 1

The Design of Problem Structure

	Proximal Grouping	
Making 100	**Yes (1)**	**No (0)**
Addition		
Yes (1)	P10: (S) 44+56+a+72+28 ⟶ (G) 100+a+100	P7: (S) 11+55+y+89+45 ⟶ (G) 100+y+100
No (0)	P13: (S) 15+87+c+66+32 ⟶ (G) 102+c+98	P14: (S) 47+33+b+52+68 ⟶ (G) 99+b+101
Multiplication		
Yes (1)	P24: (S) 25*4*b*50*2 ⟶ (G) 100*b*100	P32: (S) 10*20*a*10*5 ⟶ (G) a*100*100
No (0)	P30: (S) 8*12*d*3*32 ⟶ (G) 96*d*96	P26: (S) 4*6*c*24*16 ⟶ (G) 96*96*c

Note. S = Start state. G = Goal state.

For instance, for Problem 10 (Start state [S]: 44 + 56 + a + 72 + 28, Goal state [G]: 100 + a + 100), transforming 44 + 56 + a + 72 + 28 into 100 + a + 100 involves proximal grouping in the start state and making 100 in the goal state (i.e., 44 and 56 are adjacent to each other and make 100). In contrast to this problem, transforming 47 + 33 + b + 52 + 68 into 99

+ b + 101 (Problem 14) involves non-proximal grouping of numbers and making non-100 (i.e., 47 and 52 are non-adjacent to each other and make 99). In order to code the problem structure, we created two dummy variables; the proximal grouping of two numbers (e.g., P10: [S]: 44 + 56 + a + 72 + 28) was coded as 1 (non-proximal grouping as a reference category), and the problem-solving goal of making 100 (e.g., P10: [G] 100 + a + 100) was coded as 1 (making non-100 as a reference category).

Measures

Prior Knowledge

Students' prior algebra knowledge was measured with 11 items selected from two previously validated measures (Rittle-Johnson et al., 2011; Star et al., 2015). Each item was scored as correct (1) or incorrect (0). If a student did not attempt an item, it was scored as incorrect. An example item is $5\ (y - 2) = -3\ (y - 2) + 4$, solving for y. The Kuder–Richardson 20 coefficient for the 11 items was .70, indicating an acceptable level of reliability.

Productivity of Initial Solution Strategy

Initial solution strategies were measured by whether or not students made a productive mathematical transformation (i.e., change of expression) towards the goal state in their first transformation. All mathematical transformations made by the students were automatically logged in the database. We extracted and used students' first mathematical transformation (i.e., "first step") to measure the productivity of their initial solution strategies. We focus on students' first step because it provides insights into students' initial reaction to the problem, has impacts on their subsequent steps in reaching the goal state, and consequently influences their overall strategy efficiency on the problem.

For example, in Problem 14 (S: 47 + 33 + b + 52 + 68 ⟶ G: 99 + b + 101), the first step of transforming "47 + 33 + b + 52 + 68" into "99 + 33 + b + 68" was coded as a productive first step because the student moved closer to the goal state by combining 47 and 52 to make 99 (see Table 2). An example of a non-productive first step is transforming the start state into "80 + b + 52 + 68" by combining 47 and 33. This is coded as non-productive because "80" does not match or bring students closer to making any numbers in the goal state.

Table 2

Examples of Productivity of Students' Initial Solution Strategies

Start state	Goal state	Productive first steps	Non-productive first steps
47+33+b+52+68	99+b+101	99+33+b+68	80+b+52+68
		47+b+52+101	47+33+b+120
		33+b+99+68	47+b+85+68
		47+101+b+52	115+33+b+52
		47+52+33+b+68[a]	33+b+52+68+47
		47+b+52+33+68[a]	

[a]Transformations that involved commuting (i.e., moving numbers to be added adjacent to each other) were considered productive in bringing the student closer to the goal state.

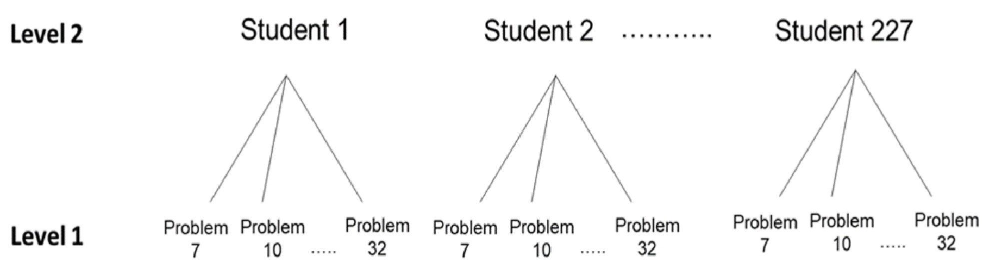
Table 2 lists examples of productive and non-productive first steps. We hand-coded the productivity of solution strategies as productive (1) or non-productive (0). The intraclass correlation coefficient of the coding was .97, indicating excellent reliability. All strategies on all problems were coded by two coders, and the discrepancies were discussed and resolved prior to analyses.

Data Analysis

We performed hierarchical binary logistic regression modeling because 1) the outcome variable (productivity of initial solution strategies) was binary, and 2) problem-level data were nested within student-level data. First, unconditional models were estimated to examine the proportion of the variance explained in the outcome variable (i.e., the productivity of students' initial solution strategies) between the schools, classrooms, and students. These intraclass correlation coefficients (ICCs) ranged between 0.00 and 0.03. Although the ICC values were low (less than 0.05), we conducted hierarchical binary logistic regression modeling because both Level-1 predictors (proximal grouping, making 100) and a Level-2 predictor (prior knowledge) were included in the models, and each student completed the same eight problems in the study (Figure 3).

Figure 3

The Hierarchical Structure of the Data



The number of problem-level data points included in the analyses was 1,816 problems, 908 for addition and 908 for multiplication. For students who tried the problems more than once, we used the data from their first attempts. We used R studio with the lme4 package for further analyses (Bates et al., 2015).

Results

Before performing data analyses, we computed frequencies of students' solution strategies for each problem (see Table 3) and descriptive statistics for prior knowledge. As shown in Table 3, more than 90% of students made productive first steps for most problems, except for problems 14, 32, and 26. The average raw (i.e., uncentered) prior knowledge score was 6.48 (*SD* = 2.51), which was distributed approximately normally with skewness of -0.16 (*SE* = 0.16) and kurtosis of -0.83 (*SE* = 0.32).

Table 3

The Frequencies of Students' Initial Solution Strategies by Problems and Problem Structure (N = 227 students)

Problem	Problem structure	Number (%) of students with a productive first step	Number (%) of students with a non-productive first step
Addition			
Problem 10 S: 44+56+*a*+72+28 G: 100+*a*+100	Proximal Grouping, Making 100	218 (96.0%)	9 (4.0%)
Problem 7 S: 11+55+*y*+89+45 G: 100+*y*+100	Non-Proximal Grouping, Making 100	210 (92.5%)	17 (7.5%)
Problem 13 S: 15+87+*c*+66+32 G: 102+*c*+98	Proximal Grouping, Making non-100	216 (95.2%)	11 (4.8%)

Problem	Problem structure	Number (%) of students with a productive first step	Number (%) of students with a non-productive first step
Problem 14 S: 47+33+*b*+52+68 G: 99+*b*+101	Non-Proximal Grouping, Making non-100	139 (61.2%)	88 (38.8%)
Multiplication			
Problem 24 S: 25*4**b**50*2 G: 100**b**100	Proximal Grouping, Making 100	209 (92.1%)	18 (7.9%)
Problem 32 S: 10*20**a**10*5 G: *a**100*100	Non-Proximal Grouping, Making 100	191 (84.1%)	36 (15.9%)
Problem 30 S: 8*12**d**3*32 G: 96**d**96	Proximal Grouping, Making non-100	213 (93.8%)	14 (6.2%)
Problem 26 S: 4*6**c**24*16 G: 96*96**c*	Non-Proximal Grouping, Making non-100	135 (59.5%)	92 (40.5%)

Main Effects for Addition Problems

First, we tested whether students were more likely to make productive first steps when the numbers to be added were adjacent to each other, when the problem-solving goal was to make 100, and when students had higher prior knowledge. Specifically, we predicted the probability of making a productive first step as a function of two problem-level predictors (Level-1; proximal grouping and making 100) and one student-level predictor (Level-2; prior knowledge) for addition problems (see Table 4). Note that all predictors, including prior knowledge, were rescaled using grand mean-centering (e.g., 0.5 for proximal grouping, -0.5 for non-proximal grouping, 0.5 for making 100, -0.5 for making non-100) to improve the interpretation of the intercept values.

Table 4

Main Effects of Three Predictors on the Productivity of Students' Initial Solution Strategies for Addition Problems (n = 908 problems)

Variable	*B* (*SE*)	*OR*
Fixed effects		
Intercept	2.45*** (0.19)	11.59
Proximal grouping	2.03*** (0.27)	7.61
Making 100	1.69*** (0.25)	5.41
Prior knowledge	0.12** (0.04)	1.12
Random effects		
	0.12 (0.35)	–
Log-likelihood	-295.5	–

$^{**}p < .01$. $^{***}p < .001$.

As shown in Table 4, the results indicated that for addition problems (all else being equal) students were more likely to make a productive first step when the numbers to be combined were proximal (B = 2.03, OR = 7.61, p < .001) compared to when the numbers to be combined were non-proximal, when the goal was to make 100 (B = 1.69, OR = 5.41, p < .001) compared to when the goal was to make non-100, and when the students had higher prior knowledge (B = 0.12, OR = 1.12, p = .007). More specifically, after holding other predictors constant, the students were (1) 7.61 times more likely to make a productive first step for addition problems when the numbers to be combined were proximal compared to non-proximal; (2) 5.41 times more likely to make a productive first step when the problem-solving goal was to make 100 compared to non-100; and (3) 1.12 times more likely to make a productive first step with a one point increase in prior knowledge.

Interaction Effects for Addition Problems

We tested an intra-level interaction of proximal grouping by making 100 (Model 1.1), as well as two cross-level interactions, proximal grouping by prior knowledge (Model 1.2) and making 100 by prior knowledge (Model 1.3; Table 5). First, the results indicated that the three main effects and the interaction between proximal grouping and making 100 (B = -1.96, OR = 0.14, p < .001) were statistically significant. Students were less likely to make a productive first step when proximal grouping and problem-solving goals of making 100 were *not* present in the problems compared to the other three types of problems (Figure 4a).

Table 5

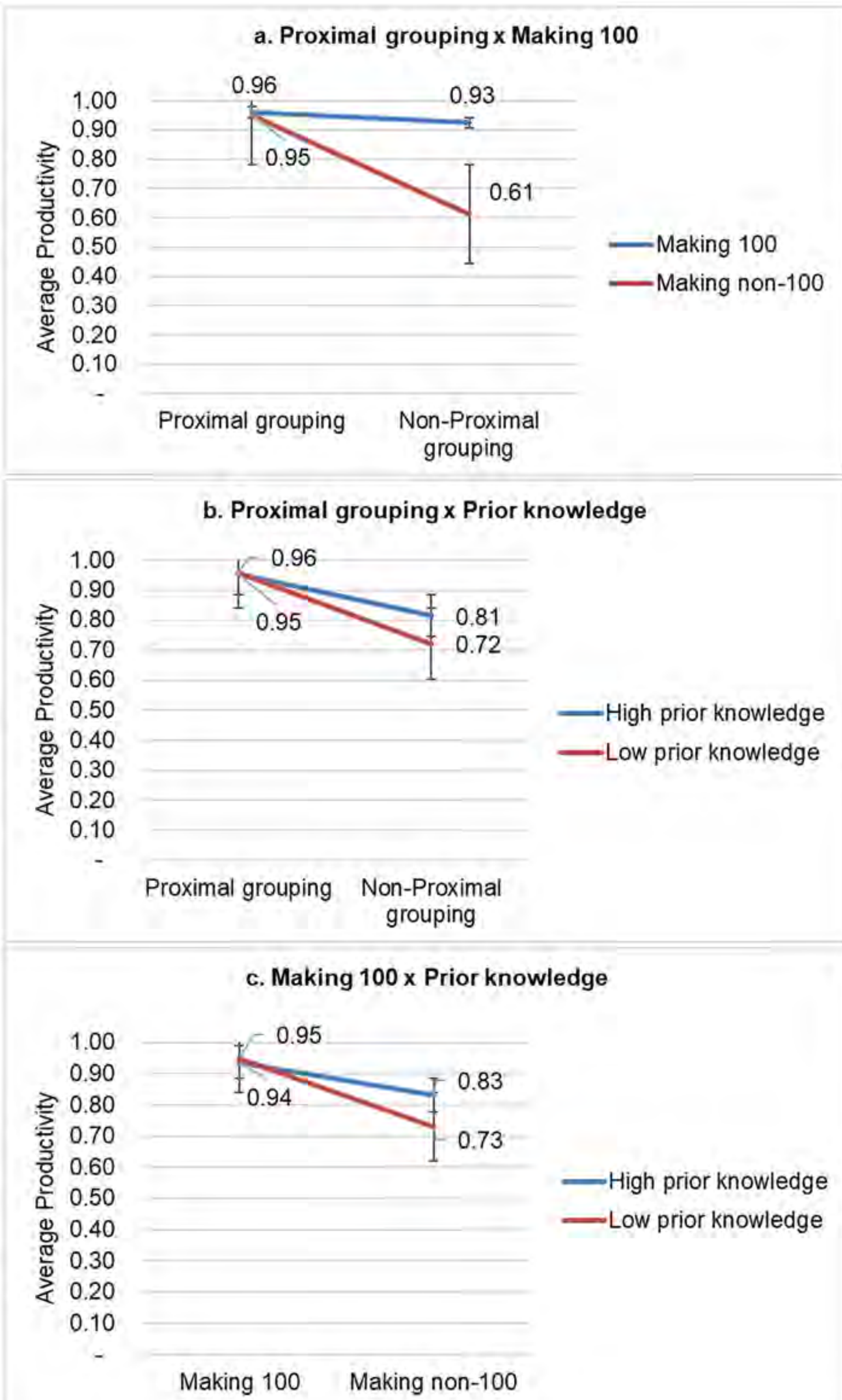
Interaction Effects of Proximal Grouping, Making 100, and Prior Knowledge on the Productivity of Students' Initial Solution Strategies for Addition Problems (n = 908 problems)

	2-way Interactions						3-way Interaction	
	Model 1.1		Model 1.2		Model 1.3		Model 1.4	
Variable	***B*** **(*SE*)**	***OR***	***B*** **(*SE*)**	***OR***	***B*** **(*SE*)**	***OR***	***B*** **(*SE*)**	***OR***
Fixed effects								
Intercept	2.42*** (0.19)	11.22	2.45*** (0.19)	11.58	2.47*** (0.20)	11.87	2.45*** (0.20)	11.57
Proximal Grouping	1.67*** (0.28)	5.29	1.99*** (0.27)	7.31	2.07*** (0.28)	7.91	1.69*** (0.29)	5.43
Making 100	1.19*** (0.28)	3.30	1.70*** (0.26)	5.50	1.65*** (0.25)	5.21	1.19*** (0.29)	3.28
Prior Knowledge	0.12** (0.05)	1.13	0.08 (0.05)	1.08	0.07 (0.05)	1.07	0.05 (0.06)	1.05
Proximal Grouping × Making 100	-1.96*** (0.56)	0.14	–	–	–	–	-1.93*** (0.57)	0.15
Proximal Grouping × Prior Knowledge	–	–	-0.14 (0.10)	0.87	–	–	-0.11 (0.11)	0.89
Making 100 × Prior Knowledge	–	–	–	–	-0.22* (0.10)	0.80	-0.23* (0.11)	0.80
Proximal Grouping × Making 100 × Prior Knowledge	–	–	–	–	–	–	-0.02 (0.22)	0.98
Random effects								
	0.28 (0.53)	–	0.15 (0.38)	–	0.17 (0.41)	–	0.39 (0.63)	–
Log likelihood	-289.6	–	-294.5	–	-292.8	–	-286.0	–

*p < .05. **p < .01. ***p < .001.

Figure 4

Interaction Plots for Productivity of First Steps on Addition Problems



Next, we added two cross-level interaction terms with prior knowledge to the model, proximal grouping by prior knowledge (Model 1.2) and making 100 by prior knowledge (Model 1.3). Except for prior knowledge, all main effects were statistically significant, whereas the interaction between proximal grouping and prior knowledge was not (B = -0.14, p = .17). In other words, the effect of proximal grouping on students' strategy productivity did not significantly differ by students' prior knowledge (see Figure 4b; to illustrate the interaction, students were divided into two groups by a median-split of the raw prior knowledge score).

In contrast, the interaction between making 100 and prior knowledge was statistically significant (B = -0.22, OR = 0.80, p = .02). Specifically, students who scored one point lower than average on the prior knowledge assessment were 20% less likely to make a productive first step when the problem-solving goal was non-100 compared to when the goal was 100 (Figure 4c). Finally, we tested a three-way interaction among proximal grouping, making 100, and prior knowledge (Model 1.4). Although the main effects (except for prior knowledge), the proximal grouping by making 100 interaction, and the making 100 by prior knowledge interaction remained statistically significant, the three-way interaction term was not significant in this model (B = -0.02, p = .93).

Main Effects and Interactions for Multiplication Problems

Next, we repeated the analyses for multiplication problems to test whether these effects were consistent across operations. Specifically, we tested the hypotheses that students were more likely to make productive first steps when the numbers to be combined were adjacent to each other, the problem-solving goal was to make 100, and when students had higher prior knowledge (see Table 6).

Table 6

Main Effects of Three Predictors on the Productivity of Students' Initial Solution Strategies for Multiplication Problems (n = 908 problems)

Variable	B (SE)	OR
Fixed effects		
Intercept	1.97*** (0.15)	7.19
Proximal grouping	1.79*** (0.23)	6.00
Making 100	0.95*** (0.20)	2.59
Prior knowledge	0.17*** (0.04)	1.19
Random effects		
	0.29 (0.54)	–
Log-likelihood	-363.6	

***p < .001.

As shown in Table 6, the log odds of making a productive first step was positively related to proximal grouping (B = 1.79, OR = 6.00, p < .001), making 100 (B = 0.95, OR = 2.59, p < .001), and prior knowledge (B = 0.17, OR = 1.19, p < .001). Similar to addition problems, after holding other predictors constant, students were (1) 6 times more likely to make a productive first step for multiplication problems when the numbers to be combined were proximal compared to non-proximal; (2) 2.59 times more likely to make a productive first step when the goal was to make 100 compared to non-100; and (3) 1.19 times more likely to make a productive first step with a one point increase in prior knowledge.

For Model 2.1, the intra-level interaction term (proximal grouping by making 100) was added to the model (see Table 7). The result indicated a significant interaction between proximal grouping and making 100 (B = -1.71, OR = 0.18, p < .001). In other words, students were significantly less likely to make a productive first step when the proximal grouping and problem-solving goals of making 100 were not present in the problems compared to the other three types of problems (see Figure 5a).

Table 7

Interaction Effects of Proximal Grouping, Making 100, and Prior Knowledge on the Productivity of Students' Initial Solution Strategies for Multiplication Problems (n = 908 problems)

	2-way Interactions						3-way Interaction	
	Model 2.1		Model 2.2		Model 2.3		Model 2.4	
Variable	***B* (*SE*)**	***OR***	***B* (*SE*)**	***OR***	***B* (*SE*)**	***OR***	***B* (*SE*)**	***OR***
Fixed effects								
Intercept	2.00*** (0.15)	7.36	1.99*** (0.13)	7.29	1.98*** (0.15)	7.25	2.03*** (0.16)	7.63
Proximal Grouping	1.69*** (0.23)	5.41	1.83*** (0.24)	6.26	1.82*** (0.23)	6.19	1.76*** (0.25)	5.82
Making 100	0.58* (0.23)	1.78	0.95*** (0.20)	2.58	0.87*** (0.20)	2.40	0.54* (0.24)	1.71
Prior Knowledge	0.18*** (0.04)	1.20	0.19*** (0.05)	1.21	0.16*** (0.04)	1.17	0.18*** (0.05)	1.20
Proximal Grouping × Making 100	-1.71*** (0.45)	0.18	–	–	–	–	-1.68*** (0.49)	0.19
Proximal Grouping × Prior Knowledge	–	–	0.05 (0.09)	1.06	–	–	0.07 (0.09)	1.07
Making 100 × Prior Knowledge	–	–	–	–	-0.17* (0.03)	0.84	-0.16 (0.09)	0.86
Proximal Grouping × Making 100 × Prior Knowledge	–	–	–	–	–	–	0.28 (0.18)	1.32
Random effects								
	0.40 (0.63)	–	0.28 (0.53)	–	0.32 (0.57)	–	0.50 (0.71)	–
Log-likelihood	-356.3	–	-363.4	–	-361.3	–	-351.6	–

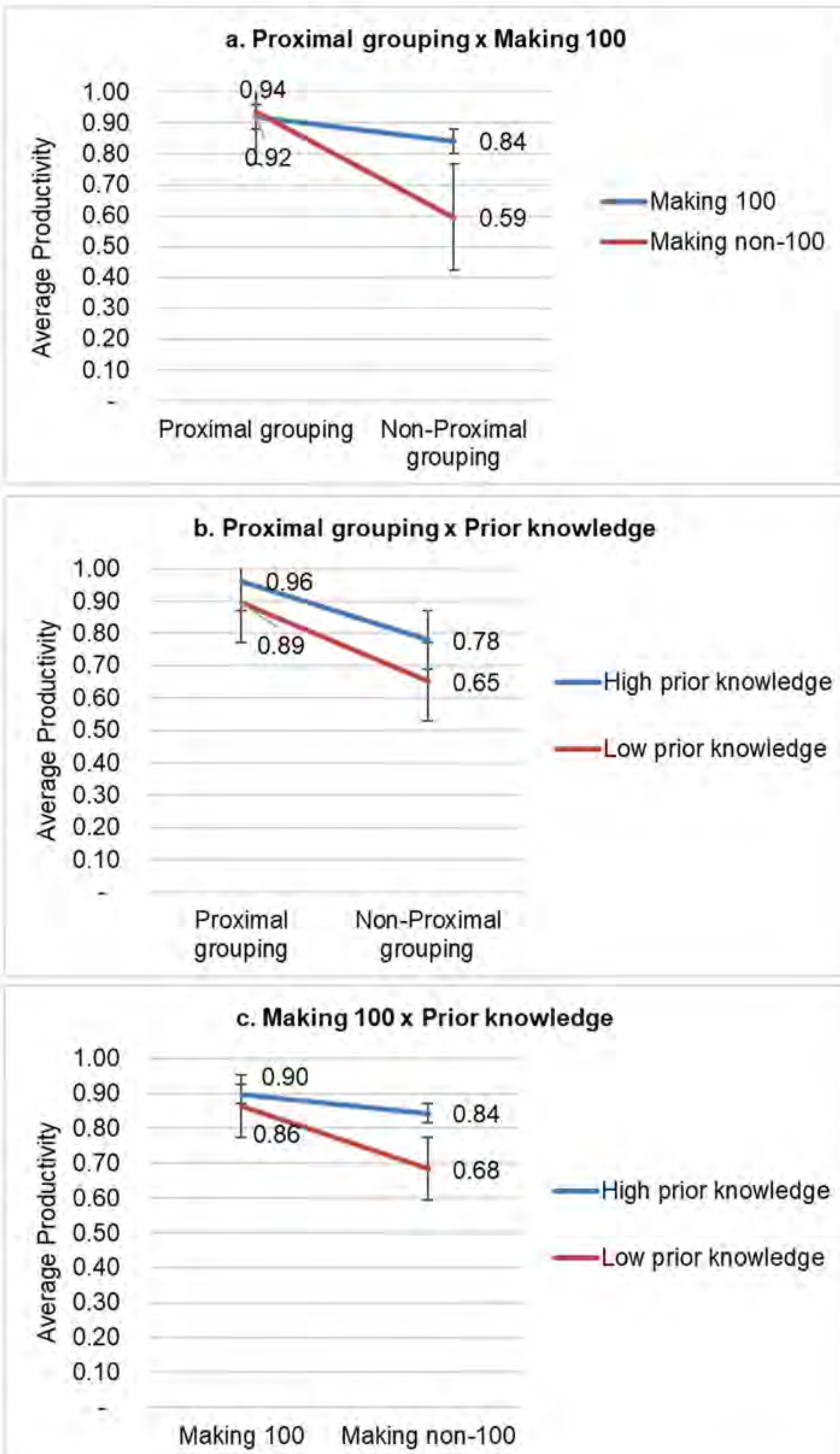
$^{*}p < .05$. $^{***}p < .001$.

Next, we added two cross-level interaction terms with prior knowledge to the model (Model 2.2 and 2.3 in Table 7). All main effects were statistically significant, whereas the proximal grouping by prior knowledge interaction was not ($B = 0.05$, $p = .56$). In other words, the effect of proximal grouping on the productivity of first steps did not significantly differ by students' prior knowledge, similar to the results for addition problems (Figure 5b; to illustrate the interaction, students were divided into two groups by a median-split of the raw prior knowledge score, akin to Figure 4b and Figure 4c for addition problems).

In contrast, the interaction between making 100 and prior knowledge was statistically significant ($B = -0.17$, $OR = 0.84$, $p = .03$). Similar to the results for addition problems, students who scored one point lower than average on the prior knowledge assessment were 16% less likely to make a productive first step when the problem-solving goal was non-100 compared to when the goal was 100 (Figure 5c). Finally, we tested a three-way interaction among proximal grouping, making 100, and prior knowledge (Model 2.4 in Table 7). Although the three main effects and one two-way interaction remained statistically significant, the three-way interaction term was not significant in this model ($B = 0.28$, $p = .13$).

Figure 5

Interaction Plots for Productivity of First Steps on Multiplication Problems



PsychOpen GOLD

Discussion

This study examined the effects of problem structures (proximal grouping of numbers and problem-solving goals to make 100) as well as students' prior knowledge on middle-school students' initial solution strategies within an online interactive mathematics game. The results indicated that proximal grouping of numbers, problem-solving goals to make 100, and students' prior knowledge were significantly associated with students' initial solution strategies. The findings of the current study were consistent with those of other studies which have shown that problem structures (Alibali et al., 2018; Geary et al., 2004; Landy & Goldstone 2010; Lemaire & Callies, 2009) and students' prior knowledge (Siegler, 1988) impact problem-solving performance.

First, for both addition and multiplication problems, as we hypothesized, students were more likely to make a productive mathematical transformation when the numbers to be combined were proximal, the problem-solving goal was to make 100 (e.g., transforming $44 + 56 + a + 72 + 28$ into $100 + a + 100$), and when the students had higher levels of prior knowledge. These findings support the Gestalt laws of perceptual organization (Hartmann, 1935)—stimuli that are proximal in terms of the spatial location are more likely to be grouped together than non-proximal stimuli. The results are also consistent with those of other studies demonstrating that problems with 10 or multiplicands of 10 are solved more quickly and accurately by adults (Aiken & Williams, 1973; Krueger & Hallford, 1984; Siegler & Lemaire, 1997), yielding support for the idea that 100 is similarly a "friendly" number for middle-school students (University of Chicago School Mathematics Project, 2014). Findings also align with previous research indicating positive associations between students' prior mathematical knowledge and their problem-solving strategies (Newton et al., 2020; Star & Rittle-Johnson, 2008; Wang et al., 2019).

Second, we tested the hypothesis that there might be significant interaction effects among the proximal grouping of numbers, problem-solving goals to make 100, and prior knowledge on students' initial strategies for solving addition problems. The results indicated significant interactions between "proximal grouping and making 100" and "making 100 and prior knowledge." Specifically, students were less likely to use productive initial solution strategies when proximal grouping and problem-solving goals of making 100 were not present in the problem (e.g., transforming $47 + 33 + b + 52 + 68$ into $99 + b + 101$) compared to the other three types of problems.

Further, students with lower prior knowledge were less likely to use productive solution strategies when the problem-solving goals were non-friendly numbers (e.g., 98, 101) compared to their peers with higher prior knowledge. Whereas students with higher prior knowledge may be fluent in solving problems involving friendly and non-friendly numbers, students with lower prior knowledge may be less fluent in solving problems with non-friendly, less practiced numbers. Considering that curricular activities often focus on addition combinations to reach friendly numbers like 100 (Pearson Education, Inc., 2008), these results suggest that teachers and researchers may need to think more about how to teach students with lower prior knowledge to use mathematical strategies for less friendly numbers (e.g., 98, 102).

Although the interaction between proximal grouping and prior knowledge was not statistically significant, the results showed that the problems that involved non-proximal grouping (e.g., transforming $47 + 33 + b + 52 + 68$ into $99 + b + 101$) were more challenging for all students regardless of their prior knowledge levels, compared to the problems involving proximal grouping (e.g., transforming $15 + 87 + c + 62 + 32$ into $102 + c + 98$). This finding suggests that the effects of proximal grouping may be pervasive, and middle-school students, regardless of their levels of prior knowledge, may be impacted by the spatial arrangement of numbers within an expression.

Lastly, we examined if these effects extended to a more advanced topic by testing them in multiplication problems. Similar to addition problems, students were more likely to make a productive mathematical transformation for multiplication problems when the numbers to be combined were proximal, the problem-solving goal was to make 100, and when the students had higher levels of prior knowledge. Moreover, there were significant interaction effects between "proximal grouping and making 100" and "prior knowledge and making 100" in multiplication problems. Specifically, students with lower prior knowledge were less likely to make a productive first step when the number goals did not support efficient solution strategies (e.g., transforming $4 \times 6 \times c \times 24 \times 16$ into $96 \times 96 \times c$), compared to students with higher prior knowledge. The parallel findings between addition and multiplication problems suggest that these effects of, and interactions with, problem structures are not limited to one operation, and may be consistent and generalized to other operations.

Together, these results provide some support for the notion that novice students are less likely to notice and leverage problem structures in mathematical problem solving (Novick, 1988). These findings also corroborate other work indicating that students tend to focus on surface attributes of problems, and it can often be challenging for them to accurately identify structural features (Sidney & Alibali, 2015). Therefore, teachers and researchers may need to further consider how to support students—in particular, students with lower prior knowledge—in noticing important patterns in problem structures, which may lead to the use of more efficient problem-solving strategies.

Limitations and Future Directions

There were at least three limitations in this study. First, our measure of initial productive problem solving was only based on students' first mathematical transformation. Future studies could investigate these effects with other measures of solution strategies (e.g., number of steps made, the sequence of students' transformations).

Second, we analyzed only students' decisions in the mathematical transformations they made, without considering the time they took to implement those transformations or other measures of mathematical skills (e.g., arithmetic skills). Another avenue for future research would be an examination of the relations between problem structure and other student behaviors (e.g., pause time before solving) or other cognitive skills that are related to mathematics performance (e.g., executive function skills, spatial reasoning skills) and their tradeoff with strategy productivity.

Third, our sample included a majority of Asian and White students as well as students in advanced mathematics classes, and thus was not representative of the U.S. population as a whole. However, even with this fairly homogenous, high-performing sample, we observed the effects of problem structures and prior knowledge. Future studies should examine the ways in which perceptual and conceptual features of problems may influence students' solution strategies across a broader sample.

Conclusion

In sum, the findings of the current study demonstrate that students' initial solution strategies vary by perceptual and number features of the problems as well as students' prior knowledge. These results suggest that it may be helpful to teach students to notice important patterns in problem structures and build upon their familiarity with 100 to use it as an anchor for decomposition in multi-digit problem solving, which may promote the use of more efficient solution strategies. Overall, these results extend past work demonstrating the effects of perceptual and number features on students' problem-solving strategies from experimental settings into a digital, authentic learning context.

Funding: The research reported here was supported by the Institute of Education Sciences, U.S. Department of Education, through Grant R305A180401 to Worcester Polytechnic Institute. The opinions expressed are those of the authors and do not represent views of the Institute or the U.S. Department of Education.

Acknowledgments: We would like to thank the teachers and students for their participation, members of the Math, Abstraction, Play, Learning, and Embodiment (MAPLE) Lab for their work, members of the Learning and Development Lab for their assistance in coding strategies, Erik Weitnauer and members of Graspable Math Inc. for programming and data support, and Neil Heffernan and the ASSISTments Team for their support.

Competing Interests: The first three authors do not have competing interests. Erin Ottmar was a designer and a developer of From Here to There! and owns a 10% equity stake in Graspable Inc. This has been disclosed to WPI's Conflict Management Committee, and a conflict management plan has been implemented.

References

Aiken, L. R., & Williams, E. N. (1973). Response times in adding and multiplying single-digit numbers. *Perceptual and Motor Skills, 37*, 3–13. https://doi.org/10.2466/pms.1973.37.1.3

Alibali, M. W., Crooks, N. M., & McNeil, N. M. (2018). Perceptual support promotes strategy generation: Evidence from equation solving. *British Journal of Developmental Psychology, 36*, 153–168. https://doi.org/10.1111/bjdp.12203

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences, 21*, 247–286. https://doi.org/10.1080/10508406.2011.611446

Bates, D., Maechler, M., Bolker, B., Walker, S., Christensen, R. H. B., Singmann, H., . . . Fox, J. (2015). Package "lme4." *Convergence, 12*(1), Article 2.

Braithwaite, D. W., Goldstone, R. L., van der Maas, H. L., & Landy, D. H. (2016). Non-formal mechanisms in mathematical cognitive development: The case of arithmetic. *Cognition, 149*, 40–55. https://doi.org/10.1016/j.cognition.2016.01.004

Campbell, J. I. (1995). Mechanisms of simple addition and multiplication: A modified network-interference theory and simulation. *Mathematical Cognition, 1*, 121–164.

Carpenter, T. P., Kepner, H., Corbitt, M. K., Montgomery, M., & Reys, R. E. (1980). Results and implications of the second NAEP mathematics assessments: Elementary school. *The Arithmetic Teacher, 27*, 10–12. https://doi.org/10.5951/AT.27.8.0010

Chan, J. Y. C., Lee, J. E., Mason, C. A., Sawrey, K., & Ottmar, E. (2022). From Here to There! A dynamic algebraic notation system improves understanding of equivalence in middle-school students. *Journal of Educational Psychology, 114*(1), 56–71. https://doi.org/10.1037/edu0000596

Geary, D. C. (2006). Development of mathematical understanding. In W. Damon & R. M. Lerner (Series Eds.) & D. Kuhn & R. S. Siegler (Vol. Eds.), *Handbook of child psychology: Vol 2. Cognition, perception, and language* (6th ed., pp. 777–810). New York, NY, USA: Wiley. https://doi.org/10.1002/9780470147658

Geary, D. C., Hoard, M. K., Byrd-Craven, J., & DeSoto, M. (2004). Strategy choices in simple and complex addition: Contributions of working memory and counting knowledge for children with mathematical disability. *Journal of Experimental Child Psychology, 88*, 121–151. https://doi.org/10.1016/j.jecp.2004.03.002

Goldstone, R. L., Landy, D. H., & Son, J. Y. (2010). The education of perception. *Topics in Cognitive Science, 2*, 265–284. https://doi.org/10.1111/j.1756-8765.2009.01055.x

Harrison, A., Smith, H., Hulse, T., & Ottmar, E. R. (2020). Spacing out! Manipulating spatial features in mathematical expressions affects performance. *Journal of Numerical Cognition, 6*(2), 186–203. https://doi.org/10.5964/jnc.v6i2.243

Hartmann, G. W. (1935). *Gestalt psychology: A survey of facts and principles*. New York, NY, USA: Ronald Press. https://doi.org/10.1037/11497-000

Hulse, T., Daigle, M., Manzo, D., Braith, L., Harrison, A., & Ottmar, E. (2019). From here to there! Elementary: A game-based approach to developing number sense and early algebraic understanding. *Educational Technology Research and Development, 67*(2), 423–441. https://doi.org/10.1007/s11423-019-09653-8

Jiang, M. J., Cooper, J. L., & Alibali, M. W. (2014). Spatial factors influence arithmetic performance: The case of the minus sign. *Quarterly Journal of Experimental Psychology, 67*, 1626–1642. https://doi.org/10.1080/17470218.2014.898669

Kirshner, D., & Awtry, T. (2004). Visual salience of algebraic transformations. *Journal for Research in Mathematics Education, 35*, 224–257. https://doi.org/10.2307/30034809

Krueger, L. E., & Hallford, E. W. (1984). Why 2 + 2 = 5 looks so wrong: On the odd-even rule in sum verification. *Memory & Cognition, 12*, 171–180. https://doi.org/10.3758/BF03198431

Landy, D., & Goldstone, R. L. (2007). Formal notations are diagrams: Evidence from a production task. *Memory & Cognition, 35*, 2033–2040. https://doi.org/10.3758/BF03192935

Landy, D., & Goldstone, R. L. (2010). Proximity and precedence in arithmetic. *Quarterly Journal of Experimental Psychology, 63*, 1953–1968. https://doi.org/10.1080/17470211003787619

Laski, E. V., Ermakova, A., & Vasilyeva, M. (2014). Early use of decomposition for addition and its relation to base-10 knowledge. *Journal of Applied Developmental Psychology, 35*, 444–454. https://doi.org/10.1016/j.appdev.2014.07.002

Lemaire, P., & Callies, S. (2009). Children's strategies in complex arithmetic. *Journal of Experimental Child Psychology, 103*, 49–65. https://doi.org/10.1016/j.jecp.2008.09.007

Marghetis, T., Landy, D., & Goldstone, R. L. (2016). Mastering algebra retrains the visual system to perceive hierarchical structure in equations. *Cognitive Research: Principles and Implications, 1*, Article 25. https://doi.org/10.1186/s41235-016-0020-9

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA, USA: Author.

National Governors Association Center for Best Practices (NGA Center) and Council of Chief State School Officers (CCSSO). (2010). *Common core state standards for mathematics*. Washington, DC, USA: Author. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* Washington, DC, USA: National Academies Press. https://doi.org/10.17226/10434

Newton, K. J., Lange, K., & Booth, J. L. (2020). Mathematical flexibility: Aspects of a continuum and the role of prior knowledge. *Journal of Experimental Education, 88*(4), 503–515. https://doi.org/10.1080/00220973.2019.1586629

Novick, L. R. (1988). Analogical transfer, problem similarity, and expertise. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 14*, 510–520. https://doi.org/10.1037/0278-7393.14.3.510

Pearson Education, Inc. (2008). *Investigations in number, data, and space: Grade 3.* Pearson Scott Foresman.

Rittle-Johnson, B., Matthews, P. G., Taylor, R. S., & McEldoon, K. L. (2011). Assessing knowledge of mathematical equivalence: A construct-modeling approach. *Journal of Educational Psychology, 103*, 85–104. https://doi.org/10.1037/a0021334

Sidney, P. G., & Alibali, M. W. (2015). Making connections in math: Activating a prior knowledge analogue matters for learning. *Journal of Cognition and Development, 16*, 160–185. https://doi.org/10.1080/15248372.2013.792091

Sidney, P. G., & Alibali, M. W. (2017). Creating a context for learning: Activating children's whole number knowledge prepares them to understand fraction division. *Journal of Numerical Cognition, 3*(1), 31–57. https://doi.org/10.5964/jnc.v3i1.71

Siegler, R. S. (1988). Individual differences in strategy choices: Good students, not-so good students, and perfectionists. *Child Development, 59*, 833–851. https://doi.org/10.2307/1130252

Siegler, R. S., & Lemaire, P. (1997). Older and younger adults' strategy choices in multiplication: Testing predictions of ASCM using the choice/no-choice method. *Journal of Experimental Psychology: General, 126*, 71–92. https://doi.org/10.1037/0096-3445.126.1.71

Star, J. R., Pollack, C., Durkin, K., Rittle-Johnson, B., Lynch, K., Newton, K., & Gogolen, C. (2015). Learning from comparison in algebra. *Contemporary Educational Psychology, 40*, 41–54. https://doi.org/10.1016/j.cedpsych.2014.05.005

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem solving: The case of equation solving. *Learning and Instruction, 18*, 565–579. https://doi.org/10.1016/j.learninstruc.2007.09.018

Torbeyns, J., Verschaffel, L., & Ghesquière, P. (2006). The development of children's adaptive expertise in the number domain 20 to 100. *Cognition and Instruction, 24*, 439–465. https://doi.org/10.1207/s1532690xci2404_2

University of Chicago School Mathematics Project. (2014). *Everyday mathematics 4: Grade 2.* Chicago, IL, USA: McGraw-Hill.

Wang, J., Liu, R.-D., Star, J., Liu, Y., & Zhen, R. (2019). The moderating effect of regulatory focus in the relationship between potential flexibility and practical flexibility. *Contemporary Educational Psychology, 56*, 218–227. https://doi.org/10.1016/j.cedpsych.2019.01.013

Journal of Numerical Cognition (JNC) is an official journal of the Mathematical Cognition and Learning Society (MCLS).

leibniz-psychology.org

PsychOpen GOLD is a publishing service by Leibniz Institute for Psychology (ZPID), Germany.