# Improving the Performance of Mathematics Teachers through Preparing a Research Lesson on the Topic of Maximum and Minimum Values of a Trigonometric Function

Hosseinali Gholami

Institute for Mathematical Research, Universiti Putra Malaysia, Serdang, Selangor, Malaysia

hossein_gholami52@yahoo.com

**Abstract:** *In each educational curriculum, trigonometry is an important subject at upper secondary schools and pre university level that apply in many other subjects such as algebra, calculus, geometry and physics. Many of students have serious problem in learning the trigonometric materials because usually mathematics educators transfer the trigonometric concepts to students through traditional methods that encourage students to memorize the trigonometric concepts. The purpose of this qualitative case study is to introduce the Lesson Study as a new teaching method based on problem solving approach in order to increase the performance of teachers in teaching trigonometry. In this study, a group of three mathematics teachers from an international pre-university centre in Malaysia and the researcher contributed in preparing a research lesson on the topic of the maximum and minimum values of a trigonometric function. Also, this research lesson improved in an existing class containing 10 students. Data collected through observations of discussion meetings and analyzed descriptively. In this research lesson, the researcher discussed teaching the maximum and minimum values of a trigonometric function through variety of solutions and likely misconceptions among students. Maybe this article helps mathematics educators to have better performance in teaching trigonometry through Lesson Study based on problem solving approach.*

**Keywords:** *Lesson Study, Misconception, Problem solving, Trigonometry*

## INTRODUCTION

Trigonometry is a complex part of mathematics that plays an important role in our daily life. Trigonometric concepts are difficult to understand by learners and usually this subject is challenging for teaching (Martin-Fernandez et al., 2019). For example, a study by Gholami et al. (2021) shows that none of the mathematics lecturers ($n = 8$) chose teaching trigonometry as their favourite subject because of the complexity of this subject. They preferred to teach courses related to the calculus, algebra, statistics and probability. In teaching trigonometry, the challenge resides in the fact that many of traditional methods of teaching, primarily emphasizes superficial skills and such methods do not allow learners to understand the trigonometric topics conceptually (Altman & Kidron, 2016). Therefore, students face difficulties in learning trigonometry through problem solving due to misconceptions about trigonometric contents (Weber, 2005). In learning trigonometry, students experience numerous obstacles due to misconceptions, for instance, students' misunderstandings with the concepts angle and angle measure at the starting point for learning trigonometry are the most basic problem among students in depth understanding of trigonometric concepts (J. Nabie et al., 2018). Discussing different methods of solving trigonometric problems and common students 'misunderstandings about them will help improve teachers' performance in the classroom. Based on Xenofontos and Andrews (2014) a mathematics task or a goal-directed activity is considered as a problem for students if this task is new and challenging to them. They further added that a mathematical exercise is not a problem because learners solve mathematics exercises by following steps they have learned. Teaching mathematics materials straight from textbooks is common in our educational institutions including schools and universities (Dhakal et al., 2020). Therefore, most of mathematics teachers still prefer to teach mathematical concepts through traditional methods by emphasizes on solving routine exercises in teaching (Voskoglou, 2019). It seems new methods of teaching such as Lesson Study requires a high level of mathematical knowledge and pedagogy to prepare suitable mathematical materials based on the ability of students. Considering appropriate activities and mathematics problems in the prepared lessons help learners to have better performance in the classes and enhance their abilities in problem solving (Gholami, Ayub, & Yunus, 2021).

The Japanese Lesson study approach, not only focuses on a team-oriented educational design and shared responsibility for the educational processes and outcomes but also clearly focuses on students' experience of learning process and not simply on the methods of teaching (Elliott, 2019; Hanfstingl et al., 2019). For example, familiarity with students' misunderstandings about different mathematical concepts provides a good opportunity for the Lesson Study team to provide appropriate research lessons. In this educational method, Lesson Study group members prepare lessons that are called research lessons in a participatory manner and after teaching in real classes, they constantly improve them (Coenders & Verhoef, 2019; Lewis et al., 2006). Therefore, Lesson Study as a kind of professional development programs improves the teaching knowledge of educators especially their pedagogical content knowledge through discussions among them regarding the students' learning (Coenders & Verhoef, 2019). This educational approach helps mathematics teachers to overcome difficulties facing students such as their misconceptions about mathematical concepts and to improve student learning (Leavy & Hourigan, 2018). Japanese Lesson Study has various models and is now spreading to educational systems of other countries in order to increases students' learning through supports teachers in improving their skills and teaching practices (Grimsaeth & Hallas, 2015). Research lesson is the most important part of Lesson Study and the procedure of preparing a research lesson is as follows (Lewis, 2002).

1) The Lesson Study group members set the goals for students' learning based on their abilities and skills
2) The members of the Lesson Study group collaborate to improve a plan for a teaching session to provide better learning situation for students

3) One of the Lesson Study group members teaches the research lesson, while the others collect data by observation.
4) In a post-lesson discussion, the members of Lesson Study group analyze their observations in order to improve the quality of research lesson.
5) If necessary, the members of Lesson Study group plan to improve their teaching practice for a new research lesson.

Discussion about the misunderstandings of students about mathematical concepts is very beneficial for educators to have effective teaching. In other words, knowing the nature of misconceptions and misunderstandings and their sources regarding various contents of mathematics that are common among students of all educational levels, helps educators to plan suitable instructions for students' learning. Understanding mathematics concepts depend on linking from the prior knowledge and new topics, which may help or hinder the process of learning. Incorrect prior knowledge regarding mathematical concepts are called misconceptions, that cause a disability to learn the new contents (Alkhateeb, 2020). Mathematics educators can change the misconceptions after brief instructions and provide an appropriate situation for conceptually learning (Durkin & Rittle-Johnson, 2015). In fact, Suitable teaching methods eliminate the mistakes and misconceptions that students have about mathematical concepts (Yilmaz et al., 2018). In teaching trigonometry, mostly misconceptions arise from the teaching method. For instance, as identified by Tuna (2013) about 90% of the novice mathematics teachers had misconceptions regarding the definition of the trigonometric concept of radian. They explained incorrect definitions for this concept such as, "the expression of degree in terms of $\pi$", "the unit of length of degree", and "I just know the formula of $\frac{D}{180} = \frac{R}{\pi}$" and “I do not know what radian is". Therefore, mathematics educators, beside the appropriate subject matter knowledge, require knowing the common misunderstandings and misconceptions that students face in a specific topic. Regarding this issue, knowing the variety of creatively solutions for trigonometric problems and students' misconceptions about them provide a good opportunity for educators to improve their teaching knowledge (Dundar, 2015). The purpose of this study is to prepare an effective research lesson for teaching a given trigonometry problem and investigate the misconceptions emerging in students learning regarding this problem in order to provide better teaching and learning situation.

## METHODOLOGY

### *Research Design and Sample*

This study was conducted during the academic year 2020. In this study, a group of three mathematics teachers (Two males and a female) from an international school in Malaysia and the researcher collaboratively planned, discussed and designed a research lesson on the topic of the maximum and minimum values of a trigonometric function. Meanwhile, all the members of the Lesson Study group were experienced teachers with at least 15 years experiences in teaching mathematics. Furthermore, 10 students (4 male and 6 female) from an existing class were participating in this study.

*Data Collection*

Finding the maximum and minimum values of a trigonometric function is an important part of trigonometry subject that apply in many trigonometric concepts such as the range of functions, drawing the graph of functions and optimization the real world problems. For example, the following problem shows an application of the maximum and minimum values of a trigonometric function in the real world.

Problem: In a four season country, the length of each day of a year calculated based on the following trigonometric function

$$L(t) = 12 + 2.4\ \sin\frac{2\pi}{365}(t-1)$$

where, $t$ represents the order of days in the year (for example, $t = 4$ means the fourth day of the year) and $L(t)$ is the length of the day in hours. Determine the length of the longest and shortest day of the year.

Based on the importance of maximum and minimum values of the trigonometric functions, the researcher studied regarding the topic "Maximum and minimum values of the function $f(x) = a\sin x \pm b\cos x$, where $a, b \in R$". Therefore, in this research lesson, the Lesson Study group members suggested three solution methods for this problem and discussed the likely misconceptions of students about these solutions.

Two weeks before starting this study, the researcher introduced the topic of this research lesson to the members of Lesson Study group and asked them to prepare suitable material for a rich research lesson. In a meeting, they planned, discussed and designed a research lesson and a teacher of Lesson Study group taught this research lesson in a class and the others observed and collected data. In a post-discussion meeting, they tried to improve the quality of this research lesson.

*Analyzing the Data*

All members of the Lesson Study group were familiar with the Lesson Study approach because they participated in an in-service program related to the Lesson Study a few months before starting this study. The researcher introduced the topic of this research lesson to teachers and asked them to share their knowledge and experience to produce a research lesson on the topic entitled "Maximum and minimum values of the function $f(x) = a\sin x \pm b\cos x$, where $a, b \in R$". During three sessions, they planned, discussed and designed a research lesson for pre-university level students. The materials in this research lesson gathered through observations of discussion meetings and teaching this research lesson for students in a real class by a member of Lesson Study group. The researcher analyzed the methods and suggestions of the Lesson Study group members descriptively to prepare a research lesson.

**FINDINGS ON THE PREPARED RESEARCH LESSON**

The members of the Lesson Study group prepared a research lesson entitled "Maximum and minimum values of the function $f(x) = a\sin x \pm b\cos x$, where $a, b \in R$" and they suggested the following solutions collaboratively. Furthermore, they referred to some common misconceptions among students about this topic.

*First solution*

We know the formula

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.$$

In this formula, for $\sin(x \pm y)$, there are two terms $\sin x \cos y$ and $\cos x \sin y$ with the following property:

The sum of squares of factors one from each of two terms is equal to one, $sin^2x + cos^2x = 1$. Similarly, the sum of squares of factors two from both terms is equal to one, $cos^2y + sin^2y = 1$. This property can be regarded as the condition for an expression of the trigonometric function $y = a\sin x \pm b\cos y$ to be converted into an expression consisting of only sine function. The method and process for converting this trigonometric function into an expression consisting of only cosine function is similar.

In the function $y = a\sin x \pm b\cos x$, we have $sin^2x + cos^2x = 1$ but the sum of squares of the first factors of terms $a\sin x$ and $b\cos x$ is $a^2 + b^2$. Since we do not know whether $a^2 + b^2$ is equal to one or not, we multiply $a$ and $b$ by a number $m$ such that $(ma)^2 + (mb)^2 = 1$.

$$(ma)^2 + (mb)^2 = 1$$

$$\Rightarrow m^2a^2 + m^2b^2 = 1$$

$$\Rightarrow m^2(a^2 + b^2) = 1$$

$$\Rightarrow m^2 = \frac{1}{a^2 + b^2} \Rightarrow m = \frac{1}{\sqrt{a^2 + b^2}}.$$

Therefore, we have

$$(ma)^2 + (mb)^2 = 1 \Rightarrow \left(\frac{a}{\sqrt{a^2 + b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2}}\right)^2 = 1.$$

Now, we can write

$$y = a\sin x \pm b\cos x$$

$$\Rightarrow \mathrm{y} = \frac{ma\sin x}{m} \pm \frac{mb\cos x}{m}$$

$$\Rightarrow \mathrm{y} = \frac{1}{m}(ma\sin x \pm mb\cos x)$$

$$\Rightarrow \mathrm{y} = \frac{1}{\frac{1}{\sqrt{a^2+b^2}}}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x \pm \frac{b}{\sqrt{a^2+b^2}}\cos x\right)$$

$$\Rightarrow \mathrm{y} = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x \pm \frac{b}{\sqrt{a^2+b^2}}\cos x\right)$$

Assume that $\frac{a}{\sqrt{a^2+b^2}} = \cos\theta$, then, $\sin\theta = \sqrt{1-cos^2\theta} = \sqrt{1-\frac{a^2}{a^2+b^2}} = \frac{b}{\sqrt{a^2+b^2}}$.

Based on the above calculations we obtain the following trigonometric function

$$y = \sqrt{a^2+b^2}(\cos\theta\sin x \pm \sin\theta\cos x)$$

$$\Rightarrow \mathrm{y} = \sqrt{a^2+b^2}\sin(x \pm \theta).$$

Since, $-1 \leq \sin(x \pm \theta) \leq 1$, then

$$-\sqrt{a^2+b^2} \leq \sqrt{a^2+b^2}\sin(x \pm \theta) \leq \sqrt{a^2+b^2}.$$

It means that

$$-\sqrt{a^2+b^2} \leq a\sin x \pm b\cos x \leq \sqrt{a^2+b^2}.$$

*Second Solution:*

We convert this function into an expression consisting of only sine function as follows

$$a\sin x + b\cos x = c\sin(k+x)$$

$$\Rightarrow \mathrm{a}\sin x + b\cos x = c\sin k\cos x + c\cos k\sin x.$$

In the above equality, for any value of $x$, the coefficients of $\sin x$ and $\cos x$ should be equal on the left and right sides. Therefore,

$$a = c\cos k$$

$$b = c\sin k.$$

In this system of simultaneous equations we have,

$$a = c\cos k \Rightarrow \cos k = \frac{a}{c}$$

$$b = c\sin k \Rightarrow \sin k = \frac{b}{c}$$

$$sin^2k + cos^2k = 1 \Rightarrow (\frac{b}{c})^2 + (\frac{a}{c})^2 = 1 \Rightarrow \mathrm{c} = \pm\sqrt{a^2+b^2}.$$

Now, according to the relations $a = c\cos k$ and $b = c\sin k$ we obtain

$$\frac{c\sin}{c\cos k} = \frac{b}{a} \Rightarrow \tan k = \frac{b}{a} \Rightarrow \mathrm{k} = tan^{-1}\frac{b}{a}.$$

Therefore, we have

$$a\sin x + b\cos x = \pm\sqrt{a^2+b^2}\sin(x + tan^{-1}\frac{b}{a}).$$

If we limit the arctan to be within

$$-\frac{\pi}{2} < tan^{-1}\frac{b}{a} < \frac{\pi}{2}.$$

Then we obtain the following relation

$$a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x + tan^{-1}\frac{b}{a}).$$

Since $-1 \leq \sin(x + tan^{-1}\frac{b}{a}) \leq 1$, we see that

$$-\sqrt{a^2+b^2} \leq \sqrt{a^2+b^2}\sin(x + tan^{-1}\frac{b}{a}) \leq \sqrt{a^2+b^2}$$

$$\Rightarrow -\sqrt{a^2+b^2} \leq f(x) = a\sin x + b\cos x \leq \sqrt{a^2+b^2}.$$

Similarly, according to this method we can find the maximum and minimum values of the trigonometric function $y = a\sin x - b\cos x$.

*Third solution:*

Definition 1:

The dot product of two vectors $\vec{u} = (a, b)$ and $\vec{v} = (x, y)$, written $\vec{u}.\vec{v}$ is given by the definition

$$\vec{u}.\vec{v} = (a, b).(x, y) = ax + by.$$

Definition 2:

Assume that the angle between two vectors $\vec{u} = (a, b)$ and $\vec{v} = (x, y)$ is $\theta$ then the dot product of these vectors is defined as

$$\vec{u}.\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

where $|\vec{u}| = \sqrt{a^2+b^2}$ and $|\vec{v}| = \sqrt{x^2+y^2}$.

Now for solution of this given problem, we consider two vectors $\overrightarrow{u_1} = (a, b)$ and $\overrightarrow{u_2} = (\sin x, \cos x)$ and find the dot product of them through two different methods based on the definitions 1 and 2.

$$\overrightarrow{u_1}.\overrightarrow{u_2} = (a, b).(\sin x, \cos x) = a \sin x + b \cos x$$

$$\overrightarrow{u_1}.\overrightarrow{u_2} = |\overrightarrow{u_1}||\overrightarrow{u_2}| \cos \theta = \sqrt{a^2 + b^2}\sqrt{sin^2\theta + cos^2\theta} \cos \theta = \sqrt{a^2 + b^2} \cos \theta$$

Since $-1 \leq \cos \theta \leq 1$,

$$-\sqrt{a^2 + b^2} \leq \sqrt{a^2 + b^2} \cos \theta \leq \sqrt{a^2 + b^2}$$

$$\Rightarrow -\sqrt{a^2 + b^2} \leq \overrightarrow{u_1}.\overrightarrow{u_2} \leq \sqrt{a^2 + b^2}$$

$$\Rightarrow -\sqrt{a^2 + b^2} \leq a \sin x + b \cos x \leq \sqrt{a^2 + b^2}.$$

Therefore, for the trigonometric function $y = a \sin x + b \cos x$ we have

$$-\sqrt{a^2 + b^2} \leq y \leq \sqrt{a^2 + b^2}.$$

Through similar process, we can show that $-\sqrt{a^2 + b^2} \leq a \sin x - b \cos x \leq \sqrt{a^2 + b^2}$.

Example 1:

Find the range of the function $f(x) = 2 + 3 \sin x - 4 \cos x$.

Solution:

We know that

$$-\sqrt{3^2 + 4^2} \leq 3 \sin x - 4 \cos x \leq \sqrt{3^2 + 4^2} \Rightarrow -5 \leq 3 \sin x - 4 \cos x \leq 5.$$

Therefore, $-3 \leq 2 + 3 \sin x - 4 \cos x \leq 7$ and the range of this function is $R_f = [-3, 7]$.

Example 2:

Determine the maximum and minimum values of the following two variables function

$f(x, y) = 3 \sin x + 4 \cos x - 3 \cos y + 4 \sin y - 4$.

Solution:

We know,

$$-\sqrt{3^2 + 4^2} \leq 3 \sin x + 4 \cos x \leq \sqrt{3^2 + 4^2}$$

$$-\sqrt{3^2 + 4^2} \leq 4 \sin y - 3 \cos y \leq \sqrt{3^2 + 4^2}.$$

Therefore,

$$-10 \leq 3\sin x + 4\cos x + 4\sin y - 3\cos y \leq 10.$$

Now, we have

$$-14 \leq 3\sin x + 4\cos x + 4\sin y - 3\cos y - 4 \leq 6.$$

It means $-14 \leq f(x,y) \leq 6$.

*Generalization of this Trigonometric Problem*

Prove that the range of the function $g(x) = a\sin kx + b\cos kx + c$ is calculated based on the rule $R_g = \left[c - \sqrt{a^2+b^2},\ c + \sqrt{a^2+b^2}\right]$.

Proof:

Firstly, we show that, $\forall a, b, k, x \in R, -\sqrt{a^2+b^2} \leq a\sin kx + b\cos kx \leq \sqrt{a^2+b^2}$. The maximum and minimum values of the function $y = a\sin kx \pm b\cos kx$ discussed as follows:

Since, $-1 \leq \frac{a}{\sqrt{a^2+b^2}} \leq 1$, we assume that $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ therefore, $\sin\alpha = \frac{\pm b}{\sqrt{a^2+b^2}}$.

$$y = a\sin kx \pm b\cos kx$$

$$\Rightarrow \frac{y}{\sqrt{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}}\sin kx \pm \frac{b}{\sqrt{a^2+b^2}}\cos kx$$

$$\Rightarrow \frac{y}{\sqrt{a^2+b^2}} = \cos\alpha\sin kx \pm \sin\alpha\cos kx = \sin(kx \pm \alpha).$$

We know, $-1 \leq \sin(kx \pm \alpha) \leq 1$, thus

$$-1 \leq \frac{y}{\sqrt{a^2+b^2}} \leq 1 \Rightarrow -\sqrt{a^2+b^2} \leq y \leq \sqrt{a^2+b^2}$$

$$\Rightarrow -\sqrt{a^2+b^2} \leq a\sin kx \pm b\cos kx \leq \sqrt{a^2+b^2}.$$

Therefore,

$$\mathrm{c} - \sqrt{a^2+b^2} \leq a\sin kx \pm b\cos kx + c \leq c + \sqrt{a^2+b^2}$$

It means

$$\mathrm{c} - \sqrt{a^2+b^2} \leq g(x) \leq c + \sqrt{a^2+b^2}$$

$$\Rightarrow R_g = \left[c - \sqrt{a^2+b^2},\ c + \sqrt{a^2+b^2}\right].$$

Example 3:

Find the range of the function $h(x) = [2\sin 4x - 3\cos 4x]$ where [] is the symbol of partial integer.

Solution:

The function $h$ is continues on the real numbers. As respect to the above theorem we have

$$-\sqrt{13} \leq 2\sin 4x - 3\cos 4x \leq \sqrt{13}$$

Therefore, we obtain

$$R_h = \{-4, -3, -2, -1, 0, 1, 2, 3\}.$$

The following four problems are suitable to discuss in the classroom, because such problems improve the ability of students in problem solving.

Problem 1:

For the function $y = 12\sin x + 5\cos x$, write the linear combination of sine and cosine as consisting of only cosine function.

Problem 2:

Find the maximum and minimum values of the function $f(x) = \frac{2+\sqrt{3\sin x - 4\cos x}}{1+\sqrt{4\sin x + 3\cos x}}$.

Problem 3:

Prove that

$$(\sin x + a\cos x)(\sin x + b\cos x) \leq 1 + (\tfrac{a+b}{2})^2.$$

Problem 4:

Prove that

$$(\sin 3x + a\cos 3x)(\sin 3x + b\cos 3x) \leq \tfrac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1}).$$

**DISCUSSION**

Based on a research by Gholami et al. (2021), some common misconceptions that students encounter regarding the problem "find the maximum and minimum values of the trigonometric function $h(x) = \sin x + \cos x$" are as follows:

a) According to inequalities $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$ we have $-2 \leq \sin x + \cos x \leq 2$ therefore, the maximum and minimum values of this function are 2 and $-2$ respectively. In this argument, the two functions $y = \sin x$ and $y = \cos x$ considered as two independent functions, whereas the values of $\sin x$ and $\cos x$ are dependent based on the formula $sin^2x + cos^2x = 1$. For instance, for the angles in the first quartile of the unit circle, the value of $\sin x$ increases when the value of $\cos x$ decreases and conversely. It means, we cannot consider $\sin x = 1$ and $\cos x = 1$ simultanously. By using the formula $sin^2x + cos^2x = 1$ it is clear that $\cos x = 0$ when $\sin x = 1$. The above argument is correct to find the maximum and minimum values of $A = \sin x + \cos y$ because the two values of $\sin x$ and $\cos y$ are independent.

b) By squaring both sides of the function $h(x) = \sin x + \cos x$ we have

$$h^2(x) = (\sin x + \cos x)^2 = sin^2x + cos^2x + 2\sin x \cos x = 1 + \sin 2x.$$

$$0 \le 1 + \sin 2x \le 2 \Rightarrow 0 \le h^2(x) \le 2 \Rightarrow 0 \le \mathrm{h(x)} \le \sqrt{2}.$$

In the above argument, the misconception is related to the concept $0 \le h^2(x) \le 2 \Rightarrow 0 \le \mathrm{h(x)} \le \sqrt{2}$ because the inequality $0 \le h^2(x) \le 2$ is equivalent to $h^2(x) \le 2$ and hence $-\sqrt{2} \le h(x) \le \sqrt{2}$.

c) $h(x) = \sin x + \cos x \Rightarrow \mathrm{h(x)} - \cos x = \sin x \Rightarrow -1 \le \mathrm{h(x)} - \cos x \le 1$
$\Rightarrow -1 + \cos x \le h(x) \le 1 + \cos x \Rightarrow -2 \le \mathrm{h(x)} \le 2$.
In this argument, obtaining the inequality $-1 \le \mathrm{h(x)} - \cos x \le 1$ is a misconception because the values $\sin x$ and $\cos x$ are dependent.

A logical argument for this problem was shown as follows

$$h(x) = \sin x + \cos x = \sqrt{2}\left(\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x\right) = \sqrt{2}(\cos 45 \sin x + \sin 45 \cos x)$$

$$\Rightarrow \mathrm{h(x)} = \sqrt{2}\sin(x + 45).$$

Since, $-1 \le \sin(x + 45) \le 1$ we have $-\sqrt{2} \le \sin(x + 45) \le \sqrt{2}$, thus $-\sqrt{2} \le h(x) \le \sqrt{2}$.

In this study, three students considered $\sin x = 1$ and $\cos x = 1$ to find $a(1) + b(1) = a + b$ as the maximum value of the function $f(x) = a \sin x + b \cos x$. Also, they argued that the minimum value of this function is $a(-1) + b(-1) = -(a + b)$. In this argument, teachers can refer to the two important misconceptions that students face. Firstly, students cannot consider $\sin x = \pm 1$ and $\cos x = \pm 1$ simultaneously and another one is related to the sign of the variables $a$ and $b$ because these variables can be negative. Therefore, based on this incorrect argument, the maximum and minimum values of the function $f(x) = a \sin x + b \cos x$ should be $|a| + |b|$ and $-(|a| + |b|)$ respectively. Students understood that the above argument is correct for the two variables function $f(x, y) = a \sin x + b \cos y$ because in this function we can consider $\sin x = \pm 1$ and $\cos x = \pm 1$ simultaneously. Therefore, the maximum and minimum values of the function $f(x, y) = a \sin x + b \cos y$ is $|a| + |b|$ and $-(|a| + |b|)$ respectively due to independency of $\sin x$ and $\cos y$.

Also, two of them rewrite the function $f(x) = a \sin x + b \cos x$ as $f(x) - a \sin x = b \cos x$, since $-|b| \le b \cos x \le |b|$, then $-|b| \le f(x) - a \sin x \le |b|$. Therefore,

$$-|b| \le f(x) - a \sin x \le |b|$$

$$\Rightarrow -|b| + a \sin x \le f(x) \le |b| + a \sin x$$

$$\Rightarrow -|b| - |a| \leq f(x) \leq |b| + |a|.$$

In the above method of solution, students considered $\sin x$ and $\cos x$, as two independent variables wrongly, whereas these two variables are dependent. Teachers explained this misconception to students using a specific function such as $g(x) = 3\sin x + 4\cos x$ and helped students to understand we cannot obtain 7 as the maximum value of this function because when we put $\sin x = 1$, the value of $\cos x$ should be only zero.

The rest of the students in the class got the right answer by considering specific situations for this problem. For example, one of the students considered equal value for both coefficients $a$ and $b$ and he found the values $|a|\sqrt{2}$ and $-|a|\sqrt{2}$ as the maximum and minimum values of the function $h(x) = a\sin x + a\cos x$ respectively.

In the Problem 1, "For the function $y = 12\sin x + 5\cos x$, write the linear combination of sine and cosine as consisting of only cosine function" some students by using the formula $\sin x = \pm\sqrt{1 - cos^2 x}$ simply changed it as $y = \pm 12\sqrt{1 - cos^2 x} + 5\cos x$. They didn't understand the concept of linear combination of two variables.

For the problem 2, "Find the maximum and minimum values of the function $f(x) = \frac{2+\sqrt{3\sin x - 4\cos}}{1+\sqrt{4\sin x + 3\cos}}$." some students found the maximum and minimum values of numerator $2 + \sqrt{3\sin x - 4\cos x}$ as follows

$$-\sqrt{3^2 + (-4)^2} \leq 3\sin x - 4\cos x \leq \sqrt{3^2 + (-4)^2}$$

$$\Rightarrow -5 \leq 3\sin x - 4\cos x \leq 5$$

$$\Rightarrow 0 \leq \sqrt{3\sin x - 4\cos x} \leq \sqrt{5}$$

$$\Rightarrow 2 \leq 2 + \sqrt{3\sin x - 4\cos x} \leq 2 + \sqrt{5}.$$

Through similar calculation, they found the numbers 1 and $1 + \sqrt{5}$ as the maximum and minimum values of denominator $1 + \sqrt{4\sin x + 3\cos x}$ respectively. They argued in order to find the maximum value of the function $f(x)$ we require to divide the maximum value of numerator to the minimum value of denominator. Therefore, the maximum value of the function $f(x)$ is $\frac{2+\sqrt{5}}{1} = 2 + \sqrt{5}$. Similarly, the minimum value of this function should be $\frac{2}{1+\sqrt{5}}$. This argument is not correct because the two values $2 + \sqrt{3\sin x - 4\cos x}$ and $1 + \sqrt{4\sin x + 3\cos x}$ are dependent. But this argument is true for the problem "find the maximum and minimum values of $C = \frac{2+\sqrt{3\sin x - 4\cos x}}{1+\sqrt{4\sin y + 3\cos}}$." because of independency of nominator and denominator of the statement $C$. There is a common misconception among students regarding this problem that is so important for teachers to know. We have the following inequalities

$$2 \leq 2 + \sqrt{3\sin x - 4\cos x} \leq 2 + \sqrt{5}$$

$$1 \leq 1 + \sqrt{4\sin x + 3\cos x} \leq 1 + \sqrt{5}.$$

By dividing all sides of the above inequalities we obtain

$$2 \leq \frac{2+\sqrt{3\sin x - 4\cos}}{1+\sqrt{4\sin y + 3\cos y}} \leq \frac{2+\sqrt{5}}{1+\sqrt{5}}.$$

It is clear that this result is not acceptable because $2 > \frac{2+\sqrt{5}}{1+\sqrt{5}}$.

The logical argument for this problem must be as follows

$$2 \leq 2 + \sqrt{3\sin x - 4\cos x} \leq 2 + \sqrt{5}$$

$$1 \leq 1 + \sqrt{4\sin x + 3\cos x} \leq 1 + \sqrt{5} \Rightarrow \frac{1}{1+\sqrt{5}} \leq \frac{1}{1+\sqrt{4\sin x + 3\cos}} \leq 1.$$

By multiplying all sides of these inequalities we can see

$$\frac{2}{1+\sqrt{5}} \leq \frac{2+\sqrt{3\sin x - 4\cos}}{1+\sqrt{4\sin y + 3\cos}} \leq 2 + \sqrt{5}.$$

Therefore, the maximum and minimum values of $C = \frac{2+\sqrt{3\sin x - 4\cos x}}{1+\sqrt{4\sin y + 3\cos y}}$ are $2 + \sqrt{5}$ and $\frac{2}{1+\sqrt{5}}$ respectively.

Another misconception is about the problem 3, "Prove that $(\sin x + a\cos x)(\sin x + b\cos x) \leq 1 + (\frac{a+b}{2})^2$.".

For this problem, the members of Lesson Study group members were faced with the following argument

$$(\sin x + a\cos x)(\sin x + b\cos x) \leq \sqrt{1 + a^2}\sqrt{1 + b^2}.$$

After that students tried to prove $\sqrt{1 + a^2}\sqrt{1 + b^2} \leq 1 + (\frac{a+b}{2})^2$ but this inequality is not correct since, by setting $a = 1$ and $b = -1$ in this inequality, we obtain $\sqrt{1 + (1)^2}\sqrt{1 + (-1)^2} \leq 1 + (\frac{1+(-1)}{2})^2 \Rightarrow 2 \leq 1$. This misconception is related to the dependency of the statements $\sin x + a\cos x$ and $\sin x + b\cos x$. In fact, for maximization of the statement $(\sin x + a\cos x)(\sin x + b\cos x)$ we cannot consider the maximum values of the statements $\sin x + a\cos x$ and $\sin x + b\cos x$ simultaneously. A logical proof for this problem that suggested by a student is as follows

Proof: If $\cos x = 0$, this inequality reduces to $sin^2 x \leq 1 + (\frac{a+b}{2})^2$, which is obviously true. We assume that $\cos x \neq 0$, thus by dividing both sides of the given inequality by $cos^2 x$ gives

$$(\tan x + a)(\tan x + b) \le \left[1 + (\tfrac{a+b}{2})^2\right] sec^2 x.$$

Now, we set $\tan x = m$, then $sec^2 x = 1 + m^2$. Therefore, the above inequality changes to

$$m^2 + (a+b)m + ab \le (\frac{a+b}{2})^2 m^2 + m^2 + (\frac{a+b}{2})^2 + 1$$

$$\Rightarrow (\frac{a+b}{2})^2 m^2 + 1 - (a+b)m + (\frac{a+b}{2})^2 - ab \ge 0$$

$$\Rightarrow \left[(\frac{a+b}{2})^2 m^2 + 1 - (a+b)m\right] + \left[\frac{a^2 + b^2 - 2ab}{4}\right] \ge 0$$

$$\Rightarrow (\tfrac{(a+b)m}{2} - 1)^2 + (\tfrac{a-b}{2})^2 \ge 0.$$

The proof is complete because all of the above statements are reversible.

The same misconception is discussable for problem 4, "prove that $(\sin 3x + a\cos 3x)(\sin 3x + b\cos 3x) \le \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})$." In this problem, we can write

$$(\sin 3x + a\cos 3x)(\sin 3x + b\cos 3x) \le \sqrt{1+a^2}\sqrt{1+b^2}.$$

Now, we should prove

$$\sqrt{1+\mathrm{a}^2}\sqrt{1+\mathrm{b}^2} \le \tfrac{1}{2}(1 + \mathrm{ab} + \sqrt{\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{a}^2\mathrm{b}^2 + 1}).$$

The above inequality is a misunderstanding, because by setting $a = 2$ and $b = -2$ we obtain

$$\sqrt{1+(2)^2}\sqrt{1+(-2)^2} \le \tfrac{1}{2}(1 + (2)(-2) + \sqrt{(2)^2 + (-2)^2 + (2)^2(-2)^2 + 1}) \Rightarrow 5 \le 1.$$

A member of the Lesson Study group suggested a logical solution for this problem as follows

$$\begin{aligned}(\sin 3x + a\cos 3x)&(\sin 3x + b\cos 3x)\\ &= sin^2 3x + b\sin 3x\cos 3x + a\cos 3x\sin 3x + abcos^2 3x\\ &= 1 + (ab-1)cos^2 3x + (\frac{a+b}{2})\sin 6x\\ &= 1 + (ab-1)\left(\frac{1+\cos 6x}{2}\right) + (\frac{a+b}{2})\sin 6x\\ &= \frac{ab+1}{2} + \left(\frac{a+b}{2}\right)\sin 6x + (\frac{ab-1}{2})\cos 6x\end{aligned}$$

$$\leq \frac{ab+1}{2} + \sqrt{(\frac{a+b}{2})^2 + (\frac{ab-1}{2})^2}$$

$$= \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1}).$$

## CONCLUSIONS

Improving the mathematical knowledge of mathematics teachers affects the quality of lesson design, teaching methods and classroom atmosphere (Copur-Gencturk, 2015). Therefore, teachers require enhancing their mathematical knowledge continually. One of the best ways regarding this issue is sharing their knowledge and experiences through Lesson Study approach. In teaching mathematics concepts through problem solving, teachers need to understand mathematical ideas regarding a problem in a deep and connected way, and further they should be familiar with different methods of solution (O. Masingila et al., 2018). In this study, the Lesson Study group members suggested three different solution methods to find the maximum and minimum values of the trigonometric function $y = a\sin x + b\cos x$ and through similar ways they found the maximum and minimum values of the function $y = a\sin x - b\cos x$. Discussion about the variety of solution methods for the maximum and minimum values of the function $y = a\sin x \pm b\cos x$ helps learners to improve their abilities in problem solving. Also, teachers generalized this given problem to finding the maximum and minimum values of the function $g(x) = a\sin kx + b\cos kx + c$ that improve the skills of educators and students in generalizing the trigonometric concepts. They enhanced the quality of this research lesson by considering some suitable problems related to this given trigonometric problem and discussing regarding the variety of students' misconceptions in order to improve the performance of educators in their teaching. In this study, the members of the Lesson Study group found that students had serious problems to solve a general problem such as "find the maximum and minimum values of the function $y = a\sin x \pm b\cos x$" and "find the maximum and minimum values of the function $y = a\sin kx \pm b\cos kx + c$" because the coefficients $a$, $b$, $c$ and $k$ is not clear for students. They can solve the problems with clear coefficients such as "find the maximum and minimum values of the function $y = 3\sin x + 3\cos x$" and "find the maximum and minimum values of the function $y = 4\sin 2x + 3\cos 2x - 5$" easily because of the clearance coefficients in these functions.

In this article, the researcher discussed about the variety of solutions for a given trigonometric problem and related misconceptions to improve the mathematical knowledge of educators. Although mathematical misunderstandings are common among students, some novice teachers are also involved. Therefore, this research lesson may help teachers to provide a better learning environment for students regarding this trigonometric problem. Meanwhile, experienced teachers can improve this research lesson based on their students' abilities to reduce the trigonometric misconceptions among students.

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