MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2022, 18(7), em2131
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/12172

Task design and enactment: Developing in-service and prospective teachers' didactical knowledge in lesson study

Paula Gomes [1*], Micaela Martins [1], Marisa Quaresma [1], Joana Mata-Pereira [1], João Pedro da Ponte [1]

[1] Instituto de Educação, Universidade de Lisboa, Lisboa, PORTUGAL

Received 7 March 2022 ▪ Accepted 18 May 2022

Abstract
Designing tasks to foster students' learning is challenging for teachers. In this article, we aim to understand how designing, enacting, and reflecting on tasks promotes the development of in-service and prospective teachers' didactical knowledge during lesson study. Data was collected through participant observation with audio recordings and document collection. Data analysis was based on a category system for task design and students' work and a teachers' didactical knowledge framework. The results suggest that, despite their different teaching experiences, there are aspects that both in-service and prospective teachers attended to when designing tasks. However, in-service teachers had clearer aims when designing tasks and changed them, after observing and reflecting on students' work. Moreover, proposing tasks and reflecting on students' work allowed both groups to rethink their practice to foster students' learning, highlighting the potential of lesson study to deepen their knowledge.

Keywords: in-service teachers, lesson study, mathematics, prospective teachers, task design, teachers' didactical knowledge

INTRODUCTION

Tasks are a fundamental basis for students' learning. Research highlights the importance of students working on tasks in which they can use different solving strategies and representations, building on their prior learning (Quaresma & Ponte, 2017; Stein et al., 2008). A particular type of such tasks are exploratory tasks (Ponte, 2005) for which students do not have an immediate solution but on which they can work using their previous knowledge.

Several investigations (e.g., Quaresma & Ponte, 2017) indicate that, when students work on exploratory tasks, they may become more involved in the learning process, resulting in more meaningful learning. Teachers have a key role in selecting and designing such tasks. However, selecting and designing exploratory tasks is challenging for teachers (Jesus et al., 2018; Santos et al., 2019). On the one hand, these tasks should be challenging, but also adjusted to the students who will work on them. On the other hand, exploratory tasks should go beyond the application of concepts and procedures, and should promote the construction of new concepts, representations or mathematical procedures.

Teachers also find a challenge to teach lessons based on those tasks. Lessons based on exploratory tasks are anchored on students' solving strategies, where moments of discussion after students' work take on particular importance, increasing teachers' in-moment decisions (Quaresma & Ponte, 2017; Stein et al., 2008). Additionally, most teachers are not familiar with lessons based on exploratory tasks (Jesus et al., 2018; Martins et al., 2021). As Stein et al. (2008) point out, leading these type of lessons requires "an extensive network of content

A preliminary version of this work was presented at the International Lesson Study Seminar in Mathematics Education organized by Grupos de Pesquisa Investigação em Ensino de Matemática da Universidade de Brasília and Prática Pedagógica em Matemática da Universidade Estadual de Campinas, Brasil at May 13th of 2021.

✉ paula.gomes@campus.ul.pt **(*Correspondence)** ✉ msterceiro@edu.ulisboa.pt ✉ mq@campus.ul.pt
✉ joanamatapereira@campus.ul.pt ✉ jpponte@ie.ulisboa.pt

Contribution to the literature

- This research indicates aspects that both in-service and prospective teachers considered when designing tasks, influenced by their teaching experiences.
- This research highlights the potential of lesson study for the development of teachers' and prospective teachers' didactical knowledge.
- Putting into practice the tasks they designed and reflecting on their influence on students' learning allowed teachers and prospective teachers to better understand how to plan and teach exploratory lessons to foster students' learning, and to rethink their teaching practices.

knowledge, pedagogical knowledge, and knowledge of students as learners" (p. 320).

Leading such lessons can be particularly challenging for prospective teachers with no teaching experience, because "without solid expectations for what is likely to happen, novices are regularly surprised by what students say and do, and therefore often do not know how to respond to students in the midst of a discussion" (Stein et al., 2008, p. 321).

Anticipating students' solving strategies and most common difficulties during lesson planning helps teachers to select the students' solving strategies that meet lesson goals, to rethink the tasks and how to support the students while they are working on the task (Fujii, 2019; Groves et al., 2016; Meiliasari, 2019). However, anticipating students' strategies and difficulties in a thorough way is often also a challenge for teachers (Groves et al., 2016).

Lesson study is a professional development process in which teachers work collaboratively to plan a lesson in detail, focusing on students' learning. During the lesson planning, they carefully design tasks and anticipate students' solving strategies and difficulties, sharing their ideas and discussing how to foster students' learning. Then, one of the teachers teaches that lesson, called the *research lesson*, and all the group observes and reflects on the students' work, taking into account the plan of the lesson (Fujii, 2019). Several investigations indicate that lesson study promotes the development of in-service and prospective teachers' knowledge, namely about task design (Barber, 2018; Leavy & Hourigan, 2016) and students' learning (Coenders & Verhoef, 2019). Lesson study is also an opportunity for teachers to understand how to plan and teach lessons based on exploratory tasks, allowing them to rethink their teaching practices, their role, and the students' role in the classroom (Martins et al., 2021; Richit & Tomkelski, 2020). However, as Fujii (2019) indicates, few investigations address in detail how teachers carry out the design of the task for the research lesson. Therefore, it is important to understand how lesson study provides opportunities for in-service and prospective teachers to develop their knowledge about task design and how to enact tasks.

In this article, we aim to understand how designing, enacting and reflecting on tasks in lesson study promotes the development of in-service and prospective teachers' didactical knowledge, that is, knowledge about teaching practice and learning of mathematics. Our research questions are, as follows:

1. What aspects do teachers attend to when they design tasks?
2. How do discussions about tasks and students' learning in lesson study promote the development of teachers' didactical knowledge?

DESIGNING TASKS AND DEVELOPING TEACHERS' KNOWLEDGE IN LESSON STUDY

Selecting and designing tasks, planning, and teaching lessons are part of the knowledge that the teacher needs to teach mathematics. Several researchers have sought to establish theoretical frameworks that allow them to describe this knowledge, among which Ball et al. (2008) and Ponte (2012). Ball et al. (2008) proposes a model where they distinguish *content knowledge* (with three subcategories) and *pedagogical content knowledge* (with other three categories. Ponte (2012) proposes a different model for the knowledge related to teaching practice and the learning of mathematics, with four dimensions that interact with each other, naming it "didactical knowledge". One of these dimensions is *knowledge of mathematics for teaching*, similar to Ball et al.'s (2008) *specialized content knowledge* and *horizon content knowledge*, which includes knowledge of mathematical concepts, procedures and representations to be taught, and the teacher's interpretation of the school subject. A second dimension is *knowledge about students and their learning processes*, which includes knowledge about students as individuals and the solving strategies they may follow, their learning processes, and the most common difficulties, similar to Ball et al.'s (2008) *knowledge of content and students*. Another dimension is *knowledge of the curriculum*, which, as Ball et al. (2008) state, concerns the overall goals and objectives of the curriculum and knowledge of various resources and forms of assessment. A final dimension is *knowledge about teaching practice*, which Ponte (2012) considers to be the core of didactical knowledge. Like Ball et al.'s (2008) *knowledge of content and teaching* and *specialized content knowledge*, it includes designing tasks to propose to students, choosing the most appropriate working mode for each task, and promoting mathematical communication. Teachers have an important role in selecting and designing mathematical tasks to propose to their students (Barber, 2018). When they select tasks, they need to consider learning goals and what students learned in previous years (Fujii, 2018, 2019). A task may be difficult for some students, while for others, it may be a simple exercise, depending on whether or not they can solve it through an immediate process (Ponte, 2005).

The work students do on solving exploratory tasks may be the basis of their learning. So, it is important that teachers design tasks where students should be able to begin their work based on what they already know. Additionally, to develop students' mathematical reasoning, they should be able to solve the tasks using several representations and different solving strategies. The students should be able to explain their work and justify their strategies or analyze justifications presented by other students (Ponte et al., 2020). However, proposing exploratory tasks poses several challenges to teachers, both in supporting students during their autonomous work, and in leading whole-class discussions. Teachers need not only to consider possible solutions for the task, but also to anticipate the representations and solving strategies that their students may use and the difficulties that may arise (Fujii, 2018; Meiliasari, 2019). Anticipating students' strategies during the lesson planning sessions "is a characteristic of task design in lesson study" (Fujii, 2019, p. 698) and may lead teachers to change the wording of the tasks whenever deemed necessary. This work is the basis for the discussions on "the flow of the research lesson" (Fujii, 2019, p. 688), which occupies a considerable time of the planning sessions, including discussions on the suitability of the task and on the preparation of the whole-class discussion. Based on the observations made during the lesson, the teachers reflect on the students' work and difficulties during the lesson, which may lead them to make changes in the task or the lesson plan.

Several studies indicate that lesson study, as a professional development process for teachers, creates environments for sharing, discussing, and reflecting on important aspects of lesson planning and classroom management (Fujii, 2018; Ponte, 2017). In lesson study, the teachers analyze the strengths and weaknesses of different types of tasks, and prepare the lesson in detail, based on the anticipation of students' work on the tasks that they designed (Barber, 2018; Fujii, 2019; Leavy & Hourigan, 2016; Martins et al., 2021). For example, Fujii (2018, 2019) refers to discussions among teachers on the choice of numbers and on the wording of questions arising from their anticipation of students' strategies and difficulties. The choice of a mathematical or non-mathematical context is another factor considered when designing tasks in lesson study (Leavy & Hourigan, 2016; Ni Shuilleabhain & Seery, 2017) as so the in real-world or in purely mathematical contexts. For instance, Leavy and Hourigan (2016) indicate that all prospective teachers participating in lesson studies that they conducted "reported the benefits of using a context to promote learning" (p. 168).

Other studies refer to the challenges faced by in-service and prospective teachers in their anticipation of students' strategies and difficulties. In the study of Groves et al. (2016), the teachers' reflections on their anticipation efforts helped them to prepare their interventions to support their students during the research lessons. They also highlight the changes made by the teachers to the lesson plan and wording of the task as a result of their reflection on the students' work. Concerning the challenges faced by prospective teachers who were asked to anticipate the students' strategies, Meiliasari (2019) concludes that this anticipation, although vague, "helped the pre-service teachers become more prepared and more confident" (p. 476).

Richit and Tomkelski (2020) claim that, in their research, by planning and teaching a research lesson, participants were led to reflect on the management of lessons based on exploratory tasks, namely on the role and learning of students and on the difficulties that may arise. Barber (2018) also mentions the learning of participating teachers in the design and management of tasks that may be solved by different representations, because of discussions in the lesson study sessions and observation of students' work in the research lesson. Regarding prospective teachers, Leavy and Hourigan (2016) indicate that there were "positive outcomes ranging from improvements in participants' mathematical knowledge and pedagogical content knowledge" (p. 162-163).

Among other learning outcomes, the authors report that prospective teachers developed their ability to identify weaknesses associated with certain procedures, leading them to be more attentive to selecting and designing tasks. In addition, Martins et al. (2021) highlight that the prospective teachers changed the wording of tasks, anticipated students' solving strategies and difficulties, and included moments of students' autonomous work and of whole-class discussion in the lesson plan, becoming more familiar with lessons based on exploratory tasks. Thus, participating in a lesson study is an opportunity for teachers and prospective teachers to develop their knowledge of task design (Barber, 2018; Leavy & Hourigan, 2016), students' strategies and difficulties (Groves et al., 2016; Meiliasari, 2019), and planning and teaching lessons based on exploratory tasks (Martins et al., 2021; Quaresma & Ponte, 2016).

METHODOLOGY

Participants

This article stands on two lesson studies, carried out in 2019/2020, in Portugal. A lesson study involved three in-service mathematics teachers (ITs), Branca, Luz, and Sofia, all from the same school, who were invited to participate. They all had been teaching for more than 25 years. The first author (researcher), who had already worked with the teachers in other formative processes, assumed the role of facilitator by preparing and leading the lesson study sessions. The other lesson study involved two prospective mathematics teachers (PTs), Mónica and Olívia, in their last year of their mathematics teacher education program, and they were the only ones who were being supervised at their university by a teacher educator who showed interest in lesson study. The PTs had no teaching experience, although they observed several lessons in the previous school year. The second author (researcher) and the teacher educator were also participants in this lesson study. They met regularly to prepare the lesson study sessions, which they both facilitated. When possible, the cooperating teacher responsible for the classes where the research lessons would be taught also participated in the sessions.

Context of the Lesson Studies

The ITs planned a research lesson to grade 11 students. A significant part of the seven planning sessions was spent on designing tasks in which students could work autonomously, explaining their ideas and justifying their reasoning. They also anticipated possible students' solving strategies, thinking about ways to support students in the difficulties that might emerge, for instance, with questions. The task was adapted from the students' textbook, where an example involving the volume of lidless boxes was presented. All teachers taught the research lesson, in their classes. After each lesson, the teachers reflected on the students' work, comparing it to what they had planned, and discussed possible changes to the task or the lesson plan.

The PTs spent four of nine lesson study sessions planning a lesson for grade 7 students. They discussed different types of tasks and possible tasks to propose in the classroom, evaluating strengths and weaknesses considering the lesson goal. The PTs also discussed the representations to foster, students' possible solving strategies and difficulties, and strategies to lead the whole-class discussion. The task was adapted from a textbook and was thoroughly discussed during three sessions. The two PTs taught the research lesson in two different classes and reflected on it, considering the students' work.

Table 1. Categories for data analysis

Category	Subcategory
Task design	Wording of questions
	Order of questions
	Representations
Students' work	Previous learning & possible solving strategies
	Common difficulties & questions to support them
	Exploratory lessons

Data Sources and Analysis

This is a qualitative and interpretative research (Bogdan & Biklen, 1994), considering the environment of the lesson study as the source of data, and focusing on the work done by the participants. Data collection includes audio recordings of the planning (Sx) and the reflection sessions, document collection (tasks discussed and adapted, lesson plans, and written reflections), and interviews with participants.

We analyzed the discussions about tasks and students' work in both the planning and reflection sessions of each lesson study and we also considered excerpts of the teachers' interviews where they talked about task design and students' work. This resulted in a preliminary system of categories, inspired by the theoretical framework (Barber, 2018; Fujii, 2019; Groves et al., 2016; Leavy & Hourigan, 2016; Martins et al., 2021; Meiliasari, 2019; Quaresma & Ponte, 2016). We considered two categories, each with four subcategories:

1. *task design*, considering context, wording and order of questions, and representations; and
2. *students' work*, including their previous learning, possible solving strategies, common difficulties, and questions to support them, and reasoning processes.

Next, we coded the data based on those categories and refined them, resulting in the categories and subcategories in **Table 1**. We merged *context* with *wording*, as the discussions about context were also about the wording of the task; we combined *reasoning processes* with *representations* because participants' discussions about representations to use considered the reasoning processes they wanted to foster; we merged students' *previous learning* and *possible solving strategies* because, when the participants discussed the students' possible solving strategies, they always had the students' previous learning in mind; we also added *exploratory lessons* as a subcategory, meaning lessons based on exploratory tasks, since both in-service and prospective teachers planned the research lessons considering this teaching approach which usually unfolds in three stages: presentation of a task, students autonomous work, whole-class discussion (Ponte, 2005; Stein et al., 2008).

Additionally, we coded the transcripts in each subcategory and analyzed them to investigate the development of knowledge of ITs and PTs, using two

1.6. After discussing the previous question, Beatriz said: "The function whose algebraic expression is $g(x) = |x - 3|$ has a minimum at $x = 3$. Will $g'(3) = 0$?

Answer Beatriz's question.

Figure 1. Wording of a question (S11)

key dimensions of teachers' knowledge: *knowledge about teaching practice*, especially about task design and enactment, and *knowledge about students and their learning processes* (Ball et al., 2008; Ponte, 2012). The data were coded separately by the first two authors of this article, who met regularly to discuss discrepancies and reach a consensus. The quality of the data analysis was ensured by discussions among all authors.

Ethics

Permissions necessary for data collection were requested from participants and school leaders. The ethical principles indicated by AERA (2011) were considered by the researchers. All participants were informed about the research goals and, after that, authorized the data collection. To ensure participants' anonymity, their names are pseudonyms.

RESULTS

Lesson Study with ITs

At the grade 11, students should be able to solve problems involving the determination of intervals of monotonicity and the extrema of real functions of a real variable by applying the derivative concept. As a starting point, the ITs decided to give the students a task that involved building lidless boxes from a sheet of paper, cutting out four squares of side x, one from each corner of the sheet. The students should establish a relationship between the monotonicity of a V function, which relates the measurement of the height of the box to its volume, and the sign of the respective derivative function. They should also determine the maximizer of the function.

Wording of questions

Considering their knowledge of their students, the ITs decided to include questions to address students' most common difficulties, such as understanding that a function may have an extreme at $x = a$, $a \in R$, without being derivable at this point. To promote the justification, the ITs decided to include the function g, whose algebraic representation is $g(x) = |x - 3|$, and discussed how to write the question, considering the information to give to the students:

> Luz: "What can be concluded about the derivative? And about the extrema?"
>
> Sofia: "And in this case, is it a null derivative?" … "Is the derivative of g at 3 null"?… "And is $g'(3)$ equal to zero?"… Or we could put it the other way around: "g has no derivative at $x = 3$".
>
> Researcher: But should we say this, or do we want them to conclude it?
>
> Sofia: Because it's more evident that [the function] has a minimum… "And is it a null derivative?" The fact that there is no derivative, or not a null derivative, does not mean it can't be an extreme (S11).

After discussing, the ITs decided to write "$g(x) = |x - 3|$ has a minimum" (**Figure 1**), aiming not to limit students' strategies and representations, thus enriching the whole-class discussion.

In the post-lesson reflection, the ITs mentioned that the wording of that question was clear, as most students realized what to answer. However, several students had difficulties in establishing a relationship between the sign of the derivative of the function and its monotonicity, so they sought reasons for this difficulty:

> When they [the students] put the graph on paper, they don't bring them together [the zero of the derivative and the maximizer of the function] … But if they first understand that the maximum of the function corresponds to the zero of the derivative, when they write it down ok then they do bring them together. If not… They have difficulty in establishing the relationship (Sofia, reflection 2).

As they believed that the students' difficulty was caused by the wording of the questions, they decided to change it. Instead asking first to draw a connection between the sign of the derivative and the monotonicity of the function, and then determining the zero of the derivative, the students should understand that the zero of the derivative is the maximizer of the function, and only then they should identify the intervals where V' is positive or negative and try to establish the requested relationship:

> For them [students] to understand that the value of x is the same, if they do this [determine the maximizer of the function], they can see that the value of x at a maximum corresponds to the zero of the derivative… They first look at the issue of 3

1.7. Let f be a continuous function on an interval $[a, b]$ and $x_0 \in]a, b[$.

Write down, providing a justification, the logical value of each of the following propositions:

I. For the function f to have an extreme at $x = x_0$ f does not need to be differentiable at $x = x_0$.

II. If $f'(x_0) = 0$ then f has an extreme at $x = x_0$.

III. If $f'(x_0) = 0,\ f'(x) > 0, \forall x \in [a, x_0[$ and $f'(x) < 0, \forall x \in]x_0, b]$ then f has a minimum at $x = x_0$.

IV. If $f'(x_0) = 0,\ f'(x) > 0, \forall x \in [a, x_0[$ and $f'(x) < 0, \forall x \in]x_0, b]$ then f has a maximum at $x = x_0$.

Figure 2. Order of questions (S12)

> and then... They'll realize that the x is the same (Sofia, reflection 2).

Thus, the task design was influenced by discussions on the questions that should be included in the task and on how they should be worded to address the students' most common difficulties, without limiting the representations and strategies they could use and without moving away from the goal of the lesson. It was also influenced by teachers' reflections about students' work, based on their observations during the research lesson.

Order of questions

In addition to discussions about the questions' wording, the ITs also discussed how they should order them:

> They have all the possible examples. To start with [the function of] the volume of the box has an extreme: the derivative is null [$at\ x = 3$] and it is extreme [of the function]. Then the other one [question] that [$x = 0$] is zero [of the derivative] but not extreme [of $f(x) = x^3$]. And then the other one [$g(x) = |x - 3|$] that has an extreme but not a derivative [at $x = 3$] (Sofia, S12).

The ITs decided to order the questions so that the students would begin by establishing a relationship between the monotonicity of function V and the sign of its derivative. They would then work on specific cases, one where there is a derivative but the function has no extremes ($f(x) = x^3$) and the other where $x = 3$ is the extreme of $g(x) = |x - 3|$, while the derivative is not set at $x = 3$. Finally, to promote a generalization, they included a question, where the students (**Figure 2**) "have to consider whether or not it is true by summarizing what they have previously seen..." (Sofia, S12).

The task design was therefore influenced by the discussion on the order of the questions, beginning with questions where the algebraic expressions of the functions were on the task and ending with a question where that was not the case.

Representations

To establish a relationship between the monotonicity of function V and the sign of the derivative, the students could use a table in which they recorded the variation of the sign of the derivative function and the monotonicity of the function. The teachers had to decide whether or not to put a table in the statement of the task for students to complete:

> Sofia: If we put it here [the table], it is as if we are already killing it.
>
> Luz: We are already conditioning… But I think that then we need something to systematize...
>
> Sofia: Yes, maybe... Not give it straightaway... Just summarize it… In other words, give it [the table] as a structure for the organization of information, don't you think? (S11).

After some discussion, the teachers decided not to include a table to avoid standardizing the representations that might emerge. However, they decided that if the students did not use the table, then they would introduce it during the whole-class discussion, as they believed it would be the best way of organizing the information and responding to these types of questions:

> If they don't [use the table], we may make a little table to clarify the idea: "OK. Therefore, what do we know about this function? … We have the value where the derivative is nullified... And we then have the issue of looking at the maximum... Where the function is increasing, where the derivative is positive…" (Sofia, S11).

These teachers' discussions influenced the task design and brought them a new perspective on the representations to include in the task, without limiting the representations that the students could use.

Students' previous learning and possible solving strategies

Considering what they had planned, the ITs identified what learning they expected the students already developed. In the grade 10, they were expected to be able to identify zeros, intervals of monotonicity and the extreme of quadratic functions, and use a graphing calculator to determine the extrema of other polynomial functions. At the present, they were expected to use the derivative rules to write the algebraic expression of the derivative of polynomial functions:

> [Reading a first version of the task] "Nuno has a counterexample that shows that the statement is false… And refers to the real function of the real variable $f(x) = x^3$. Who is right? Why?" They have to think here. They are already familiar with the derivative rules... They immediately get the zero of the derivative… (Sofia, S11).

To decide on how they would phrase the questions, the teachers also identified contents that would be studied on in grade 12, such as how to relate the second-order derivative to the concavity of the graph of a function.

> [We should include in this task] Something that summarizes the relationship of monotonicity, the sign… Because this is also important for the 12th [grade]. There's a part there where we introduce concavity and the inflection points (Luz, S10).

Thus, the ITs evidenced their knowledge of the mathematics curriculum, not only on what the students were expected to know, but also on what the students were expected to learn in the following year, which influenced the task design.

With this knowledge, the ITs anticipated the students' possible solving strategies. For example, to the relationship between the monotonicity of function V and the sign of the derivative, students could use a verbal representation or a table.

After observing the lesson, the teachers reflected on the representations used by the students, namely the table in which they would record the variation of the sign of the derivative function and the monotonicity of the function:

> Luz: That student went up to the board to draw the table, and that [x values] was $-\infty$ to $+\infty$… And so, I asked: 'is this in the context of the problem?' They immediately said "no".
>
> Sofia: They should do a complete analysis and then intersect with the domain, or contextualize right from the beginning. [Students have] to be aware that they may do it both ways. These students understood because there they did it one way, and their peers presented it in another way (reflection 3).

Luz spoke about the answers of the students who drew the table considering the set of real numbers, and Sofia added that when the students presented and explained their work to the class, they could see different strategies could be used to solve the task. The teachers' reflections led them to look again at the task they designed, recognizing its potential not only for students being able to use different representations, but also for presenting and explaining them.

Students' common difficulties and questions to support them

The ITs also considered the difficulties that students might have when working on the task. For example, they anticipated that the symbolic language used in question 1.7 (**Figure 2**), would be difficult for some of the students. Instead of changing the wording of the question, they thought about how they could support them.

Just as they had anticipated, they realized that, during the lesson, some of the students "were a little overwhelmed by all these symbols… Because the interpretation is not that obvious" (Sofia, reflection 3).

However, the teachers believed that their planned interventions helped the students to overcome their difficulties:

> In their initial readings, "I haven't got a clue about what is written here". It's more formal, isn't it? … I think here it helped to have said: "So, let's have a good look. In line with the previous examples…" Because they don't always connect what has been done before with what comes later… After some discussion they managed to get it (Sofia, reflection 3).

According to Sofia, the suggestion to return to the answers they had given to the previous questions helped the students to overcome their difficulties, without having to change the wording of the task.

They also analyzed the difficulties and errors that they did not anticipate:

> Luz: I noticed with my class… [On the side of the base of the box] $16 - x$. They only remove it [x, the height of the box] from one side...
>
> Sofia: There's always someone in the groups who says: "$16 - x$". And someone else saying: "No". Even within the group, in the first discussion many use the $16 - x$ and then in the group someone says: "No. You have to remove..."
>
> Luz: They're aware that something must be removed, but they only take it from one side.
>
> Sofia: That's their initial instinct, but then there's always someone in the group to correct it. "No. You have to remove x from each side." There they get to $2x$ (reflection 3).

Listening to students' explanations during the lesson gave the teachers a new insight into the communication among them, as they were able to observe how discussions among the students during their autonomous work helped them to identify and correct the errors in their answers.

Reflecting on the work they carried out in the lesson study, the ITs acknowledged that anticipating students' strategies and difficulties influenced their work and learning, while helping the teacher to manage students' work on the task, even when they anticipate that the wording of the questions may give rise to difficulties.

Exploratory lessons

The work of the teachers in the lesson study gave them a broader view of the planning and teaching of exploratory lessons. They acknowledged that when students work on tasks such as the one they planned, learning is more meaningful, compared to more teacher-centered teaching approaches:

> It became somewhat more ingrained, more consolidated… You are not telling the student how things are done, the student is the one who creates, engages… The fact that it takes them some time is also the time required for them to engage… Because much of the time when we are telling them, they can see it, but it doesn't mean very much to them, right? … And I think that their learning becomes more consolidated… Understood in a deeper way (Sofia, interview).

Referring to propose exploratory tasks in their subsequent teaching practice, the teachers discussed the importance of planning its presentation, considering the target students and how they engage with and react to the task:

> Luz: I am now going to try this [a task to work on transformations of the graph of quadratic functions], which we did last year… I have a feeling that it will be different… Because they have more difficulties… Even the simple fact of working in groups… Sometimes this may also be a determining factor for things to go well.
>
> Paula: But as you already have the experience, if you were to plan the lesson from scratch… would it also be different?
>
> Luz: I'd already thought about that… It's completely different…
>
> Sofia: The introduction of the task itself may be more detailed, for example. If they have more difficulty in communicating, you may give them more detailed preparation there… So that they feel more at ease and can work autonomously. (interview)

Thus, the task design and lesson observation, as the ITs' reflections on that work, led them to rethink the tasks they propose to their students and how they teach the lesson.

Lesson Study with PTs

To introduce the topic of functions in grade 7, the cooperating teacher decided to begin reviewing direct proportion as a relationship between two variables. At this point, the students were expected to already be able to recognize and interpret this relationship using tables, graphical representations, and algebraic expressions. They were also expected to be able to determine values using various procedures such as proportional relationships, rule of three, and multiplicative factors.

Wording of questions

In the initial interview, the PTs expressed concern regarding students' motivation for mathematics lessons. In Olívia's words: "One [of my main concerns] will really be… motivating the students, getting the students' attention" (Olívia, initial interview).

Consequently, the PTs suggested that the context of the task for the research lesson should be closely connected to the students' reality: "Here [in the context of the school] it is more [the sale of] codfish and [the work of] codfish trawlers..." (Mónica, S4).

Then, the PTs sought to find out more information about the socio-economic context of the students and decided to adapt the original task (which illustrated a situation with the cost of cakes) to the context of fishing and selling codfish, since this is a common professional activity in the region.

After designing the first version of the task, considering the insights by the discussions during lesson study sessions, the PTs decided that some of the questions needed to be reworded, stating that a question wording influences the students' work and, consequently, the learning goal. Particularly, to use the word "reasoning" could lead students to structure their work on proportions which would limit the strategies they could follow: "Should we use the word 'reasoning' or 'determine the value of'? Because by saying that it is reasoning, the word already leads them towards proportions" (Mónica, S5).

This concern with the students' interests influenced the task design, adapting the task to their context (fishing and selling codfish), and considering numbers that were in line with the situation (the cost of codfish boxes, obtained through contact with a local wholesaler). The wording of the task was also influenced by discussions on the mathematical terminology that would shape the students' work.

Order of questions

To meet the learning goals, the group felt it was necessary to rethink the order of the questions (**Figure**

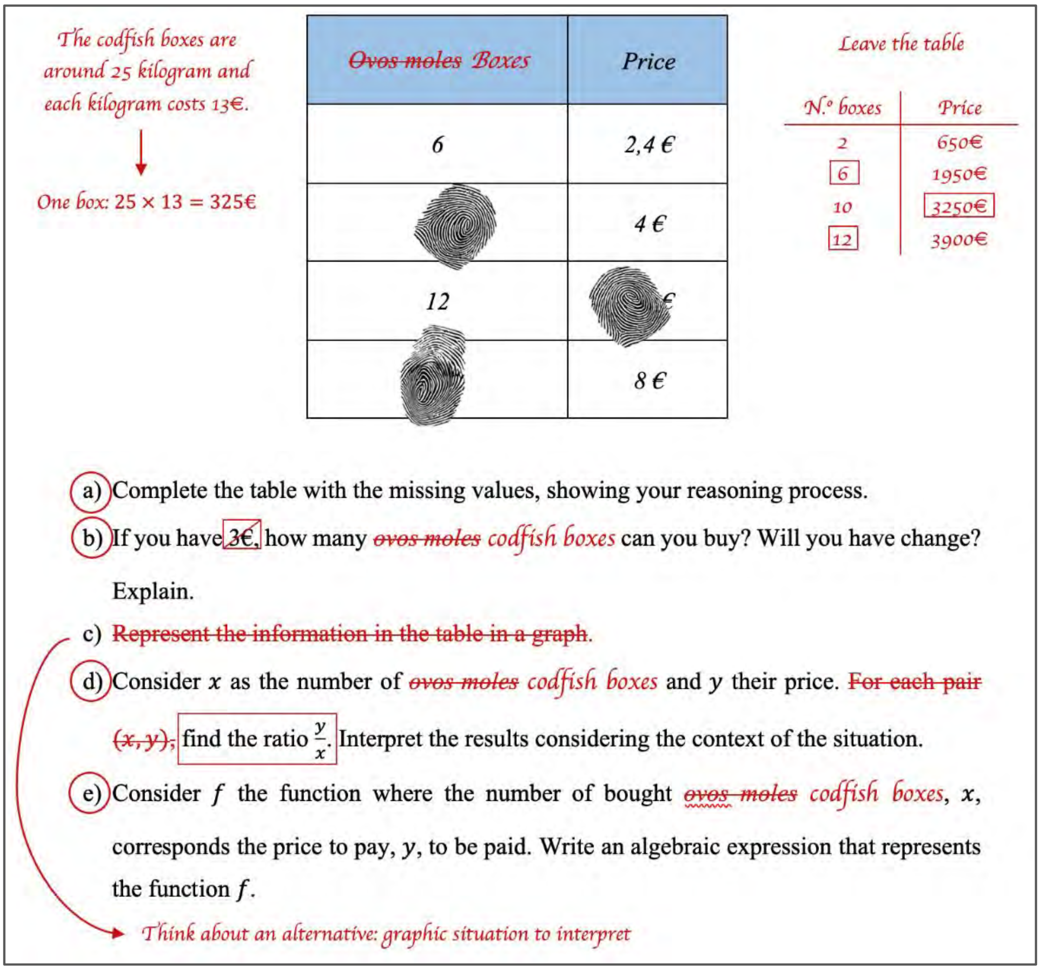


Figure 3. Order of questions (S4)

3). In the first version of the task, the students would begin by completing a table that related the number of codfish boxes to their cost (referred as *price* by PTs). Then, they would make a graphical representation of the situation, to finally write the algebraic expression representing the situation.

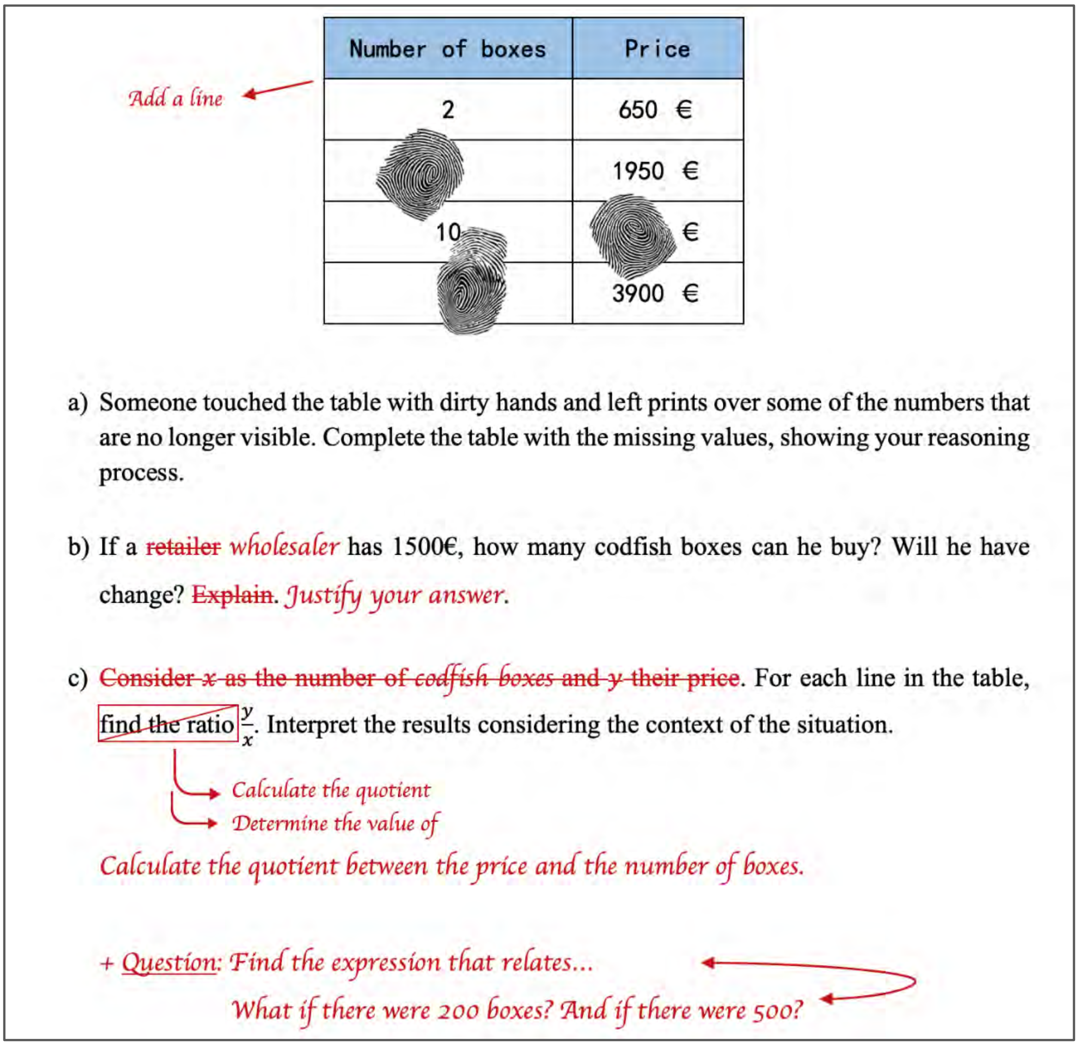
The PTs considered changing the order of the questions, as they wanted the students to identify the proportional ratio through the multiplicative factors that students would use to complete the table. After completing the table, the PTs would ask them to calculate the quotient between the two variables for all the values presented in the table. Then, the students could conclude that this quotient was constant, leading them to the value of the direct proportion constant (**Figure 4**). To help students write the algebraic expression relating the two variables, the group added another question in which they had to determine the cost of 200 and 500 boxes of codfish. Considering the values in **Figure 3**, to calculate the cost of 200 and 500 boxes, students could use proportion constant, which would direct them to write requested algebraic expression and, consequently, to promote generalization.

The PTs seemed to understand the importance of considering the order of the questions in the task, since it influenced the solving strategies and representations that the students might use. In this case, by proposing a graphical representation before an algebraic expression, the students could be led to write the algebraic expression without using multiplicative factors, which was not the PTs' intention.

Representations

The discussion about the representations to foster substantially influenced the task design, namely the order of questions. Initially, the PTs intended the students to explore the different ways of representing a function and considered the table, the graphical representation, the arrow diagram and the algebraic expression. However, the group decided to revise direct proportion as a relationship between two variables, and the PTs said that the students should explore tabular and graphical representations:

> Olívia: [Let's choose the] Cartesian graph… Because the arrow diagram is what they work with most…
>
> Mónica: The graph and the table… Is the best way for them to relate the variables…? As a basis for them to get to the algebraic expression as well… Which will foster generalization (S4).



Add a line

Number of boxes	Price
2	650 €
	1950 €
10	€
	3900 €

a) Someone touched the table with dirty hands and left prints over some of the numbers that are no longer visible. Complete the table with the missing values, showing your reasoning process.

b) If a ~~retailer~~ wholesaler has 1500€, how many codfish boxes can he buy? Will he have change? ~~Explain~~. Justify your answer.

c) ~~Consider x as the number of codfish boxes and y their price~~. For each line in the table, ~~find the ratio~~ $\frac{y}{x}$. Interpret the results considering the context of the situation.

→ Calculate the quotient
→ Determine the value of

Calculate the quotient between the price and the number of boxes.

+ Question: Find the expression that relates...

What if there were 200 boxes? And if there were 500?

Figure 4. Order of questions (S5)

To Mónica, after working on tabular and graphic representations, it would be easier to write the algebraic expression that related the number of codfish boxes to their cost. However, the group considered that asking the students to do a graphical representation would take up too much time and would not allow solving and discussing the remaining questions. Hence, the PTs decided to change the order of the questions (**Figure 4**) to help students to write the algebraic expression, showing a concern with promoting generalization. Additionally, they discussed the potential of using multiple representations, favoring tabular and algebraic representation, but proposing the graphical representation at the end of the task.

In the reflection, the PTs referred to the students' difficulties in writing the algebraic expression and compared with what they had anticipated:

> We did not anticipate, for example, that they would be able to generalize orally, but that they would have difficulties in writing mathematically. Algebraically (Mónica, reflection 2).

> [The students] were unable to translate into mathematical language, but wrote in everyday language (Olívia, reflection 2).

After observing how students expressed the relationship between the variables using verbal representation, the PTs paid more attention to the representations that students used to justify their reasoning, although they had not been able to write the algebraic expression. The PTs also valued the use of different representations when designing the task, leading them to change the order of the questions, considering the established learning goal and the duration of the lesson.

Students' previous learning and possible solving strategies

In the planning sessions, the PTs mentioned the importance of considering the learning the students should already have to properly design the task. According to Olívia: "the students' previous learning is important… Because it is the starting point to solve that task" (Olívia, S2).

The PTs demonstrated knowledge of the mathematics curriculum, namely in relation to the knowledge of students, considering the learning goals:

> [In the previous grade, students usually] work with proportion and have to find the proportion constant. Therefore, they work without variables… And sometimes do trial and error ... (Mónica, S4).

Thus, the discussions on the students' usual ways of working (procedures and representations) in direct proportion situations influenced the order and the wording of the questions (**Figure 4**), and therefore, the task design.

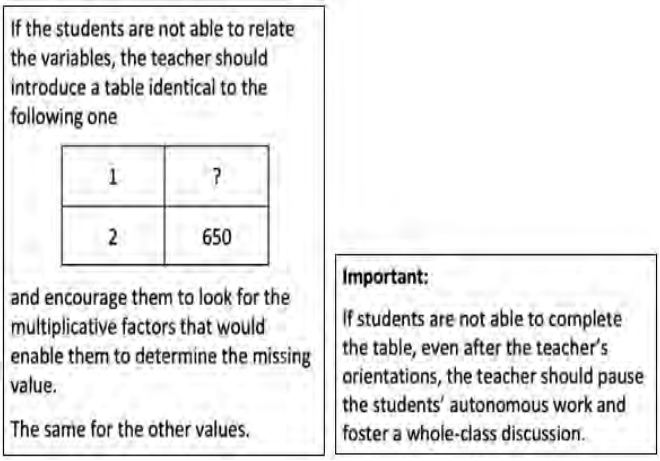
With this knowledge, the PTs autonomously anticipated the possible solving strategies students might use for each of the questions. In their analysis, the researcher, teacher educator and cooperating teacher concluded that the PTs did it in detail. In the planning sessions, the PTs had the opportunity to expand their knowledge on the strategies that the students could follow, based on the discussions regarding students' expected learning and the difficulties they might experience, namely the use of multiplicative factors, a solving strategy that had not been anticipated beforehand.

When reflecting on the lesson, the PTs mentioned students' difficulties in using proportions and the rule of three, which is learning they would have been expected to have already acquired. Mónica also referred to the fact that it would have been important to include a connection in the final synthesis between this learning and the work carried out during the lesson.

In terms of writing the algebraic expression, Olívia found that a group of students gave an answer that had not been anticipated and that should be accepted as a valid strategy, although the expected algebraic representation had not been used:

> Two groups wrote in everyday language that the cost of the codfish was obtained by multiplying 325 by the number of boxes. They were unable to translate this into mathematical language, but they did write it in everyday language. We also have to value this! (Olívia, reflection 2).

The work of the PTs in anticipating solving strategies was quite extensive. However, when observing the lesson, they identified an unanticipated answer and valued the different students' solving strategies, which provided them with a new insight into managing the task and the lesson.



If the students are not able to relate the variables, the teacher should introduce a table identical to the following one

1	?
2	650

and encourage them to look for the multiplicative factors that would enable them to determine the missing value.

The same for the other values.

Important:

If students are not able to complete the table, even after the teacher's orientations, the teacher should pause the students' autonomous work and foster a whole-class discussion.

Figure 5. Teacher's questions to support students (lesson plan)

Students' common difficulties and questions to support them

In addition to the students' solving strategies, the PTs also anticipated their potential difficulties when working on the task and, accordingly, included some supporting orientations in the lesson plan. For example, they considered that the students might find it difficult to complete the table relating the number of codfish boxes to the amount the wholesaler should pay. To help them overcome this, the teacher would refer to the multiplicative factors or consider only the first two lines of the table in order to simplify the information given (**Figure 5**).

The PTs also anticipated that the students might have difficulties in interpreting the values of the quotients between the two variables, considering all the values in the table, in the context of the situation. Intending to support them, the PTs suggested that the teacher use the price of a box to help the students make a connection with the found values (**Figure 6**).

When the PTs reflected on the students' work in the research lesson, they valued the anticipation of the students' difficulties and the preparation of teacher's interventions:

> These [anticipated] questions were confirmed and the students needed a lot of guidance… [They] managed to grasp the key ideas through the sentences that had previously been planned… That turned out to be an asset (Mónica, written reflection).

In Mónica's opinion, thinking about how the teacher could support the students during the lesson appears to have been crucial to their work on the task. Thus, the results point to a positive appraisal of the work involving anticipation and reflection on the task design and the students' work.

Teachers' answers and aspects to pay attention to

The teacher guides the students to draw a table identical to the task, adding a column to register what is asked in the question, in this case, $\frac{y}{x}$. Students should be encouraged to calculate the ratio for all the lines in the table.

Concerning the interpretation of the result, the teacher can ask students the price of one codfish box, encouraging them to compare this value with those obtained in the table, and leading them to conclude that the requested ratio is the price of one codfish box.

Figure 6. Teacher's questions to support students (lesson plan)

Exploratory lessons

Following the teacher educator's suggestion, the lesson would be planned according to the three stages of an exploratory lesson. The first tasks proposed by the PTs did not allow to use different solving strategies that could be discussed with the whole class. When they analyzed these tasks, the PTs realized that they needed to make adaptations, thus indicating their understanding that the tasks students are required to solve are related to the work they carry out. Therefore, the first lesson study sessions discussed the structure of an exploratory lesson and on the type of task that promotes the contrast among different solving strategies and representations.

Mónica created a task that allowed the students to use different solving strategies and representations, thereby enriching the whole-class discussion: "We want them [the students] to relate the variables and proportion ratio again, so… Making a [graphical] representation or analysis, will already lead them to the [whole-class] discussion..." (Mónica, S4).

Olívia, in turn, selected a task from a textbook that gave students the opportunity to relate different ways of representing a direct proportion situation (tabular, graphical and algebraic), although not allowing for different solving strategies.

Analyzing the proposed tasks, discussing their strengths and weaknesses, the PTs acknowledged the value of the tasks as a starting point for the students' work, as Olivia referred during the reflection phase: "The choice of the task is important, mainly because… It enabled the students to have an active role in the whole-class discussion" (Olívia, reflection 2).

This work appears to have been important for the PTs to develop their knowledge about the tasks which, in turn, influenced their knowledge regarding the organization of an exploratory lesson. At the end, Mónica and Olívia appeared to value this kind of lesson:

> I think this is the direction we should take and move away from traditional teaching… I think the students learn a lot (Olívia, final interview).
>
> The students develop great critical thinking, as it is easier for them to question their peers' strategies than their teacher's (Mónica, written reflection).

The discussions during the planning and reflection were opportunities for the PTs to rethink the planning and teaching of the lessons that promote student learning, with particular emphasis on exploratory tasks.

DISCUSSION

This study suggests that there are aspects that both ITs and PTs considered in task design and enactment. However, having different teaching experiences, their reasons for considering such aspects are distinct. Concerned about the students' learning, the ITs decided to include questions to address students' most common difficulties and discussed how to word them to foster justification. After observing students' work, they considered that the wording of two questions created them unnecessary difficulties and decided to reword the questions for students to promote the intended learning (as in Fujii, 2018, 2019). Differently, the PTs showed a great concern for students' motivation and decided to adapt the task to a context closely related to their reality. Thus, when discussing the wording of the questions in order to not limit students' possible strategies, albeit in different ways, the two groups developed their *knowledge about teaching practice*. In addition, the ITs also deepened their *knowledge about the students and their learning processes*.

The ITs decided that the students should begin working on particular cases to address their misconceptions and, at the end, they would include a question to encourage students to make generalizations. In a different way, aiming for students to explore different representations, the PTs decided to begin by proposing to them to fill in a table, leading the students to generalize, using the value of the direct proportion constant. Thus, both groups ordered the questions to develop students' capacity to generalize.

Regarding representations, based on their experience, the ITs considered the tabular representation as the most effective to establish the sought relationships. However, discussing whether to include a table in the task, they decided not to do so, to keep open the strategies that the students could follow. The PTs, for their part, decided to include questions where students should complete a table or write an algebraic expression of a function. Considering the goal and the available time for the lesson, they decided not to explore the graphical representation, although they recognized its value. Therefore, discussing the representations to include in the task and the representations that students could use was an opportunity for both groups to develop their *knowledge of teaching practice*. While valuing different representations, possibly due to lack of experience, the PTs did not anticipate that students could use the verbal representation instead of writing an algebraic expression of the function. Thus, analyzing the representations that the students used, allowed the PTs to develop their *knowledge about students and their learning processes*.

When they designed the tasks, both groups also considered the students' work, as their previous learning and possible solving strategies. The ITs, with greater knowledge of the curriculum and teaching practice, also attended to what the students were expected to learn and the representations they would use in the following year. When the ITs reflected on the research lesson, they discussed the potential of the task allowing a great range of solving strategies and representations that students could use, while the PTs focused on the unanticipated strategies. This work favored the development of ITs and PTs' *knowledge about students and their learning processes*.

Another aspect considered by both groups was anticipating students' difficulties. Considering their teaching experience, the ITs paid particular attention to the question in which the students were supposed to generalize, because they were supposed to do so based on the work that they did on several questions, which they were not usually asked to do. Reflecting on the lesson, the ITs mentioned that their interventions effectively helped the students solve the task and generalize. Reflecting on students' difficulties, the ITs also valued students' interactions while working on the task as a way to overcome them. Regarding the PTs, several studies point out they face challenges anticipating students' solving strategies and difficulties (Fujii, 2018; Meiliasari, 2019; Santos et al., 2019). However, they were able to anticipate almost all students' solving strategies and difficulties considering the *knowledge about students* that they developed observing the cooperating teacher practice. This highlights the collaborative work between the PTs, based on the lessons they observed. Reflecting on the lesson, both the ITs and PTs concluded that anticipating students' difficulties and preparing questions to support them, helps the teacher to manage students' work on the task. Therefore, this activity in lesson study influenced the task design, as Fujii (2019) and Meiliasari (2019) also point out and created opportunities for both groups to deepen their *knowledge about teaching practice* and about *students and their learning processes*.

Finally, both groups reflected on leading the exploratory lessons. The ITs and PTs realized that the tasks were, indeed, the basis for the students' work and that their learning is more meaningful, compared to teacher-centered teaching approaches. Reflecting on their students' strategies and difficulties, the ITs also discussed how they could manage exploratory tasks in other classes after the lesson study (as in Ni Shuilleabhain & Seery, 2017). In the case of the PTs, the development of their knowledge was enhanced by analyzing different types of tasks in the planning sessions, with an emphasis on exploratory tasks, as also concluded by Martins et al. (2021). This analysis was not necessary with the ITs as they had already worked with different types of tasks, including exploratory tasks.

The ITs had more teaching experience and knowledge about teaching practice and students' learning than the PTs, influencing the way they discussed the task design and enactment, and consequently, the knowledge that they developed. The ITs thought about the potential of the task to promote students' learning, helping them to overcome their usual difficulties. When the ITs designed the task, they also considered what they expected students to learn in the following year, which was not discussed by the PTs, showing their knowledge about the mathematics curriculum. The PTs had opportunities to develop their knowledge from working between themselves, but mainly through discussions with the teacher educator, the cooperating teacher, and the researcher, who had all teaching experience. Therefore, the ITs and the PTs were able to expand their knowledge on the design and management of tasks, in tandem with their knowledge about students and their learning processes.

CONCLUSION

Teachers face several challenges in designing tasks that allow students to use various solving strategies or representations as starting points for whole-class discussions. Our research highlights the potential of lesson study in the development of teachers' and prospective teachers' knowledge, allowing them to rethink their teaching practices and to better understand how to design and enact exploratory tasks. Answering research question 1, we conclude that when designing tasks, both ITs and PTs paid particular attention to the wording and order of the questions, as well as to the representations that they should include in the statement of the task or that students could use. Other aspects that both groups attended to when designing tasks were students' previous learning and possible solving strategies, the difficulties students might have in solving the task, and the interventions that could be made to help students overcome their difficulties.

The lesson study allowed the participants to put into practice the tasks that they designed, which favored their reflections about how the aspects that they attended to in designing the tasks influenced students' learning. Answering research question 2, we conclude that the discussions in the planning sessions allowed ITs and PTs to develop their *knowledge about the students and their learning processes*, namely the difficulties they might have and the strategies and representations they might use. Moreover, the ITs' and PTs' reflections on the research lesson allowed them to broaden their knowledge by analyzing the strategies that the students followed and the difficulties they had. Both groups were also able to develop their *knowledge about teaching practice*, not only when designing the task and planning the lesson, but also when reflecting on the students' work. In particular, they had the opportunity to discuss and reflect about designing tasks to propose to the students and about planning and leading different moments of an exploratory lesson.

Our research stresses the reflection phase as a fundamental part of the development of teachers' and prospective teachers' knowledge, pointing out the importance of creating opportunities for them to reflect on students' learning and in their teaching practice critically and systematically.

The PTs were able to anticipate students' work in detail, which may be a result of the lessons they observed before planning the research lesson. However, the cooperating teacher did not attend the lesson study sessions, which influenced task design. The task and the lesson were prepared without knowing the particular characteristics of the classes, as their usual difficulties or misconceptions, which is a limitation of this study.

The PTs were concerned with students' motivation, and they came up with several ideas to promote it, a concern that was not explicit in this ITs group. It may be very beneficial for PTs to participate in joint lesson studies with ITs (Coenders & Verhoef, 2019) as they could further develop their knowledge through the experiences shared by the ITs, who have more in-depth knowledge about teaching practice and about students and their learning processes. Additionally, joint lesson studies may also have advantages for the ITs since they could get in touch with the different ideas and concerns of the PTs, such as students' motivations and interests or the use of innovative materials and technologies that they tend to be more familiar with.

It is important to create national policies that support teachers' professional development by considering their needs and giving them the time and conditions to work collaboratively with other teachers and with prospective teachers. In addition, disseminating the work done in lesson study by PTs during their teacher education courses is also important for their professional development (Zhang, 2015) and may encourage their involvement with the research community.

Our research shows that the discussions in the planning and reflection phases were opportunities for participants to deepen their knowledge, particularly about tasks and students' learning, and these discussions were enhanced by the enactment of tasks. It also makes visible task design aspects in lesson study, which may contribute to teacher educators and facilitators to value this professional development process and be attentive to issues regarding how to structure and conduct lesson study.

Author contributions: PG & MM: wrote the first version of the theoretical framework and the research methodology, made the data collection and preliminary analysis of data and **PG, MM, MQ, JM-P, & JPP:** discussed all parts of the article, reviewed the paper, and contributed for the final version of the paper. All authors have agreed with the results and conclusions.

Funding: This work is supported by national funds through FCT–Fundação para a Ciência e Tecnologia by a grant to Paula Gomes (SFRH/BD/145118/2019) and Micaela Martins (SFRH/BD/143869/2019), and by a Stimulus of Scientific Employment program to Marisa Quaresma (2020.02874.CEECIND). This paper received national funding from the FCT–Fundação para a Ciência e a Tecnologia, within the scope of the UIDEF–Unidade de Investigação e Desenvolvimento em Educação e Formação–UIDB/04107/2020, and through the Project REASON (PTDC/CED-EDG/28022/2017).

Ethics approval: Permissions necessary for data collection were requested. The ethical principles indicated by AERA (2011) were considered by the researchers. All participants were informed about the research goals and, after that, authorized the data collection. To ensure participants' anonymity, their names are pseudonyms.

Declaration of interest: No conflict of interest is declared by authors.

Data sharing statement: The data collected for this study may be made available by the corresponding authors, Paula Gomes and Micaela Martins, in a request with adequate justification.

REFERENCES

AERA. (2011). Code of ethics. *Educational Researcher*, *40*(3), 145-156. https://doi.org/10.3102/0013189X11410403

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407. https://doi.org/10.1177/0022487108324554

Barber, K. (2018). Developing teachers' mathematical-task knowledge and practice through lesson study. *International Journal for Lesson and Learning Studies*, *7*(2), 136-149. https://doi.org/10.1108/IJLLS-09-2017-0042

Bogdan, R., & Biklen, S. (1994). *Investigação qualitativa em educação* [*Qualitative research for education*]. Porto Editora.

Coenders, F. G., & Verhoef, N. C. (2019). Lesson study: Professional development (PD) for beginning and experienced teachers. *Professional Development in Education*, *45*(2), 217-230. https://doi.org/10.1080/19415257.2018.1430050

Fujii, T. (2018). Lesson study and teaching mathematics through problem solving: The two wheels of a cart.

In M. Quaresma, C. Winsløw, S. Clivaz, J. P. Ponte, A. Ní Shúilleabháin, & A. Takahashi (Eds.), *Mathematics lesson study around the world* (pp. 1-21). Springer. https://doi.org/10.1007/978-3-319-75696-7_1

Fujii, T. (2019). Designing and adapting tasks in lesson planning: A critical process of lesson study. In R. Huang, A. Takahashi, & J. P. da Ponte (Eds.), *Theory and practice of lesson study in mathematics* (pp. 681-704). Springer International Publishing. https://doi.org/10.1007/978-3-030-04031-4_33

Groves, S., Doig, B., Vale, C., & Widjaja, W. (2016). Critical factors in the adaptation and implementation of Japanese lesson study in the Australian context. *ZDM-Mathematics Education*, *48*(4), 501-512. https://doi.org/10.1007/s11858-016-0786-8

Jesus, C. C., Cyrino, M. C., & Oliveira, H. (2018). Análise de tarefas cognitivamente desafiadoras em um processo de formação de professores de matemática [Analysis of cognitively challenging tasks in a process of mathematics teacher education]. *Educação Matemática Pesquisa: Revista Do Programa de Estudos Pós-Graduados Em Educação Matemática* [*Mathematics Education Research: Journal of the Postgraduate Studies Program in Mathematics Education*], *20*(2), 21-46. https://doi.org/10.23925/1983-3156.2018v20i2p21-46

Leavy, A. M., & Hourigan, M. (2016). Using lesson study to support knowledge development in initial teacher education: Insights from early number classrooms. *Teaching and Teacher Education*, *57*, 161-175. https://doi.org/10.1016/j.tate.2016.04.002

Martins, M., Mata-Pereira, J., & Ponte, J. P. (2021). Os desafios da abordagem exploratória no ensino da matemática: Aprendizagens de duas futuras professoras através do estudo de aula [The challenges of the exploratory approach in mathematics teaching: Preservice teachers' learning through lesson study] *Bolema: Boletim de Educação Matemática, 35*(69), 343-364. https://doi.org/10.1590/1980-4415v35n69a16

Meiliasari, M. (2019). Mathematics pre-service teachers' anticipation of students' responses: A case study of lesson study for pre-service teachers. In Y. Rahmawati, & P. C. Taylor (Eds.), *Empowering science and mathematics for global competitiveness* (pp. 472-478). CRC Press. https://doi.org/10.1201/9780429461903-65

Ni Shuilleabhain, A., & Seery, A. (2017). Enacting curriculum reform through lesson study: A case study of mathematics teacher learning. *Professional Development in Education*, *44*(2), 222-236. https://doi.org/10.1080/19415257.2017.1280521

Ponte, J. P. (2005). Gestão curricular em matemática [Curriculum management in mathematics]. In GTI (Ed.), *O professor e o desenvolvimento curricular* (pp. 11–34). APM.

Ponte, J. P. (2012). Estudiando el conocimiento y el desarrollo profesional del profesorado de matemáticas [Studying the knowledge and professional development of mathematics teachers]. In N. Planas (Ed.), *Teoría, crítica y práctica de la educatión matemática* (pp. 83–98). Graó.

Ponte, J. P. (2017). Lesson studies in initial mathematics teacher education, *International Journal for Lesson and Learning Studies*, *6*(2), 169-181. https://doi.org/10.1108/IJLLS-08-2016-0021

Ponte, J. P., Quaresma, M., & Mata-Pereira, J. (2020). Como desenvolver o raciocínio matemático na sala de aula? [How to develop mathematical reasoning in the classroom?]. *Educação e Matemática, 156*, 7-11.

Quaresma, M., & Ponte, J. P. (2016). Comunicação, tarefas e raciocínio: Aprendizagens profissionais proporcionadas por um estudo de aula [Communication, tasks and reasoning: Professional learning provided by a lesson study]. *Zetetike, 23*(2), 297-310. https://doi.org/10.20396/zet.v23i44.8646540

Quaresma, M., & Ponte, J. P. (2017). Dinâmicas de aprendizagem de professores de matemática no diagnóstico dos conhecimentos dos alunos num estudo de aula [Learning dynamics of mathematics teachers in the diagnosis of students' knowledge in a lesson study]. *Quadrante, 26*(2), 43-68.

Richit, A., & Tomkelski, M.L. (2020). Secondary school mathematics teachers' professional learning in a lesson study. *Acta Scientiae* [*Journal of Science*], *22*(3), 2-27. https://doi.org/10.17648/acta.scientiae.5067

Santos, L., Oliveira, H., Ponte, J. P., & Henriques, A. (2019). Pre-service teachers' experiences in selecting and proposing challenging tasks in secondary classrooms. In *Proceedings of the 11th Congress of European Research in Mathematics Education* (pp. 3762-3769).

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313-340. https://doi.org/10.1080/10986060802229675

Zhang, Y. (2015). Sustaining lesson study in schools with positive leadership. *International Journal of Lesson and Learning Studies, 4*(2), 140-154. https://doi.org/10.1108/IJLLS-07-2014-0018