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Discourse analysis on understanding the differential concept of high school students in a dynamic geometry environment

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Abstract

This study investigated the cognitive and social processes through which high school students acquire the differential concepts through communication in a dynamic geometry environment through some cases. Additionally, we observed how a dynamic geometry environment affects these processes. To achieve this objective, eight students were recruited by using the pre-test results and divided into an experimental group and a control group. The learning environment for the two groups was designed to be the same except for jointly manipulating *Geogebra* on the laptop in the experimental group. Students' discourse was analyzed using an analysis framework that applied both Piaget's (1959) linguistic categories and Rutherford's (2011) neo-Piagetian model. We found that the dynamic geometry environment improved communication among students and their achievement levels. In particular, graphic and dynamic representations created by dragging stimulated students' interest and increased communication among them. Although the quality of communication was somewhat different in the discourse analysis of the two groups, each participant's role was confirmed in the co-construction of knowledge among all cases of eight students. We expect that precise verbal information on various representations of mathematical content in the process of understanding the concepts of students could be an opportunity to prepare educational environments corresponding thereto.

Keywords: case study, differential concept, discourse, dynamic geometry environment, high school students

INTRODUCTION

The role of technology in mathematics education is growing. To generate and enhance mathematical understanding, Freudenthal (1991) invoked the appropriate use of calculators and computers. In 2002, to promote equal opportunity in school mathematics, the National Council of Teachers of Mathematics (NCTM, 2000) encouraged the use of technical engineering. They contended that technical engineering tools and environments can provide all students student the opportunity to explore complex problems and mathematical ideas. Reflecting this trend, many researchers have investigated the effects of dynamic geometry environments on math teaching and learning (e.g., Hollebrands, 2003; Mariotti, 2000; Sinclair & Moss, 2012; Sinclair & Yurita, 2008). Thus, we were interested in how a dynamic geometry environment affects the processes in which students understand the concepts of differential.

Differentiation is a field widely applied in, for example, natural science and engineering, and the differential coefficient is its central concept. High school students know that the differential coefficient is the slope of the tangent at a given point, but most do not fully understand the concept of the tangent (Park, 2017). In addition, students tend to focus only on solving routine differential coefficient calculation problems and therefore have difficulty with the conceptual understanding of differentiation (Oh, 2018). Recent studies in South Korea have made various attempts, including using engineering tools, to enhance students'

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Contribution to the literature

- This study tried to explore the cognitive and social processes through which high school students acquire the differential concepts through communication in a dynamic geometry environment.
- This study shows how a dynamic geometry environment affects cognitive and social processes.
- This study considered rationale for the applicability of Piaget's (1959) speech category and Rutherford's (2011) neo-Piagetian model and applied the language analysis framework that applied two models together.

understanding of the differential concept (e.g., Hwang & Kim, 2016; Kang, 2012; Lee et al., 2014; Oh, 2018). However, according to our review of the literature, studies have rarely investigated how a dynamic geometry environment affects students' communication in the learning process to understand differential concepts. This gap in the literature encouraged us to conduct a study on our topic of interest, and we took note of Sfard's (2007) discourse theory for the analysis of communication that occurs in a class.

Many studies have demonstrated that Sfard's (2007) discourse theory can be applied in mathematics education (e.g., Gucler, 2013; Kim et al., 2012; Lewis, 2017; Sinclair & Moss, 2012; Sinclair & Yurita, 2008). The focus of this discourse theory is the analysis of the process by which students' colloquial discourses are transformed into mathematical discourses. However, some studies related to this theory have not presented a framework for analyzing students' language in the course of teaching and learning. To fill this gap in the literature, Sinclair and Moss (2012) analyzed the level-specific language discourse by applying Van Hieles' theory of geometry levels, and Kim et al. (2012) developed structured questionnaires to classify the level of infinite concept.

In this context, two studies, Piaget (1959) and Vygotsky (1987), of the relationships between thought and language are noteworthy. Devries (1997) posited that Piaget's (1959) theory could be fused with Vygotsky's (1987) theory through the social expansion of child research. Rutherford (2011) proposed a neo-Piagetian model that extended Piaget's (1959) cognitive perspective to a socio-cultural perspective in the human socialization process. In addition, researchers have highlighted the need for empirical studies on the cognitive and social dimensions of students' knowledge construction and their interaction patterns and influences (e.g., Bormanaki & Khoshhal, 2017).

In this study, we intended to analyze the discourse of high school students understanding the concepts of differential in a dynamic geometry environment. The following two problems were set for discourse analysis. First, how are the cognitive and social processes in which high school students acquire differential concepts through communication in a dynamic geometry environment? Second, how does a dynamic geometry environment affect the processes by which students understand the concepts of differential? To solve these problems, we considered the rationale for the applicability of Piaget's (1959) speech category and Rutherford's (2011) neo-Piagetian model and applied the language analysis framework that applied two models together.

THEORETICAL BACKGROUND

Piaget's Speech Category

Piaget (1959) classified the external expression of children's speech into two types, namely, egocentric speech and socialized speech, depending on whether the listener's perspective is considered. He categorized egocentric speech into repetition, monologue, and dual or collective monologue and socialized speech into adapted information, criticism, command-request-threat, question, and answer. In this functional classification of children's speech, the branching point of the categorization of egocentric speech and socialized speech is a dual or collective monologue and adapted information. A dual or collective monologue is an immediate response to external stimuli, without understanding or participation, and does not consider the perspectives of others. For example, when a child appears to be talking but only asserts oneself and the conversation is disconnected. Adapted information corresponds to a case of exchanging thoughts with others, conveying interests, affecting behavior, and cooperating for a common purpose. From this point on, children begin to consider others' perspectives, and their language is socialized.

Mayer (2005) described Piaget (1959) as the first to record and analyze children's peer conversations within an established social context. Mayer (2005) viewed that Piaget (1959) attempted to interpret children's speech function in relation to the accompanying behavior, rather than the mere feature of children's uttered phrases. In other words, Mayer (2005) explained that Piaget (1959) introduced the natural scientists' method to the study of children's behavior and language and adapted it to that purpose. Vygotsky (1987) suggested that the greatest strength of Piaget's (1959) theory is its basis on an evolutionary perspective and pure empiricism and proposed that the factual basis of Piaget's (1959) theory was presented in studies of children's use of language.

Based on this background, in this study, we applied the natural scientific and measurable functional classification of children's speech, the basis of Piaget's (1959) theory, to discourse analysis.

Rutherford's Neo-Piagetian Model

Various attempts have been made to apply Piaget's (1959) theory to teaching and learning. Rutherford (2011)'s neo-Piagetian model is one of these studies. A prerequisite for these attempts is the consideration of social relations, that is, socio-cultural context. Rutherford (2011) constructed a theory of assimilation and accommodation in the cognitive and cultural domain (Bormanaki & Khoshhal, 2017). Bormanaki and Khoshhal (2017) considered that in addition to the cognitive development of the ego, contextual and socio-cultural factors play a notable role in cognitive development. They viewed foreign language learning, especially reading comprehension, as related to contextual and socio-cultural factors. For example, when readers read text related to English literature, they understand the text well if they are familiar with the culture and social context of the English-speaking world. In this case, readers can assimilate and accommodate the cultural information in the text, achieve cultural equilibration, and further their cognitive development. Thus, Bormanaki and Khoshhal (2017) proposed the neo-Piagetian model as a theory that could play an important role in the field of teaching and language learning.

Rutherford (2011) explained the process of an individual's adaptation and adjustment to society as cognitive social assimilation and cognitive social accommodation based on Piaget's (1959) theory of assimilation and accommodation. Rutherford (2011) defined cognitive assimilation as the *growth* of the schema for the adaptation of the internal system, cognitive accommodation as the *change* of the schema for external (environmental) adaptation, social assimilation as the cultural *growth* through adaptation of various cultural groups for adaptation in context, and social accommodation as the cultural *change* for adaptation of beliefs and traditional practices of various cultural groups. Thus, the model proposed by Rutherford (2011) is that an individual's personality develops through cognitive assimilation, cognitive accommodation, social assimilation, and social accommodation through linguistic semiotic mediation between individuals.

Rutherford's (2011) model suggests that communication is possible by using the symbol of language in society and that the social ego is formed through this use. Thus, in this study, we considered that Rutherford's (2011) neo-Piagetian model might reveal in the communication process the cognitive aspects of students' changes in mathematical concepts and the meaning of social interaction or context.

Table 1. Framework for students' language analysis

Category			Code
Piaget's speech	Egocentric speech	Repetition	R
		Monologue	M
		Dual monologue	DM
	Socialized speech	Adapted information	AI
		Criticism	CR
		Command-request-threat	CT
		Question	Q
		Answer	A
Neo-Piagetian model	Cognitive assimilation		NCA
	Cognitive accommodation		NCR
	Socio-cultural assimilation		NSA
	Socio-cultural accommodation		NSR

METHODS

Framework and Perspectives for Students' Language Analysis

The primary interest of this study was to assess in what language format students express their thought changes such as assimilation and accommodation on the differential concepts in a dynamic geometry environment. We attempted to use an analysis framework that applied both Piaget's (1959) speech category and Rutherford's (2011) neo-Piagetian model, to analyze the language in students' discourses (**Table 1**). We coded the eight categories of Piaget's (1959) speech and the four dimensions of the neo-Piagetian model in **Table 1**.

Notably, assuming that there is a limit to finding verbal evidence of assimilation and accommodation, the following four perspectives were noted. First, we attempted to explain the concept of understanding a process of an individual or collective by assimilation and accommodation. In particular, in the process of understanding a concept by a student, we regarded it to be socio-cultural if he/she was absorbed into culture through cooperation or agreement with others, for example, a teacher or fellow students. Second, verbal evidence of assimilation and accommodation were designated as "grow" and "change," respectively, based on Rutherford (2011). However, the distinction between assimilation and accommodation might be ambiguous, and the linguistic expressions of students that result from assimilation and accommodation might not manifest. Assuming that all events presuppose a change in thinking, on the basis of the overall context of discourse, we attempted to classify each event into assimilation and accommodation. Third, because the same word could have different meanings or a different word could have the same meaning, we observed and classified the language expression of each student. Fourth, we reclassified Piaget's (1959) speech categories into four dimensions in the neo-Piagetian model.

Table 2. Class topics and environment

Group	Environment	Topics	Time (min)
Experiment	Cooperative learning through tasks with *Geogebra* without a teacher	• Average rate of change • Differential coefficient • Derivative	40
Control	Cooperative learning through tasks without a teacher		

The advantages of the language analysis framework in **Table 1** to which both theories are applied are summarized in the following three points. First, in Piaget's (1928) language category, egocentric speaking was mainly used by Piaget (1959) to show that children's egocentric speaking decreases in the process of transitioning from the pre-operational stage to the concrete operational stage, but we regarded AI, A, CR, etc., as linguistic tools that play a role in communication. This is because AI, A, CR, etc., implies that the student considers the intentions of others or conveys his/her own intention. Thus, in this study, AI, A, CR, etc., were used to analyze quantitative and qualitative changes in practical communication. Second, although Piaget's (1959) speech category considered only the aspect of subjective socialization in which students consider each other, Rutherford (2011)'s classification of cognitive and socio-cultural categories on assimilation and accommodation made it possible to express the sociological influence on the students in every language category. It can be seen as a further expansion of the scope of analysis of the Piaget's (1959) speech category. Third, the language analysis framework in **Table 1** was able to show concretely the case in which students acquired the geometric meaning (the slope of the tangent line) of the differential (instantaneous rate of change) in the dynamic geometric environment.

A few of examples of the language analysis framework applied in this study are as follows. First, the student C2 who said "if you get $f'(x)$ first, you get $2x + 3$" is presenting the result related to the derivative of the function and giving the information to other students. This student's language was classified as AI that conveys math-related information to other students. At the same time, in this process, the student recalled the concept of differentiation and applied it to the function, so it was classified as NCA.

Let's apply it to another example: "When this point is near to this point, it is tangent. Is this (2, 1)? Then, when (4, 7) approaches (2, 1), the slope of the tangent becomes equal. (4, 7) overlaps with (2, 1) and this line overlaps with the tangent". Student E3, who said this, looked at the computer and visually recognized the process of changing the graph from the average rate of change to the instantaneous rate of change, and subjectively understood the geometric meaning of the instantaneous rate of change. And she shared the knowledge she had acquired to other students, so we classified this language as AI. At the same time, we judged that the student changed the concept of instantaneous rate of change by subjective judgment based on dynamic graph expression rather than logically expanding the concept of differentiation, and we reclassified this verbal expression as NCR. On the other hand, since these students' conceptual images were formed based on dynamic visual images rather than a formal definition of differentiation, it implies the occurrence of misconceptions related to the differential concept in the future. Nevertheless, we interpreted the dynamic graph representation as affecting students' understanding of the concept of differentiation.

Class Topics and Environment

In mid-October 2018, we separated eight participants into two groups of four and then conducted the experiment. Each class was held for 40 minutes for each group. Both groups conducted classes that completed the same tasks, without an explanation or intervention from a teacher (**Table 2**).

In the experimental group class, students manipulated *Geogebra* by jointly using one laptop. They had experience using cell-phones to draw graphs in their first-year math classes; thus, they did not object to using *Geogebra* and had no difficulty manipulating it. To exclude the overlapping phenomenon of students' change by a teacher and students' change by using *Geogebra,* and to reveal the effects of *Geogebra*, the teacher did not intervene in the cooperative learning process and merely observed the students. *Geogebra*'s features related to dragging and dynamic graph representation are common features of other dynamic geometry programs. Therefore, in this study, *Geogebra*, a geometric dynamic software familiar to students, was used.

Test Tool

We used six questions to evaluate how well students understood three class topics (**Table 3**). Two problems were presented per topic. One problem in each topic was easily solved by calculation, and the other problem required simple application ability. We extracted the six problems from a calculus I textbook. The pre- and post-test tools comprised the same problems. We assessed that the pre-test did not affect the post-test because students who insufficiently understood differential concepts were selected and the post-test was conducted approximately one month after the pre-test. In addition, the interviews indicated that the students did not recognize the pre-test and post-test problems as the same.

Table 3. Test tool

Topics	Problems
Average rate of change	1. In the following function $f(x) = x^2 + x$, calculate the average rate of change when the value of x changes from 1 to 3. 2. For a function $f(x) = x^2 - 3x$, the average rate of change is 0 when the value of x changes from a to $a + 2$, then calculate value of the constant a.
Differential coefficient or instantaneous rate of change	3. Calculate the differential coefficient at $x = 1$ in the function $f(x) = x^2 - x + 1$. 4. For a function $f(x)$, obtain the extreme values of $\lim_{h\to 0}\frac{f(1+5h)-f(1)}{h}$ when $f(1) = 5$.
Derivative function	5. Use definition of a derivative function to find derivative of function f(x) $= x^4$. 6. Find the rest of the polynomial $x^7 - x^3 + 3$ divided by $(x + 1)^2$.

Participants

To select participants, we conducted a pre-test with 100 high school sophomores in D-city, South Korea, who had learned differential concepts in September 2018. On the basis of the test results, we first selected students who required additional learning on the differential concept; next, eight students who wanted to voluntarily participate in the experiment were finally selected. All the students provided information on their grades, tendencies, and attitudes toward math, and their math teachers provided information on the students' communication abilities. On the basis of this information, we divided students into an experimental group and a control group, each comprising four students at similar levels in every aspect.

The profiles of the four students in the experimental group are as follows. E1 was a high-achieving student. She generally enjoyed studying math, and her scores were higher in math than in other subjects. She was introvert but kindly answered her friends' questions on problems that they did not understand. E2 was a relatively high-achieving student. She was usually quiet but did her best to participate. She had low confidence in math; thus, she carefully studied and often asked her friend questions. She solved the textbook problems several times and studied hard enough so that she could solve every problem in the textbook before the examination. E3 was a mid-achieving student. She encouraged and led her friends effectively as the leader of her class. She did not like math but wanted to be good at it. E4 was a relatively low-achieving student. She actively participated in class but did not study math much. Her math grades were lower than those of other subjects. She had poor comprehension ability but attempted to ask questions and understand the answers.

The profiles of the students in the control group are as follows. C1 was a high-achieving student. She had a meticulous, sincere personality, and actively participated in class. She also effectively shared what she understood with her friends. She thought that her scores in math were lower than those in other subjects; thus, she invested in studying math. C2 was a relatively high-achieving student. She was sincere, diligent, and hardworking. She liked math and had higher scores in math than in other subjects. She was introvert but answered her friends kindly if they asked about problems. C3 was a mid-achieving student. She was an active class leader and got along well with her friends. She did not like math and did not ordinarily study math. C4 was a relatively low-achieving student at school. She liked to hang out with friends and wanted to be good at math but was not. Although she participated actively in class and understood concepts when they were introduced, because she did not subsequently review the concepts, she forgot them.

In terms of communication, we compared the two groups and expected them to show similar communication patterns as follows. In the experimental group, E3 would take a leading role in the dialog because she effectively led students. E1 and E2 had high grades; thus, they would mainly discuss and answer their friends' questions. E4 would continue her group's conversation by asking about what she did not understand. In the control group, C3 would lead the conversation because she was an excellent class leader. C1 and C2 would comment on or explain the learning content because they had high grades. C4 usually participated in class and expressed her opinions; thus, if she did not understand, she would ask her friend questions and encourage them to provide explanations.

Task Material

The task material comprised three parts—average rate of change, instantaneous rate of change, and derivative concepts—based on a calculus I textbook. The task for the average rate of change was organized into three subtasks:

1. indicating the change in x and y as symbols,
2. defining the average rate of change, and
3. solving basic examples.

The task for the instantaneous rate of change was organized into three subtasks:

1. defining the instantaneous rate of change or the differential coefficient at a point,
2. understanding its geometry meaning through graphs, and
3. solving basic examples.

The task for derivative concepts comprised two subtasks:

Table 4. Frequency by language category

Category		Experimental group						Control group					
		E1	E2	E3	E4	Total		C1	C2	C3	C4	Total	
R	NCA	-	-	-	-	-	-	-	-	-	1	1	4
	NCR	-	-	-	-	-		-	-	-	-	-	
	NSA	-	-	-	-	-		1	-	1	1	3	
	NSR	-	-	-	-	-		-	-	-	-	-	
M	NCA	-	-	-	1	1	2	-	-	-	2	2	8
	NCR	-	-	-	-	-		-	-	-	-	-	
	NSA	-	1	-	-	1		1	-	2	3	6	
	NSR	-	-	-	-	-		-	-	-	-	-	
DM	NCA	1	2	-	-	3	11	-	-	-	2	2	16
	NCR	-	-	-	-	-		-	-	-	-	-	
	NSA	1	2	1	2	6		3	1	4	6	14	
	NSR	-	1	-	1	2		-	-	-	-	-	
AI	NCA	7	4	4	10	25	34	6	2	1	1	10	11
	NCR	-	-	1	-	1		-	-	-	-	-	
	NSA	2	1	1	3	7		-	-	1	-	1	
	NSR	-	-	1	-	1		-	-	-	-	-	
Q	NCA	2	7	29	6	44	45	7	1	14	5	27	27
	NCR	-	-	-	-	-		-	-	-	-	-	
	NSA	-	-	1	-	1		-	-	-	-	-	
	NSR	-	-	-	-	-		-	-	-	-	-	
A	NCA	10	5	2	6	23	69	10	4	4	1	19	41
	NCR	-	-	-	-	-		-	-	-	-	-	
	NSA	11	13	8	13	45		7	6	4	5	22	
	NSR	1	-	-	-	1		-	-	-	-	-	
Total		35	36	48	42	161		35	14	31	27	107	

1. defining the concept of derivative and
2. solving basic examples.

In the case of subtasks that define and explain mathematical concepts, we left blanks so that the students directly filled in the formulas, terminologies, and symbols. For the example subtasks, only the problems were presented, and the students were asked to solve them.

Data Collection and Analysis

All classes were recorded by camcorder, with the consent of the students, and transcribed. The transcript included various verbal expressions that could reveal students' behavior, attitudes, and feelings. Individual interviews were conducted with all four students in the experimental group, and interviews in the control group were conducted only when confirmation was required. Individual interviews were recorded and the recordings transcribed after receiving students' consent. In the interviews with the experimental group, we asked students whether *Geogebra* and cooperative learning were helpful. The purpose of these questions was to assess whether students' understanding of differential concepts changed and they encountered any difficulties while communicating with each other. We also attempted to determine students' perspectives of learning using *Geogebra*. Students' language expressions in the transcript were coded based on the framework in Table 1 and four perspectives in the analysis. Based on the coded data, we calculated the categorical frequency of each student's language. We examined language frequency and class scenes to assess how students' understanding of differential concepts changed. Triangulation verification was conducted by discussing the data in the process of analysis.

RESULTS AND DISCUSSION

Analysis by Language Category

Table 4 presents our classification results of eight students' speech expressed during their cooperative learning, extracting speech only related to differential concepts. Comparing the discourse between the experimental group and the control group, we identified four characteristics of communication. First, the total speech frequency of the experimental group (161) was approximately 50% more than that of the control group (107). This finding demonstrates that students' class participation increased in a dynamic geometry environment where the teacher's intervention was excluded. Second, the ratio of speech categories, AI, Q, and A, to actual communication functions was 92% in the experimental group and 74% in the control group. In addition, the speech frequency of NSA and NSR meaning cooperation or agreement with others was 64 in the experimental group and 46 in the control group. These results show that the interaction among students in the dynamic geometry environment generated more than in an environment that did not. Third, in both groups, students' speech corresponding to CR and CT was not observed. This finding reveals that most students avoided words or actions that threaten their peers with criticism or commands during cooperative learning. Fourth, for the control group, no linguistic evidence affected communication related to accommodation that indicated students' changes; in the experimental group, we observed five corresponding pieces of evidence. The biggest difference between the two groups was to use *Geogebra*; thus, we interpreted this result to be due to environment in which rich visual mediators can be used. The evidence for this interpretation is presented in detail in class scenes.

We now examine the two groups' speech characteristics in detail. In the experimental group class, there were 175 speech states by students, 161 of which related to differential concepts. **Table 4** highlights three notable findings for the experimental group. First, of the 161 speech states, 96 concerned cognitive assimilations, one involved cognitive accommodation, 60 concerned socio-cultural assimilations, and four involved socio-cultural accommodations. In addition, 53 socio-cultural assimilations related to speech states (88%) manifested as practical communication AI, Q, and A. This finding indicates that the amount of communication in the experimental group was higher than that in the control group. Second, unlike in the control group, cognitive and socio-cultural accommodation states occurred in the experimental group. This phenomenon can be demonstrated to stem from a conceptual change in students as a result of their practical communication. Third, in terms of communication, E3 and E4 accounted for a relatively high proportion of speech states. This finding could be attributed to E3 having low grades but exhibiting leadership as one of the class leaders and E4 having low grades but a desire to complete tasks.

In the control group class, there were 131 speech states by students, and 107 of these related to the differential concepts. First, among the 107 speech states, cognitive assimilation occurred 61 times and socio-cultural assimilation occurred 46 times. In addition, socio-cultural assimilation states involved practical communication AI, Q, and A 23 times (50%). When expanded to all speech states, the ratio decreases to 24%. This finding demonstrates that communication in the control group lessons was less successful than that in the experimental group. Second, cognitive and socio-cultural accommodation did not occur in the control group. We interpret that this finding occurred because of students' non communicative speech states R, M, and CM or that the correction and the knowledge development did not manifest in discourse. Third, in terms of communication, C1 and C3 accounted for a relatively high proportion of the speech. This finding can be attributed to C1's high performance and activeness and C3's leadership as one of the class leaders. Additionally, compared with C4, C2's grades were higher but her frequency of speaking was lower, which could be attributed to C2's tendency not to actively participate.

Detailed Language Analysis by Class Topic

We analyzed students' language by class topic. No differences in how students communicated in the average rate of change and the derivative classes were observed. However, we found qualitative differences between two groups in the instantaneous rate of change class. Because these differences could not be analyzed based on achievement results, we determined that these conversational changes (accommodation) would affect the understanding of the differential concept. Therefore, we additionally performed the class scene analysis only for the instantaneous rate of change.

Average rate of change class

Table 5 presents our classification results of eight students' speech expressed in the average rate of change classes. This class produced two important findings. First, in both groups, only assimilation states occurred. We interpret this phenomenon as a result of no students who had difficulty with the average rate of change because they had already learned the differential concepts. In particular, C3, C4, and E4 felt comfortable learning the average rate of change, although all their answers to questions on this concept on the pre-test were wrong. Regarding these findings, C3, C4, and E4 respectively stated the following in the interview: "I learned it so long ago, so I couldn't remember it at first. I didn't know what to get, so I couldn't solve it, but I thought about it while studying together;" "I forgot, but it was so easy to look again;" "In the first test, I didn't know the term the average rate of change, but in the second test, it was easy to know the formula." In addition, most students had good results on the post-test, except that E4 incorrectly answered question 2. These findings reveal that students can understand the concept of average rate of change rather easily. Second, the incidence of actual communication AI, Q, and A was higher in the experimental group (33%) than in the control group (23%), and the actual frequency of speech states was higher in the experimental group than in the control group. These findings can be seen because the students communicated through computers. Notably, in the video, the students in the experimental group discussed computers at the beginning of the class.

Now, we discuss both groups' speech characteristics in more detail. In the experimental group's average rate of change class, there were 66 speech states by students, 63 of which related to the average rate of change. As

Table 5. Frequency of average rate of change

Category		Experimental group						Control group					
		E1	E2	E3	E4	Total		C1	C2	C3	C4	Total	
M	NCA	-	-	-	1	1	1	-	-	-	1	1	7
	NSA	-	-	-	-	-		1	-	2	3	6	
DM	NCA	-	1	-	-	1	2	-	-	-	-	-	6
	NSA	1	-	-	-	1		1	-	3	6	6	
AI	NCA	3	1	1	4	9	11	1	-	-	-	1	2
	NSA	-	-	-	2	2		-	-	1	-	1	
Q	NCA	1	4	12	4	21	22	3	-	7	3	13	13
	NSA	-	-	1	-	1		-	-	-	-	-	
A	NCA	4	2	2	1	9	27	5	2	2	-	9	19
	NSA	5	5	3	5	18		3	3	1	3	10	
Total		14	13	19	17	63		15	5	14	13	47	

Table 6. Frequency of instantaneous rate of change

Category		Experimental group						Control group					
		E1	E2	E3	E4	Total		C1	C2	C3	C4	Total	
R	NCA	-	-	-	-	-	-	-	-	-	1	1	4
	NSA	-	-	-	-	-		1	-	1	1	3	
M	NCA	-	-	-	-	-	1	-	-	-	-	-	-
	NSA	-	1	-	-	1		-	-	-	-	-	
DM	NCA	1	1	-	-	2	8	-	-	-	1	1	6
	NSA	-	1	1	2	4		1	-	2	2	5	
	NSR	-	1	-	1	2		-	-	-	-	-	
AI	NCA	4	3	2	6	15	20	4	2	1	1	8	8
	NCR	-	-	1	-	1		-	-	-	-	-	
	NSA	2	-	1	-	3		-	-	-	-	-	
	NSR	-	-	1	-	1		-	-	-	-	-	
Q	NCA	1	2	14	2	19	19	4	-	2	2	8	8
A	NCA	5	3	-	5	13	31	3	2	1	1	7	16
	NSA	3	5	5	4	17		3	2	3	1	9	
	NSR	1	-	-	-	1		-	-	-	-	-	
Total		17	17	25	20	79		16	6	10	10	42	

Table 4 shows, this 63 comprised 41 cognitive assimilation states and 22 socio-cultural assimilation states. The frequency of socio-cultural assimilation-related speech states that involved actual communication AI, Q, and A was 21 (96%), and this ratio declined to 33% when expanded to all speech states in the entire class. In the control group, there were 61 speech states by students, 47 of which related to the differential concepts. This 47 comprised 24 cognitive assimilation states and 23 socio-cultural assimilation states. The frequency of socio-cultural assimilation states that involved practical communication AI, Q, and A was 11(48%), and this ratio declined to 23% when expanded to all speech states in the entire class.

Instantaneous rate of change class

Table 6 presents our classification results of eight students' speech expressed in the instantaneous rate of change classes. We found a qualitative difference in the discourse between the two groups. Only assimilation states occurred in the control group, both assimilation and accommodation states occurred in the experimental group. In the discourse of the control group, communication between students occurred through literal transmissions (scene 2; C-74, 76, 80, 82, 85, 87). However, in the experimental group, knowledge was doubly mediated through literal expressions (scene 1; E-62, 66, 67) and graphs (scene 1; E-101). In addition, students' communication (scene 1; E-104~110) became more active through direct manipulation activities (scene 1; E-103), allowing students to concentrate on the co-constructing of knowledge. This phenomenon resulted in actual communication and enabled them to influence each other, resulting in cognitive and socio-cultural accommodation states. As the analysis of the class scenes demonstrates, this change could be interpreted as being caused by the effects of the dynamic geometry environment. In other words, the dynamic geometry environment facilitated the occurrence of richer visual mediators than those in the control group, leading to discourse changes. Consequentially, the rate of actual communication states was higher in the experimental group (28%) than in control group (21%).

We now discuss both groups' speech characteristics in more detail, including the class scenes where the difference in the communication methods between the two groups was noticeable. First, in the experimental group, there were 87 speech states by students, 79 of which related to the instantaneous rate of change. As **Table 6** demonstrates, this 79 involved 49 cognitive assimilation states, 1 cognitive accommodation state, 25 socio-cultural assimilation states, and 4 socio-cultural accommodation states. In particular, socio-cultural accommodation states appeared only in instantaneous change rate class of the experimental group. We also found that 22 (76%) of the socio-cultural assimilation and socio-cultural accommodation states were related to actual communication function AI, Q, and A.

Scene 1 (**Table 7**) revealed that students in the experimental group communicated through not only the transformation of literal expressions but also the dynamic movement of graphs. Hence, they co-constructed knowledge through richer visual mediators than students in the control group did. In addition, the enlargement of a statement that described the relationship between the differential coefficient and the slope of the tangent was confirmed in this discourse, as the speech of E4 showed.

In E-62-67, students communicated through verbal representations and transformations of literal

Table 7. Scene 1

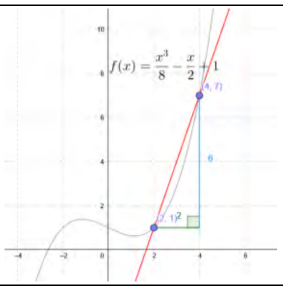
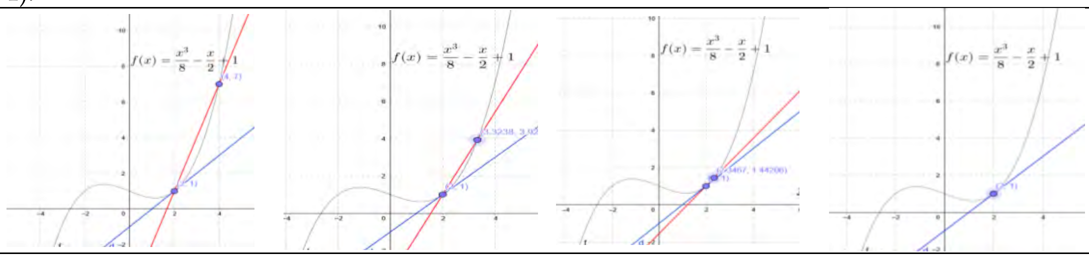
		··· Middle omission ···		
E-62	E3	Extreme value of average rate of change when $\Delta x \rightarrow 0$?	Q	NCA
E-63	E4	Extreme value?	Q	NCA
E-64	E2	Limit if it is extreme?	Q	NCA
E-65	E4	Oh, you can put a 'lim' in front of the average rate of change.	A	NCR
E-66	E1	When the limit Δx goes to 0, $\frac{\Delta y}{\Delta x}$.	AI	NSA
E-67	E3	Oh! Then you can put the 'lim' on the expression above $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$.	AI	NSA
E-68	All	Yes. That's right.	A	NSA
		··· Middle omission ···		
		 All four students were talking to each other looking at the average rate of change graph drawn using Geogebra on a laptop.		
E-101	E1	But when I look for a slope, I get an x increment over y increment. So, slope of straight line passing through two points is $f(b)-f(a)$ over $b-a$, which is same as average rate of change.	AI	NCA
E-102	All	Ah! That's right!	A	NSA
E-103	E1	I can move this. If I get close to this point …	AI	NCA
		At this time, E1 drew a tangent line on (2, 1) using Geogebra. Then she dragged point (4, 7) to (2, 1).		
				
E-104	E4	The line is getting closer.	AI	NCA
E-105	E2	The red line and the blue line get the same.	AI	NCA
E-106	E3	Ah-ha! When this point is near to this point, it is tangent. Is this (2, 1)? Then, when (4, 7) approaches (2, 1), the slope of the tangent becomes equal. (4, 7) overlaps with (2, 1) and this line overlaps with the tangent.	AI	NCR
E-107	E1	Yes.	A	NSA
E-108	E4	Ah~! The slope of tangent is the same.	DM	NSR
E-109	E2	Oh! Same as the slope of the tangent at point (2, 1).	DM	NSR
E-110	E3	Oh, so the extreme value of the average rate of change is not equal to 0, is it equal to the tangent slope? The slope of the tangent is equal to the differential coefficient at $x=2$.	AI	NSR
E-111	E3	Well, let's get our concept organized and go over.	CT	

Table 8. E4's active communication

	
Students in the experimental group communicating with each other while watching the process by which a straight line is a tangent on the laptop	E4 actively communicating as a participant by pointing to the tangent line with her hand

expressions. In E-65, E4 attempted to attach the "lim" expression to the average rate of change which qualifies as cognitive accommodation even if it did not result in fully understanding the concept of limit. This shows that she transformed the concept of average rate of change into that of instantaneous rate of change. Although she used these verbal expressions, it is difficult to say that she understood the concept of limit. These changes were objectified through a common identification process in E-66~E-68.

Subsequently, students examined the average rate of change graph drawn by using *Geogebra* on the laptop together and confirmed that the slope of the graph was the same as the average rate of change (**Table 8**). The scene 1 demonstrated that the movement of the straight-line graph played a conceptual mediating role in the students' acquisition of the concept of instantaneous rate of change.

In the case of E4, although she did not fully understand the concept of average rate of change and instantaneous rate of change on the pre-test, communication by using average rate of change, the "limit" expression transformation (E-65), and the visual representation of the movement of the straight-line graph enabled her to understand the concept of instantaneous rate of change.

At E-103, E1 used *Geogebra* to drag point (4, 7) to point (2, 1), and all students were concentrating and communicating. From E1's Drag, E3 (E-106, E-110), E4 (E-108), and E2 (E-109) consciously accepted that the average rate of change transforms into the instantaneous rate of change and is geometrically equal to the slope of the tangent line. This experience in the dynamic geometry environment produced changes in students' attitudes or feelings of empathy, observed by the acceptance through "ah!" However, understanding the concept by using these visual representations can lead to the formation of false concept images (Tall & Vinner, 1981). Because these the false concept images can later influence the formation of formal differential concepts taught in universities (Robert & Speer, 2001; Sierpinska, 1987, 1990), teachers should pay attention to the teaching and learning of differential concepts.

Next, in control group, there were 50 speech states, 42 of which related to instantaneous rate of change concept. 42 speech states involved 25 cognitive assimilation states and 17 socio-cultural assimilation states (**Table 5**). Nine socio-cultural assimilation states (53%) were related to actual communication AI, Q, and A.

In scene 2 (**Table 9**), the biggest feature revealed in this group was that students only communicated through the transformation of literal expressions. They used literal expressions as visual communication mediators in the co-construction of knowledge. Nevertheless, the narratives of the instantaneous rate of change were abundant, except regarding the geometrical meanings.

Table 9. Scene 2

		⋯ Middle omission ⋯		
C-73	C1	Now, shall we solve the problem?	Q	
C-74	C2	You have to calculate the differential coefficient at $x = 2$, so you have to find $f'(2)$.	A	NCA
C-75	All	Yeah.	A	NSA
C-76	C2	If you get $f'(x)$ first, you get $2x + 3$.	AI	NCA
C-77	C1	Uh, you know we haven't studied differential calculus yet, right?	Q	
C-78	C2	Oh.	A	
C-79	C3	She's been rehearsing. You can do that, go on.	DM	
C-80	C2	Yes. First, if you get $f'(x)$, you get $2x + 3$, so if you put $x = 2$ into it, the answer is 7.	AI	NCA
C-81	All	That's right.	DM	NSA
C-82	C3	Then, I'll try to solve it once in the original way. $f'(2) = \lim_{\Delta x \to 0} \frac{f(2+\Delta x)-f(2)}{\Delta x}$, and if you assign $x^2 + 3x$ to $f(x)$ and solve it $\lim_{x \to 2}(7 + \Delta x)$, the answer is 7.	AI	NCA
C-83	C4	The answer is the same as what C2 did earlier.	AI	NCA
C-84	C1	Who else did it?	Q	

Table 9 (continued). Scene 2

C-85	C1	I solved it with $f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$, and when I solved it, the numerator was factorized, resulting in $x-2$.	AI	NCA
C-86	C4	Oh! Factorization.	R	NCA
C-87	C1	Then, we have $x+5$ remaining. You can put 2 in x, right? So the answer is 7.	AI	NCA
C-88	C4	Wow ~! We solved it in three ways. If I solved it alone, I would have only one thing, but together I found three methods!	M	

Table 10. Frequency on derivative

Category		Experimental group						Control group					
		E1	E2	E3	E4	Total		C1	C2	C3	C4	Total	
M	NCA	-	-	-	-	-	-	-	-	-	1	1	1
DM	NCA	-	-	-	-	-	1	-	-	-	1	1	4
	NSA	-	1	-	-	1		-	1	1	1	3	
AI	NCA	-	-	1	-	1	3	1	-	-	-	1	1
	NSA	-	1	-	1	2		-	-	-	-	-	
Q	NCA	-	1	3	-	4	4	-	1	5	-	6	6
A	NCA	1	-	-	-	1	11	2	-	1	-	3	6
	NSA	3	3	-	4	10		1	1	-	1	3	
Total		4	6	4	5	19		4	3	7	4	18	

In C-74, C2 experienced a recognitive and regenerative assimilation, expressing the meaning of instantaneous rate of change through a literal expression. Subsequently, in C-76 and C-80, she expressed the concrete meaning of $f'(x)$ by using a literal expression $2x+3$ and found the result value 7 through the substitution of $x=2$.

The speech of C2 demonstrates that the regenerative assimilation repeated. Unlike C2, who attempted to solve by using the differential formula, C3 solved by using the definition and showed that the answers by the formula and by the definition were the same (C-85). In C-85 and C-87, C1 attempted to solve by using another differential definition and confirmed that this produced the same answer.

In conclusion, students in the control group identified the meaning of instantaneous rate of change by using three expressions, two definition approaches, and one derivative formula. However, no evidence is observed that students spoke of or constructed the geometrical meaning of instantaneous rate of change.

From the results of the pre-test and post-test, we found that C1 and C2 understood the instantaneous rate of change concept but that C3 did not. This finding suggests that although C3 calculated the instantaneous rate of change by definition, she did not understand its meaning.

Derivative class

Table 10 presents our classification results of eight students' speech expressed in the derivative classes. We produced two important findings. First, both groups experienced assimilation, and neither experienced accommodation. Because students had learned the differentiation method before the experiment, our class observation results demonstrated that both groups solved problems by using the differential formula and that that phenomenon did not cause cognitive conflict. Second, the incidence of socialized speech states was higher in the experimental group (63%) than in the control group (17%), and the frequency of speech states was a little bit higher in the experimental group. This finding suggests that *Geogebra* continues to be a driver of student communication.

We no discuss both groups' speech characteristics in more detail. In the experimental group, there were 22 speech states, 19 of which were related to the concept of derivative. 19 speech states involved six cognitive assimilation states and 13 socio-cultural assimilation states. Among all socio-cultural assimilation states, 92% were socialized speech states that involved actual communication AI, Q, and A. In the control group, there were 20 speech states, 18 of which were related to the concept of derivative. 18 speech states comprised 12 cognitive assimilation states and six socio-cultural accommodation states. Among all the socio-cultural assimilation states, three (50%) were socialized speech states that involved actual communication AI, Q, and A.

Test and Interview

Table 11 presents the results of pre-test and post-test. Although C3, C4, E3, and E4 did not demonstrate comprehension of the average rate of change on the pre-test, they did on the post-test. Although C2, C3, C4, E2, E3, and E4 did not demonstrate comprehension of the instantaneous rate of change on the pre-test, they did on the post-test, although they used the formula. Most students did not demonstrate comprehension of the derivative on the pre-test, but for all of them, their comprehension levels improved on the post-test.

Table 11. Results of pre-test and post-test

Group		Pre-test						Post-test					
		1	2	3*	4	5*	6	1	2	3*	4	5*	6
Experiment	E1	○	○	●	○	-	-	○	○	◎	○	◎	○
	E2	○	○	-	-	-	-	○	○	●	○	●	-
	E3	○	-	-	-	-	-	○	○	●	○	●	-
	E4	-	-	-	-	-	-	○	○	●	○	●	-
Control	C1	○	○	●	○	●	-	○	○	◎	○	●	-
	C2	○	○	-	-	-	-	○	○	●	○	●	-
	C3	-	-	-	-	-	-	○	○	●	-	●	-
	C4	-	-	-	-	-	-	○	○	●	○	●	-

Note. 3*/5* ◎: Using the definition; ●: Using the formula

Notably, E1 solved the problem related to the derivative by using the definition and improved more than other students did.

Notably, concluding that the two groups' results on the written tests differ has limitations. However, the analysis of the scenes showed that the difference in the visual mediator increased students' qualitative and quantitative communication in the process of understanding the differential concepts. In the experimental group interviews, all students responded positively to the question on the helpfulness of *Geogebra*. Specifically, the results confirmed that the process of students' dragging objects and visually checking their movement was helpful. In addition, all students in the experimental group reported that colleagues' explanations and cooperation helped them learn. As was observed in the prior results, the analysis of the scenes demonstrated that the difference in the visual mediator increased students' qualitative and quantitative communication in the process of understanding of the differential concepts. These results were consistent with the interview results, that is, the groups exhibited positive attitudes toward the class.

CONCLUSIONS AND SUGGESTIONS

We investigated the cognitive and social processes in which high school students acquire differential concepts through communication in a dynamic geometry environment through some cases. Additionally, we observed how a dynamic geometry environment affects these processes. To achieve this objective, we recruited eight students on the basis of the pre-test results and divided them into an experimental group and a control group. We designed the learning environment to be the same except for manipulating *Geogebra* on the laptop jointly in the experimental group. The two groups' discourse was analyzed by applying an analysis framework that applied both Piaget's (1959) speech category and Rutherford's (2011) neo-Piagetian model. On the basis of our findings, we drew the following conclusion and proposed the following suggestion.

First, our discourse analysis revealed that the dynamic geometry environment improved communication among students and increased their achievement levels, and the class scene and the interview analysis showed that it positively affected their comprehension of the concept of differential. We found that graphic and dynamic representations created by dragging stimulated students' interest and increased communication among them. In particular, cognitive and socio-cultural accommodation states appeared only in the instantaneous change rate class of the experimental group. This finding aligns with those of Sinclair & Moss (2012) and more clearly demonstrated the qualitative difference between the communication of the control and experimental groups.

Second, the scene analysis and interview results regarding the instantaneous rate of change class confirmed that the main features of the dynamic geometric environment, visual expressions, and manipulation activities including drag, positively influenced students' understanding of concepts. Because the instantaneous rate of change has an aspect that is difficult to abstract only through students' real experiences by including the concept of limit, there is a high possibility of causing cognitive conflict in their process of understanding this. However, we found that the manipulation activities involving graphical representation and dragging in the dynamic geometry environment served as the mediator of indirect experience, acting as a catalyst for the co-construction of knowledge through communication.

Third, although the quality of communication was somewhat different through the discourse analysis of the two groups, each participant's role was confirmed in the co-construction of knowledge among all students. This finding suggests that individual student discourse influences other peers in the communication process through participation, perceived change, expansion of concepts, and correction of perceptions. In particular, in the absence of teacher intervention, students' attitude changes occurred in the experimental group in methods that facilitated communication through the influence of rich visual mediators by using *Geogebra.*

Finally, Sfard (2008) divided human thought processes expressed in language into invisible and visible dimensions and claimed that the study of invisible, higher thought processes such as Piaget's (1959) reflective abstraction process would be difficult. These arguments imply that it would be difficult to find verbal evidence for assimilation and accommodation states in the processes of class and in individual students. Despite these limitations, in this study, we attempted to demonstrate that by applying Rutherford's (2011) neo-Piagetian model to the discourse analysis of students in high-school differential concept classes, the invisible aspects of human thinking can be analyzed qualitatively and quantitatively. An example of qualitative analysis is to investigate when and what type of assimilation and accommodation states occurred in the process of understanding the concept of differential in high school students, and an example of quantitative analysis is to measure the frequency of such assimilation and accommodation states. Although the results of this study are difficult to generalize to some cases, we expect that precise verbal information on various representations of mathematical content in the process of understanding the concepts of learners could be an opportunity to prepare educational environments corresponding thereto. Therefore, we suggest that the attempts to analyze the discourse on the students' understanding process of various mathematical concepts will be made continuously.

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