MODESTUM
EURASIA Journal of Mathematics, Science and Technology Education, 2022, 18(5), em2111
ISSN:1305-8223 (online)
OPEN ACCESS
Research Paper
https://doi.org/10.29333/ejmste/12034

Doing mathematics with music – Creating epistemic environments

Ana Cristina Azevedo da Silva [1*], Maria Cecília Costa [2,3], Joaquim Bernardino Lopes [2,3]

[1] Celorico de Basto School Group, PORTUGAL
[2] UTAD, University of Trás-os-Montes and Alto Douro, Vila Real, PORTUGAL
[3] CIDTFF–Research Center in Didactics and Technology in Training of Trainers, Aveiro, PORTUGAL

Received 9 January 2022 ▪ Accepted 14 April 2022

Abstract

It is rare to turn music into a mathematical object in an educational context, although the benefit of the articulation between mathematics and music is recognized. The present study focuses on the epistemic environment created for students to do maths with music. The methodological approach had two components: an exploratory one to study the relationships between the epistemic environment experienced by students and the epistemic levels at which they do mathematics, and a quasi-experimental one to assess the effectiveness of intervention in student learning. Three groups took part in the study. The "doing maths with music" approach is more effective than conventional ones whether, or not, students have in-depth musical knowledge. On the other hand, educational artefacts used by students allowed them to deal with music at various epistemic levels, with relevant relationships between the quality of students' epistemic activity and the profile of epistemic levels.

Keywords: artefacts, epistemic levels, instrumental orchestration, learning, performance, teaching

INTRODUCTION

Are music and science different types of intelligence or are they two manifestations of common ways of thinking? Focusing on scientists who were musicians and the ways in which they used their musical knowledge to inform their scientific work, Root-Bernstein (2001) argues that music and science are two ways of using a common set of "tools" that unifies all disciplines. It explores the notion that creative individuals are often polymaths, who think transdisciplinary (Root-Bernstein, 2001). The mathematician, poet and musician Sylvester (1864) believes that the soul of each one (mathematics and music) is the same. Musician feels mathematics and mathematician thinks music: music is the dream; mathematics is professional life (thought a century ago).

Several studies present the beneficial effects of integrating music in an educational context, at the level of behavioral control (Hallam & Price, 1998), as well as the implications for the students' mathematical learning, when they, from the first years of life, have musical training (Chao-Fernández et al., 2017; Gardiner et al., 1996; Graziano et al., 1999). Some of the studies already integrate music in an educational context, in parallel with mathematics (Elofsson et al., 2018; Viladot et al., 2018).

Other studies show experiences in an educational context in which music begins to play a relevant role and not just an external component in the mathematics teaching process (An et al., 2008; Quinn et al., 2019). It is in this line of approach that the present study is identified. Music as a working basis for teaching mathematics, a starting point. From music to make the teaching process focused on the student, who is the agent of their own learning (there being a need to use artefacts for this).

Pythagoras was the first to scientifically demonstrate the relationship between mathematics and music (11th century BC). Pythagoras created an artefact, the monochord, and from it he established relationships between the length of the extended string and the sound emitted when it was played. The artefact used allowed us to relate musical intervals and introduce the concept of fractions. The present study also uses artefacts (musical and others) that are transformed into epistemic tools, through an orchestration of instruments (Silva et al., 2021).

✉ acas27@gmail.com **(*Correspondence)** ✉ mcosta@utad.pt ✉ blopes@utad.pt

Contribution to the literature

- Turn music into a mathematical object in an educational context through epistemic environment as a context encompassing different artefacts intentionally orchestrated.
- The intervention "doing maths with music" is effective in terms of students' mathematical learning and the differences between students with and without in-depth musical knowledge does not affect this effectiveness.
- Educational artefacts used by students allowed them to deal with music at various epistemic levels, with relevant relationships between the quality of students' epistemic activity and the profile of epistemic levels.

In this previous study (Silva et al., 2021), we intended to understand how artefacts are orchestrated when interventions are made where mathematics is done with music, as well as to understand how these artefacts become epistemic tools for students. The results suggest that it is possible to do mathematics with music, using artefacts orchestrated with each other. They also suggest that orchestrated artefacts allow creating a context in which student learning is active, where the artefact has the status of an epistemic tool.

In the present study we intend to understand the connections between the epistemic activity verified when students are subjected to interventions where a set of artefacts are articulated and the profile variability of epistemic levels of student work. Furthermore, it is intended to understand how this learning is reflected in terms of mathematical performance, that is, if the chosen approach is effective in terms of mathematical learning.

The study carried out is supported using artefacts, since it recognizes the implications for students' learning of mathematics, as is the case in the study by Quinn et al. (2019). The artefacts used have different characteristics and are articulated with each other in an intentional way. This articulation between artefacts allows the creation of an epistemic environment to do mathematics, with music as the starting point (the mathematical object), thus creating the possibility for students to experience different degrees of proximity (epistemic levels) with the referent used, the music.

The objectives of this study are:

1. Identify didactically relevant patterns in the relationships between the experienced epistemic environment and the epistemic levels at which students do mathematics.
2. Test the effectiveness of the teaching approach adopted in the performance of mathematical learning resulting from the epistemic environment experienced by students in an educational context, where mathematics is done with music and whether this effectiveness depends on the students' musical knowledge.

STATE OF ART

Integrating Music into Effective Math Learning Curriculum

The human sense of hearing has remarkable pattern-recognition powers, according to chemist Robert Morrison, but listening has been largely ignored as a means of looking for patterns (Peterson, 1985). Even Einstein never thought of equations; he sensed or visualized the responses and then converted them at a later stage to communicate to others (Hadamard, 1945). Auditory sensory experiences have been little explored in the field of learning and teaching mathematics, although they may contribute to the semantic content of mathematical notions, considering the multimodal nature of human cognition (Thayer-Morel et al., 2018).

Two thousand years after Pythagoras (the first to report the close connection between mathematics and music), great mathematicians and musicians emerged: Marin Mersenne, Descartes, Fermat, and Napier. All contributed intensely to musical understanding, but it was Mersenne who left a valuable legacy in his work "Harmonie Universelle", dated 1637 (Mersenne, 1637). The list of mathematicians-composers, along with other scientist-composers (endocrinologists, physicians, surgeons, cardiologists, chemists, physiologists, astronomers, geologists, microbiologists, zoologists, experimental psychologists, epidemiologists, among others) has increased over time: MA Balakirev (1837-1910), Ernest Ansermet (1883-1969), Joseph Schillinger (1895-1943), Iannis Xenakis (1922-2001), and Diana S. Dabby (contemporary) (Root-Bernstein, 2001).

Several studies refer to the benefits of music in students' mathematical learning (An et al., 2008; Chao-Fernández et al., 2017; Gardiner et al., 1996; Graziano et al., 1999; Hodges, 2005; Viladot et al., 2018). On the other hand, Turkka et al. (2017) report that the integration of art in the classroom does not include expressions of emotions normally associated with art. Art integration should address the role of emotions more explicitly, rather than expecting it to happen naturally.

Teachers can use music to improve children's understanding of more complex math concepts and skills. Children need to collect meaningful experiences to develop thinking and problem-solving skills. The idea of a standard is powerful and is not only essential for math and music, but also covers all other areas of the curriculum (Edelson & Johnson, 2003).

Investigations that focus on the potential of music for teaching mathematics continue to emerge. For example, Quinn et al. (2019) were dedicated to the study of the transformations of trigonometric functions in secondary education. Wilhelmi and Montiel (2019) extend this type of approach to future teachers during their initial training. Mannone (2019) emphasizes the aesthetic pleasure that can be observed in sciences seen as more "creative" and thus helping students to be motivated to face the difficulties that may arise during the study of sciences traditionally seen as more rigid.

The present study differs from the literature in that it is uncommon to turn music into a mathematical object in an educational context, although there are studies that already use music for mathematics teaching purposes (An et al., 2008; Quinn et al., 2019; Thayer-Morel et al., 2018). In the case of An et al. (2008), the mathematical object in music is identified, more specifically in the study of a musical composition, and from there, the statistical study of this composition is carried out. In the case of Quinn et al. (2019), there is identification of mathematical objects in music and this identification is done using a task where it is intended to understand trigonometric transformations (concepts such as period and frequency, through the production of a touch for a smartphone). However, these mathematical concepts are already known by the students. The task, using music, serves to understand the concepts previously studied. Which differs from our present study, since in the teaching approach adopted by us, students work math new concepts, that is, the concept is worked on, without definitions, without mathematical language, only with what emerges from the manipulation of the artefacts presented. In the case of Thayer-Morel et al. (2018), there are several similarities with our study, in the sense that resources were created (using sounds and music) for use in an educational context. These resources went through an iterative evaluation process until they reached the final version. In addition, they are also accompanied by didactic guides. There is identification of mathematical objects in sounds and music.

Instrumental Orchestration to Do Maths with Music

It is intended to do maths with music. "Doing maths" by students is an approach to mathematics with a playful and serious attitude, with a focus on working at the frontier of knowledge, "breaking maths", where making mistakes is also a crucial aspect for learning mathematics (Mun & Hertzog, 2018).

Epistemic tools are contextual artefacts manipulated to investigate and evaluate ideas to build knowledge (Sezen-Barrie et al., 2020). In the chosen approach, doing maths with music, these are the tools that will allow you to do maths, having music as its object.

Lopes and Costa (2019) consider that an artefact is an entity or product of human creation. For example, a piece of wood is not an artefact, it is an object. However, it can be used as a hammer, becoming a tool. Likewise, when an artefact is used to solve a problem, it becomes a tool. If its use in an educational context allows the creation of an epistemic context where student learning has a high degree of autonomy and intentionality, the artefact becomes an epistemic tool (Lopes & Costa, 2019).

Although they play an important role, the use of artefacts by mathematics teachers in their teaching practices is still not fully utilized, even in cases where teachers know how to work easily with these artefacts (Lopes, 2019). The classroom must provide a context, an activity to the student so that he is guided to act, think and communicate. The use of artefacts incorporated in activities for teaching mathematics should aim at the active participation of the student in the construction of their knowledge, leading them to reflect on the action they are taking (Lerman, 2001).

Instrumental orchestration is defined as an action plan, participating in a system of didactic exploration that an institution (the school institution, in this case) organizes with the aim of guiding students in instrumented action (Guin & Trouche, 2002). More specifically, an instrumental orchestration is defined as a systematic and intentional organization by the teacher, using various artefacts in a learning environment, to guide students (Drijvers et al., 2010). The same authors distinguish three elements within instrumental orchestration: a didactic configuration, a mode of exploration and a didactic performance. A didactic configuration as an arrangement of artefacts in the environment, in other words, a configuration of the environment and the artefacts involved in it. Using the musical metaphor, the didactic configuration can be compared to choosing the musical instruments to be included in the band and organizing them in space so that the different sounds result in a certain song, which in the mathematics classroom can be summarized to a solid and convergent mathematical discourse. The exploration mode is how the teacher decides to explore a didactic setting for the benefit of their didactic intentions. This includes decisions about how the task is presented and worked on, about possible functions of the artefacts to be mobilized, and about the schemes and techniques to be developed and established with students. Didactic performance involves ad hoc decisions made during teaching about how to perform in the chosen didactic setting and mode of exploration: which question to ask, what to do with a student's particular opinions, how to deal with an unexpected aspect of the task (Drijvers et al., 2010).

It is uncommon to have an artefact to make music a mathematical object in the teaching and learning of mathematics in the classroom and a constellation of artefacts (duly orchestrated) to create an epistemic environment favorable to student's learning. As already mentioned before, some studies are already moving in this direction. It is in this type of approach that the present study fits and where in fact an artefact (guide) was used to allow the creation of this epistemic environment. In the next section we go deeper into the concepts of epistemic environment, necessary for students to do mathematics with music in the sense indicated above, and of epistemic level and the relationship between them.

Epistemic Environment and Epistemic Levels

Muis and Duffy (2013), from other authors, defined the epistemic environment as a context encompassing different epistemic factors and processes that influence the way knowledge is understood and constructed. In an educational context, the same authors specify that the epistemic environment can be shaped in classroom practices through activities, discourse, curriculum and materials. According to this definition of an epistemic environment, teaching practices that create an epistemic environment can have several configurations. In any case, an epistemic environment lacks the construction of a context in which various mathematical activities can take place to build and justify mathematical knowledge in a school context using artefacts (Goizueta, 2019; Hershkowitz et al., 2001).

The epistemic aspects of mathematics learning are still poorly understood (Kämäräinen et al., 2019). However, there is some consensus that epistemic processes in mathematics learning involve abstraction and epistemic work by students who lack tasks and artefacts used with tools (Hershkowitz et al., 2001; Kämäräinen et al., 2019; Monaghan et al., 2016). Epistemic work is recursive in that it is the result of previous activities and the constructions achieved can themselves become artefacts for further actions (Hershkowitz et al., 2001; Svahn & Bowden, 2021). This characteristic makes epistemic work intrinsically linked to the context in which it takes place (Hershkowitz et al., 2001). In this sense, the role that the peers of the group of students who are dealing with a task is fundamental for each member of the groups to learn and build mathematical knowledge in a school context (Kämäräinen et al., 2019). Another important aspect for having an epistemic environment is the way in which the material environment is organized to allow the construction of knowledge and can be used to make understandable both the actions and requests for help from the students and the presentation of the problem by the teacher (Svahn & Bowden, 2021). This characteristic of an epistemic environment implies the followings:

1. Artefacts function as epistemic tools to determine the problem to be solved from the student's point of view (Svahn & Bowden, 2021).
2. The different artefacts are intentionally orchestrated by the teacher and play an important role in changing the teaching and learning of science and mathematics (Drijvers et al., 2010, 2020; Guin & Trouche, 2002; Lopes & Costa, 2019).
3. Teachers allow students the opportunity to assume greater responsibility in the construction of mathematical knowledge in the classroom (Stroupe et al., 2019) so that they have autonomy and feel "inhabitant" of their environment, directing their learning itself, making choices about which resources will be employed (McLaughlan & Lodge, 2019).
4. The teacher proposes challenges to students since Asterhan and Schwarz (2016) demonstrated its beneficial effect. However, the attribution of challenges has as an indispensable component: the teacher's intervention throughout the students' activity to avoid prolonged moments of idleness, non-validated achievements, off-task involvement, and other critical moments in social interaction (Schwarz et al., 2018).

An epistemic environment characterized as mentioned above brings out the epistemic quality that Hudson (2019) considers to be the quality of what students come to know, make sense of and be able to do in terms of (mathematical) knowledge. The changes that are referred to in the definition of the epistemic environment may refer to changes in epistemological beliefs about how mathematical knowledge is constructed (Muis & Duffy, 2013), or even in the academic performance of students (Cartiff et al., 2021; Muis & Duffy, 2013).

The epistemic environment created from properly orchestrated artefacts, under the teacher's attention, leads to a differentiated effective curriculum. Epistemic environments neither block constructive alignment nor require teachers to renounce the direction of the learning process. Rather, epistemic environments require teacher to take a more dynamic approach to ensure that constructive alignment occurs (McLaughlan & Lodge, 2019).

The expression "epistemic level" appears in the literature with at least three meanings: the quality of the epistemic activity (e.g., Mishiwo, 2021; Schwarz et al., 2018), the degree of proximity to real life in the coming and going between mathematical knowledge and real life (e.g., Branchetti & Morselli, 2019; Ndemo & Mtetwa, 2021; Švaříček, 2019), or the epistemic view of knowledge itself (Lee et al., 2021). In this work we adopt the second meaning in line with what is used in other areas of knowledge, for example science education (Kelly & Takao, 2002). Furthermore, this understanding

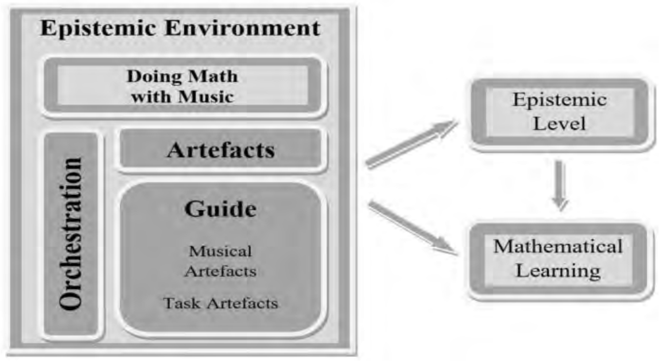


Figure 1. Study outline

of the notion of the epistemic level is more consistent with concept of epistemic environment presented above.

Kelly and Takao (2002) constructed a model of epistemic levels to explain, in the context of scientific arguments, how students used evidence. If the evidence was used in the form of unrelated facts and without any conceptualization, the epistemic level is low. By contrast, if the evidence is used in a context of robust conceptualization (supported by theories) and abstraction, the epistemic level is high. Therefore, the epistemic level is the greater or lesser distance in relation to the referent used as the basis for the construction of mathematical knowledge. Greater variability in the epistemic levels at which the epistemic activity takes place tends to promote more effective learning (Branchetti & Morselli, 2019).

Branchetti and Morselli (2019) consider that the students' mathematical activity occurs in a back-and-forth process between mathematics and real life and, therefore, it can occur in different degrees of proximity to real life. Thus, we define an epistemic level as the degree of proximity to real life (versus the degree of abstraction and justification of mathematical construction) in which mathematical activity takes place with a view to the construction and justification of mathematical knowledge in a school context. Therefore, students' epistemic activity during maths activity can occur at different epistemic levels. In the same mathematical activity, there may be variability of epistemic levels (Branchetti & Morselli, 2019; Švaříček, 2019), or there may be, at the same time, a progressive tendency to increase the epistemic level as mathematical knowledge is built by students (Ndemo & Mtetwa, 2021; Švaříček, 2019).

It is intended to understand the role that the educational artefacts used by students, when they do mathematics with music in an educational context. In particular, it is intended to know how the epistemic environment is related to the epistemic levels with which students deal with music as a mathematical object and how different profiles of variability of epistemic levels provide more sophisticated levels of mathematical learning (**Figure 1**).

Table 1. Number of students by groups

	G.Exp.1	G.Exp.2	G.Co
N	17	19	18

Research Questions

With this study we intend to answer the following research questions:

1. **QI$_1$**: What links exist between epistemic activity and the variability profile of epistemic levels, when in an educational context students do mathematics with music as their object?
2. **QI$_2$**: Is the intervention "doing math with music" effective in terms of students' math learning? Are there differences between students with and without in-depth musical knowledge?

To answer these questions, it is necessary:

1. Identify the moments when students "do" maths with music.
2. In these moments, see which artefacts are used and how (characterize the different ways of using artefacts).
3. Relate the use of artefacts with the epistemic levels allow to experience.
4. Relate the epistemic environments experienced with the levels of learning performance.

INTERVENTIONS IN EDUCATIONAL CONTEXT

Study Design

Two methodological approaches were adopted: an exploratory component (to answer the first research question–QI$_1$) and a comparative, quasi-experimental component evaluating the effect of two variables (answering the second research question–QI$_2$). The variables to consider are: V1–class with and without in-depth musical knowledge; V2–class with and without intervention having music as a mathematical object using mediating artefacts.

Three groups of the seventh year of schooling took part in the study, in a basic and secondary school, in the North region of Portugal. Two classes were part of the experimental group and a third class the control group (**Table 1**). In two of the groups–experimental groups (G.Exp.1 and G.Exp.2)–interventions such as doing mathematics with music were carried out with a given instrumental orchestration. The classes in the experimental group have different characteristics: G.Exp.2 is a class that follows the national mathematics curriculum, articulated with music education; G.Exp.1 is a class that follows the national math curriculum. In the control group (G.Co) there were no interventions under the chosen approach.

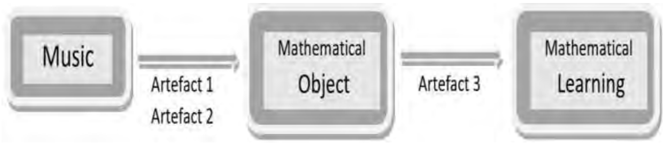


Figure 2. Scheme that operationalizes the way of doing mathematics with music

Table 2. Set of artefacts

Intervention artefacts			
Artef-M	**Artef-T1**	**Artef-T2**	**G**
Digital piano (mobile application)	Task-artefact "Completando a partitura de uma música" (Completing a sheet music)	Task-artefact "Resolvendo equações" (Solving equations)	Guide n.º 3
Reserve artefacts (**Artef-R**)			
Recorded audios (various interventions)			
Smartphone+Speaker (sound link mini–Bose)			

The interventions in the classroom, in the experimental groups, followed the scheme in **Figure 2** (Silva et al., 2021).

The interventions were carried out within the scope of the approach: doing mathematics with music, using properly articulated artefacts. To orchestrate the instruments, the methodology used in the previous study (Silva et al., 2021) was followed, that is, the instrumental orchestration of the artefacts had the following features:

1. A musical artefact easily manipulated by each student;
2. A task artefact to transform the enjoyed music into a mathematical object; and
3. A task artefact allowing the mathematical object identified in the music to be transformed into mathematical knowledge.

In a first step, artefacts were used to allow the student to identify the mathematical object in the music. In a second stage, the artefact had to allow mathematical learning, from the previously identified mathematical object, in a scientific approach, using reason and logical knowledge. The constructed artefacts (and combined with other existing ones) were based on the theme Equations: notion of equation; solution of an equation; solving linear equations. It should be noted that "knowing how to read music" is not a necessary condition for interventions to take place with the benefit of students. The artefacts, in intervention, include a way to solve this musical gap that students can demonstrate.

In the interventions, different artefacts were articulated: intervention artefacts and reserve artefacts. Intervention artefacts are a set of artefacts constructed / selected for a particular intervention. Reserve artefacts are a set of artefacts that will only be used if necessary, replacing some intervening artefacts. The intervention has a set of indexed artefacts, as summarized in **Table 2**.

The intervention starts with music. This is where the intended mathematical object is identified. Generally, two artefacts are used simultaneously: a musical artefact (Artef-M) and a task artefact (Artef-T1). The task artefact (Artef-T2) follows the context created by the previous artefacts. To articulate the artefacts described above (Artef-M, Artef-T1 and Artef-T2), a fourth artefact is added: guide (G). See the task artefact (Artef-T1 and Artef-T2) on the **Appendix A**.

The guide is the main artefact because it articulates all the other artefacts, so it is always used together with one or more artefacts, that is, it accompanies the entire intervention. All indications for the entire intervention are part of the guide: in addition to the time allotted for each artefact (or set of artefacts), the intention of each of the tasks is highlighted, that is, what are expected to students to be able to learn at the end of each challenge (or set of challenges) presented. Indications are given, namely the intention to enjoy the music (fundamental in the teaching approach adopted), so that each challenge must be fully explored by the students (with or without the intervention of the teacher) and only then move on to the next, that is, the sequence of challenges is crucial for learning to occur and for it to be epistemic.

The guide consists of three parts: before class, during class, and after class. Pre-class includes target audience, duration, content, and required material. One of the materials needed is Artef-M (digital piano), which is installed, initially, on the student's mobile phone or tablet. The in-class phase includes the five steps of classroom intervention. The after-class phase includes the analyzes to be carried out on the work carried out.

Focusing on the phase during the class, the first step is to present to the students, in a global way, the intervention that is intended to be carried out, its objectives and how it will work. In the second stage, Artef-T1 is distributed: "completando a partitura de uma música" (completing a sheet music) (**Table 2**), allowing students to explore the challenges with the application on their mobile phone, manipulating Artef-M: digital piano (mobile application). Artef-T1 comprises four challenges/tasks. The first two challenges (Task1 and Task2) consist of finding a missing note in the sheet music (Happy Birthday to You) and for that to happen, the following rules are indicated. The need to follow a set of rules to solve a challenge is a constant in Artef-T1, since the objective is that this same principle is used later in Artef-T2, where there will already be rules that allow solving equations. If the students do not know the melodies of the songs presented, the audios can be used, using the reserve artefacts (Artef-R), guiding the students in case they have difficulties (**Table 2**). Also, in Artef-T1, there is another challenge (task 3) where students are invited to complete a table, which aims to equal the duration of a set of rhythmic figures. Once again, a set of rules is described to follow, reinforcing the need to follow rules.

Table 3. Characterization of effective curricula

Curriculum effective	Groups	V1	V2	Learning environment characterization: Proposed tasks, 1st phase	Learning environment characterization: Proposed tasks, 2nd phase
CE_I	G.Exp.1	No	Yes	8 tasks (Integrating the tasks artefacts)	FT (worksheet)
CE_II	G.Exp.2	Yes	Yes		
CE_III	G.Co	No	No	Presentation of definitions/examples+ Exercises similar to the examples	

The third stage consists of interacting with students, reflecting on the proposed challenges, exploring the concept of equation in a holistic way-identification of the mathematical object. It is identified in the challenges of completing the sheet music that the unknown note, in mathematics, is called incognita; when the note is discovered, we say that we have found the solution (that is, we discover the value of the unknown that allows us to match the sheet music to the intended music); to equality, where there is at least one unknown, we call equation.

In the fourth stage, the students are presented with the mathematical challenges that appear in Artef-T2 "resolvendo equações" (solving equations) (**Table 2**). Five more challenges are presented. The first (task 4) is identical to the last challenge (task 3) of Artef-T1, as the same table structure is presented, but without musical figures–with numbers and letters, and you are asked to find the unknown value to obtain a true equality. In the next challenge (task 5), two rules are given that guarantee equality (still presented in the form of tables identical to task 3 and task 4): rule1–we can add or subtract the same amount on both "sides" of the equality; rule 2–we can multiply or divide by the same amount (except zero) on both "sides" of the equality (at this stage, the mathematical terminology "equation member" is not yet used). Task 6, task 7, and task 8 are challenges with increasing degree of difficulty, where it is intended that they use the skills acquired during the previous challenges.

At the end (step five), a synthesis / conclusion is made with the students, remembering how the mathematical object was identified in the music, as well as the mathematical learning carried out later. It ends by inviting students to play the songs proposed in the initial challenges.

Effective Curriculum

Regarding the nature of the groups (three classes) and the variables under study (V1-class with and without in-depth musical knowledge; V2-class with and without intervention having music as a mathematical object using mediating artefacts), the work developed involved three distinct epistemic paths, which resulted in three different effective curricula, as can be seen in **Table 3**. The effective curriculum I (CE_I) of the G.Exp.1 which is characterized by the implementation of the intervention in a class with little musical knowledge (only those from the national curriculum worked in the subject of music education, in the 5th and 6th grades); the effective curriculum II (CE_II) of the G.Exp.2 which is characterized by the implementation of the intervention in a class with enough musical knowledge (those developed, either at the theoretical level or in the performance of a musical instrument, in articulation with the national curriculum); the effective curriculum III (CE_III) of G.Co is related to a class where there was no place to implement the intervention (did not follow the approach adopted in the interventions made to the other two classes) and equally with little musical knowledge.

The three groups went through two phases. In a first phase, G.Exp.1 and G.Exp.2 were subjected to interventions in the chosen approach (doing math with music). In this first phase, all groups, although in different ways (**Table 3**) approached the topic of equations (notion of equation, root or solution of an equation, equivalent equations, principles of equivalence of equations, solving first degree equations without denominators and without parentheses). In a second phase, the three classes reinforced the knowledge of the first phase, adding the solution of equations with parentheses, using a worksheet (FT).

Data Collection

During the interventions, in a total of four classes, audio recordings, photographic records of the moments of the class, records on the board and copies of the students' notebooks were made. The data were used to prepare a complete multimodal narration (NM) of each of these classes according to the protocol presented by Lopes et al. (2014). This instrument organizes and aggregates the data collected in teaching practice and facilitates research work, as it brings together in a single document the various aspects that can be observed in the classroom (Lopes et al., 2014).

The pre-tests (theme equations) were applied in the three groups: the control group (G.Co) and in the experimental groups (G.Exp.1 and G.Exp.2). The application of the pre-test took place before the beginning of the theme. The post-tests were applied to the three groups and took place after the completion of the worksheet.

Table 4. Identification of the evaluation object of each question (pre- and post-test)

Question	Evaluation object
Q_1	Notion of equation.
Q_2	Solution of an equation.
Q_3	Translate a problem of low difficulty into an equation (unknown with a coefficient of one) & its respective resolution.
Q_4	Translate a problem of average difficulty into an equation (unknown with coefficient other than one) and its respective resolution.
Q_5	Translate a problem of high difficulty into an equation (involves more than one term with unknown with coefficient other than one) and its respective resolution.

Table 5. Criteria for evaluating artefacts as epistemic tools

Criteria	Description
C1.1	Students solve the challenge/task autonomously.
C1.2	Students solve the challenge/task with some autonomy.
C1.3	Students do not solve the challenge/task with autonomy.
C2	Students move from a musical representation to a mathematical representation (there is identification of the mathematical object).
C3	Students show conversions between mathematical languages (natural, symbolic, algebraic, and graphic).
C4	Students demonstrate mathematical learning.
C5	Students demonstrate mathematical learning beyond the concepts that were the subject of the intervention.

Table 6. Degrees of quality of mathematical epistemic activity

Degrees of quality of mathematical epistemic activity	Criteria involved
Low	C1.3
Poor	(C1.1 or C1.2) and (C2 or C4)
High	(C1.1 or C1.2) and [(C3 and C4) or (C2 and C5)]
Very high	(C1.1) and (C3 and C4)

Table 7. Epistemic levels

Epistemic levels	Description
EL1	Enjoyment of music without mathematical concepts
EL2	Music as a mathematical object without conceptualization
EL3	The mathematical object is worked with mathematical conceptualization
EL4	The mathematical object is worked with high abstraction and generality

Table 4 identifies the object of evaluation of each question. The degree of difficulty of the questions (Q_i) is made in an increasing way, that is, it starts with Q_1, lower difficulty and increases the difficulty until Q_5.

Data Analysis

In this study, two methodological approaches were adopted: an exploratory component and a comparative, quasi-experimental one, evaluating the effect of two variables. Data analysis, in each methodological approach adopted, is found in the following subsections.

Qualitative analysis

In order to answer the first research question (QI_1-What links exist between epistemic activity and the variability profile of epistemic levels, when in an educational context students do mathematics with music as their object?), an exploratory component was methodologically adopted. To characterize, in the experimental group, the epistemic activity and to understand how artefacts are used and how they are transformed into epistemic instruments, criteria reported by Silva et al. (2021) were used. The criteria/indicators serve to characterize the quality of mathematical epistemic activity (**Table 5**).

By grouping some of the previous indicators, four degrees of the quality of mathematical epistemic activity were defined (**Table 6**).

Based on ideas and results about epistemic levels (Branchetti & Morselli, 2019; Kelly & Takao, 2002; Ndemo & Mtetwa, 2021; Švaříček, 2019), four epistemic levels were defined for experimental groups, adopting concept of epistemic level for mathematics on two ideas: degree of abstraction and proximity of the mathematical object to the starting musical object (**Table 7**).

Quantitative analysis

To answer the second research question (QI_2-Is the intervention "doing maths with music" effective in terms of students' mathematical learning? Are there differences between students with and without in-depth musical knowledge?), a comparative, quasi-

Table 8. Performance levels by item

Items	Performance levels
Item 1	Four: From 0 to 3
Item 2	Four: From 0 to 3
Item 3	Seven: From 0 to 6
Item 4	Eight: From 0 to 7
Item 5	Nine: From 0 to 8

experimental component was methodologically adopted evaluating the effect of two variables. The variables to consider are: V1–class with and without in-depth musical knowledge; V2–class with and without intervention having music as a mathematical object using mediating artefacts.

To understand the relationship between the different epistemic paths and learning, the answers given by the students were categorized by performance levels in the pre and post tests, based on Lopes et al. (2011). The following steps were taken:

1. Student performance levels before and after the interventions were assessed using a pre-test and a post-test, previously validated by two specialists in science education;
2. The tests consisted of five items;
3. The pre-test was applied, in the three groups G.Exp.1, G.Exp.2 and C.Co, before the beginning of the teaching of the theme equations. The post-test was applied to the three groups at the end of the total teaching of the topic; and
4. Collection of all answers (pre- and post- tests) given by students and later grouped by performance levels (**Table 8**).

Table 9 shows an example of categorization of responses grouped by performance levels for the item 1.

In order to understand the data obtained in the pre- and post-tests, non-parametric statistical tests were applied: Kruskal Wallis test and Wilcoxon test.

As the test performance scale used is qualitative ordinal, neither means (but medians) nor ANOVA can be used. Therefore, non-parametric tests must be used. The groups G.Exp.1, G.Exp.2, and G.Co are independent samples and, therefore, the comparison between them can be made using the Kruskal Wallis statistical test (allows us to assess whether there are significant differences between the three groups). In each of the three groups (between the pre-test and the post-test) the criterion of paired samples can be applied, and the Wilcoxon test can be used. It allows assessing whether the evolution in each group is statistically significant.

RESULTS

Characterization of the Intervention

In order to characterize the intervention in the two experimental groups, the time used by each artefact throughout the interventions was recorded (**Table 10**).

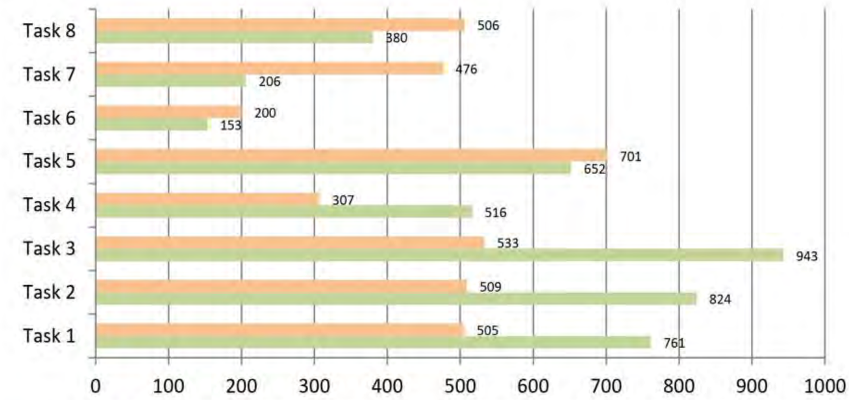
The time used in each challenge/task in both groups was also counted (**Figure 3**). In the first four tasks,

Table 9. Performance levels used in item 1 (example)

Item 1		1. The figure shows a balance in balance, in which each banana has *b* grams. a. Write a condition that translates what you observe in the balance of the previous figure, considering that it is in balance.
CP	Performance level	Examples of student responses
0	No answer or wrong answer with a senseless justification.	"An equation is an equality that contains at least one unknown"; "a=b"; "5 bananas"; "600" "It is not in balance because the weight weighs more than bananas"
1	It recognizes that it is facing a balanced situation.	"The scale shows that bananas have 600 grams"; "Bananas weigh as much as 600g"; "600g = 600g" "The banana is balanced with 600 grams on each side and is balanced because bananas weigh 600g" "Balance on both sides"
2	Recognizes the components of balance, recognizing that there is an unknown part (notion of unknown).	"5x"; "b=600" "bananas=600";
3	It presents a complete condition of the equality represented on the balance.	"5 bananas=600g" "5b=600" "5b=100+500" "5x=600"

Table 10. Use of artefacts in intervention, as a percentage

	Intervention artefacts					Time (total)
	G	Artef-M	Artef-T1	Artef-R	Artef-T2	
G. Exp. 1	100%	29.0%	54.8%	0.4%	40.2%	98m 56s
G. Exp. 2	100%	25.5%	45.5%	8.8%	54.5%	76m 26s



	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6	Task 7	Task 8
G.Exp.2	505	509	533	307	701	200	476	506
G.Exp.1	761	824	943	516	652	153	206	380

Figure 3. Time (in seconds), by challenge/task, in the two experimental groups

G.Exp.1 spent more time than G.Exp.2., reversing this trend in the remaining four tasks.

The class with poor musical knowledge (G.Exp.1) took more time to solve the tasks where the artefact was used, task directly linked to the musical artefact (first three tasks), that is, these students need more time to make the music as a mathematical object. In general, the class with in-depth musical knowledge (G.Exp.2) uses less time to complete tasks where musical artefacts are involved. In tasks 6, 7, and 8 (with more mathematical elaboration) G.Exp.2 devotes more time to them. The total time indexed to each group differs, but this difference comes from the extra time needed by the group without mathematical knowledge (G.Exp.1) to complete the tasks where musical artefacts are involved.

Relationship Between Epistemic Activity and Epistemic Level

The identified epistemic activity was listed by challenge/task worked and by each group (G.Exp.1 and G.Exp.2) (they allow identifying when students use an artefact as an epistemic tool), considering the predefined criteria. The degrees of epistemic quality (low, poor, high, and very high) are also identified, by task and groups, and whether mathematical learning took place (**Table 11**).

In the tasks 1, 2, and later, 3 the students work with mathematical concepts using the musical artefact (and others in articulation with it), which later lead to the identification of the mathematical object in the music (**Table 11** and **Appendix A**). In the first two, where students are invited to find a missing note in the score of a song known to everyone, students already start working on mathematical concepts such as the unknown and solution of an equation. This type of work is intensified in task 3. Students, in this task, are invited to fill in a table, equating the beats of the notes on the left side with the beats on the right side of the table (since the notes are used ligatures, the sound of both sides will be the same, although the beats of the notes used are not the same). This notion of equality between two parts, two sets, two sounds, which the students rely on to answer to the challenge, is considered as identifying a mathematical object in music.

The same type of table is presented in task 4, but this time mathematical language is already introduced (since there has already been done the identification of the mathematical object in the music), so the table is already filled with mathematical expressions and the students are challenged to fill the same in order to match the two columns of the table. In this challenge, students already begin to use mathematical language such as unknown, solution, equation, solving an equation, but without the formality of an equation, that is, still making use of tables with two columns where equality is intended (we keep some proximity of the mathematical object to the starting musical object). Even in the following challenges, the tables remain. It is only in the sixth task that the tables are abandoned, and the equation begins to be written as an equality between two expressions where there is at least one unknown.

The quality of the epistemic activity is the same in practically all the tasks in the two groups, except in task

Table 11. Quality grades of epistemic activity

		Intervention-Artefacts by challenge/task															
		G; Artef-M; Artef-T1		G; Artef-M; Artef-R; Artef-T1		G; Artef-T1		G; Artef-T2		G; Artef-T2		G; Artef-T2		G; Artef-T2		G; Artef-T2	
		Task 1		Task 2		Task 3		Task 4		Task 5		Task 6		Task 7		Task 8	
		G.Exp.1	G.Exp.2	G.Exp.1	G.Exp.2	G.Exp.1	G.Exp.2	G.Exp.1	G.Exp.2	G.Exp.1	G.Exp.2	G.Exp.1	G.Exp.2	G.Exp.1	G.Exp.2	G.Exp.1	G.Exp.2
Epistemic activity	C1.1	x	x		x	x	x	x	x			x	x				
	C1.2			x						x	x			x	x		x
	C1.3															x	
	C2	x	x	x	x	x	x										
	C3											x	x	x	x		x
	C4							x	x	x	x	x	x	x	x		x
	C5					x											
DQMEA	Low															x	
	Poor	x	x	x	x		x	x	x	x	x						
	High					x								x	x		x
	Very high											x	x				
Mathematical learning						x		x	x	x	x	x	x	x	x		x

Note. DQMEA: Degrees of quality of mathematical epistemic activity

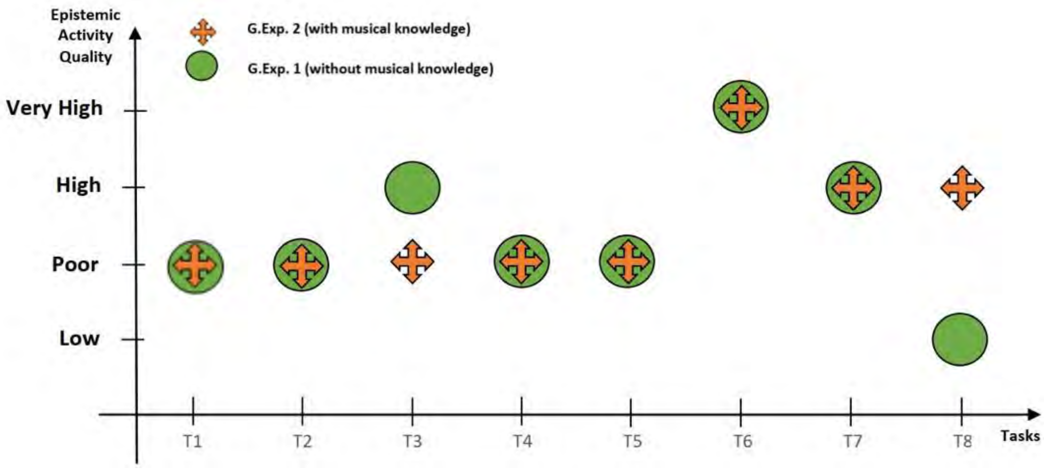


Figure 4. Variation in the quality of epistemic activity during tasks

3 and task 8, it is worth noting that in task 8 the G.Exp.1 has low epistemic activity (**Figure 4**).

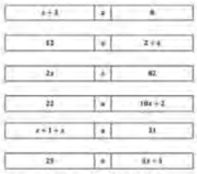
In the most difficult task (task 8), the groups differ greatly in the quality of the epistemic activity: G.Exp.1 has low quality and G.Exp.2 has high quality. Bearing in mind that what differs in groups is having (or not) in-depth musical knowledge, G.Exp.2 students show that they are able to transform the mathematical object identified in music into mathematical knowledge, even in more difficult tasks.

If we compare **Figure 3** and **Figure 4**, the quality of the epistemic activity in tasks with a lot of mathematical elaboration (task 6, task 7, and task 8) is influenced by the time dedicated to the task: G.Exp.2 works on task 8 with higher quality epistemic than G.Exp.1. G.Exp.1 (with less musical knowledge) needs more time to transform the music into a mathematical object (from task 1 to task 4).

In task 3 both G.Exp.1 and G.Exp.2 solve the task autonomously and identify the mathematical object in the music, as can be seen in the excerpt of the multimodal narrative (G.Exp.1) presented in **Figure 5**.

The quality of the epistemic activity is different, since the G.Exp.1 demonstrates mathematical learning beyond the concepts that were the subject of the intervention. The students, in addition to working on the concept of equation, ended up approaching the concept of geometric progression and solving questions that derived from task 3. We can see this in the excerpt from the multimodal narrative of G.Exp.1 (**Figure 6**).

(13:44) Student – That's easy!
Teacher- Let's go! Everyone looking at the sheet.
(14:02) Teacher – So… part B of the assignment: solving equations. In mathematics, the unknown note, in that first part… of part A… the note that we didn't know, we'll call it unknown, ok?!
(14:18) Student – How?
(14:19) Teacher – Unknown.
(14:21) Student – Is it important to know this?
(14:22) Teacher – Yes! This is the math part… let's call it "names" which corresponds to the math of what we saw with the music.
(14:35) Teacher – Then it says like this: when the note is discovered, we say that we have found the solution! He is well?!...
(14:43) Teacher – In the case of Task 1… it was F, number 4; and the second was...
(2:48) Students – The 6.
Teacher – The 6 that was A, right?
Teacher – Solution the first: F. Second solution: A. Or 4 or 6, as you wish!
(14:58) Teacher – Then look at what comes next, which is very important (…) We discovered the value of the unknown that allows us to match the intended music to the original, not to an invented music! So, we must equal two things.
(15:19) Teacher – Then… to the equality, where there is at least one unknown, we call equation, ok?
(15:32) Student – Yes!
(15:33) Teacher – Finally… the unknown is represented by a letter. In mathematics, people usually use x… which is to put in place of the number I'm looking for! He is well? And when we find the number, we have the solution.
(15:48) Teacher – Solving an equation is finding the value of the unknown to obtain true proposition. Shall we solve equations?
Students quickly fill in the table (task 4).
(16:03) Teacher – So Marco... what is the value of that unknown, so that this side is equal to the other side?
(16:13) Student – Can I say teacher?



Teacher – I said Marco. Marco, shall we go?
(Other Students want to say, but the teacher insists it's Marco).
(16:34) Marco – 5+3=8
Teacher – So… how much is the unknown? What is the solution, the value of the unknown?
(16:41) Marco – 5.
Teacher – 5. So the answer is x = 5. Why? Because it's like Marco said: 5 + 3 = 8. That's the reason. But what is the solution? x = 5. Ok?

Figure 5. Excerpt from multimodal narrative (students of group Exp.1 identify mathematical object in the music, task 3)

(11:04) Teacher – Then explain your reasoning, go!
Student – If… one sixteenth is a hemidemisemiquaver, we need 16 (the previous student says… I know!)
(11:18) Teacher – Say it!
Student – We add 16 + 16 + 16 + 16 and it gives 64.
(11:24) Teacher – Yeah! Now think with me: 4 … half… 2… half… 1… her reasoning (another student) was different. It was to make halves. When it gets here to one… then we have:
(11:51) Student – ½
Teacher – Half of ½ …
Students – ¼ .
(11:57) Teacher – I need 4 semiquavers to get to 1 beat, right? And I need two quavers to get to one beat… right? How much is half of ¼ ?
(12:06) Student – 1/8.
(12:07) Teacher – Half of 1/8? ... (writes on board 1/16)... right?
(12:17) Teacher – So I need two of these (teacher points to quaver) to get one, right? And I need 4 of these (points to semiquaver) to get one, right?
(12:27) Teacher – That is, when I'm in the hemidemisemiquaver, if I add 16 hemidemisemiquavers, i get 1 beat. As I want to have 4 beats… I have to make 16 x 4… which gives 64. In other words, if I want to replace a semibreve only with hemidemisemiquavers… I need 64!
Student – I told you!
(12:59) Teacher – This was up in the air, but after all it went down, it's back on earth!"

Figure 6. Excerpt from multimodal narrative (students of the group Exp.1 demonstrate mathematical learning beyond the concepts that were the subject of the intervention, task 3)

Considering the epistemic levels experienced by the groups (EL1, EL2, EL3, and EL4), **Figure 7** depicts the variation in epistemic levels by task and by experimental groups (G.Exp.1 and G.Exp.2).

The epistemic level in G.Exp.2 is increasing with one inflection and the epistemic level in G.Exp.1 has three inflections and there is no upward trend.

Then it seems that both experimental groups manage to appropriate music as a mathematical object and manage to conceptualize it mathematically. However, G.Exp.2 still manages to work the mathematical object with high abstraction.

The control group (G.Co) has no variability in the epistemic level with which it deals with the mathematical object. It always works with mathematical conceptualization and, therefore, always at the EL3 epistemic level. In the experimental groups (G.Exp.1 and G.Exp.2) there is variation in epistemic levels during the challenges posed. After the eight challenges/tasks, the path merged to the control group path (worksheet - FT), with level EL3.

In the last task, task 8, students are already in the final stage of the intervention. Groups differ greatly about the epistemic level at which they work with the mathematical object. The G.Exp.1 did not work the mathematical concept, as can be seen from the excerpt of the multimodal narrative presented in **Figure 8**. G.Exp.2 works the mathematical object with a high level of abstraction and generalization.

Table 12 shows the quality of the epistemic mathematical activity versus the respective epistemic

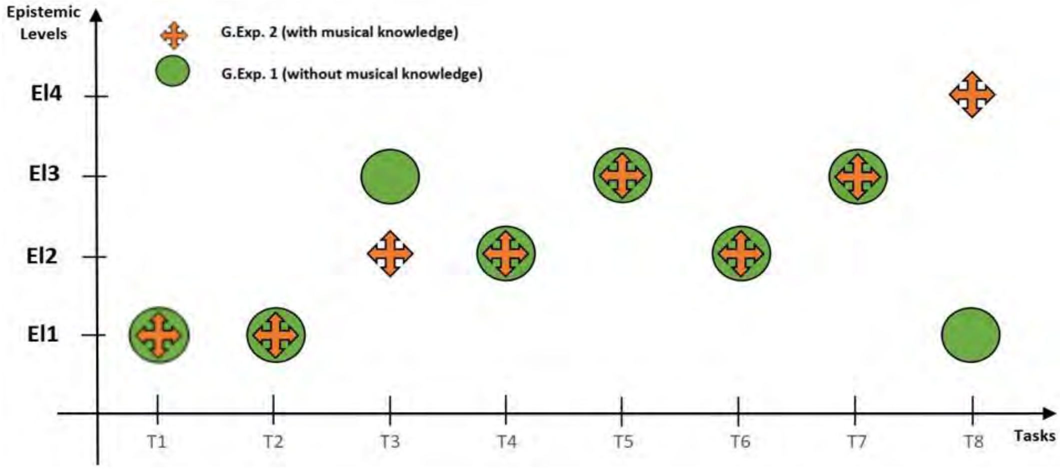


Figure 7. Variation of epistemic levels during tasks

(9:34) Teacher – Now say it: Think of a number.
Student – I already thought.
(9:42) Teacher – x! Now what do you want, tell me?
Student – multiply by 4.
(9:46) Teacher – Multiply by 4… and then what?
Student – Multiply by 10.

$4x$
$40x \quad (4x \times 10)$

Teacher – Multiply by 10. And then? We're already at 40x, right?
(10:02) Teacher – You want to get the x out of here. And then, remember? Divide by the number you thought of. How much?

$\frac{40x}{x}$

Student – Gives 40.
Teacher – Everyone is already in 40.
Students – Yeah!... Right now!
(Some Students get lost in the calculations).
(10:13) Teacher – Think of a number! Think of a small number to be easy!
(10:15) Student – The 1!
Teacher – 1 x 4?
Students – 4
Teacher – 4 x 10?
Student: 40
(10:20) Teacher – 40! Right? And now… divide by the number you thought… wasn't it 1?
Students – Yes!
Teacher – 40 divided by 1?
Student - 40.
Student – Fire! It was the first number I thought of!
(10:29) Teacher – This is not magic!
(There are still unbelieving students)

Figure 8. Excerpt from multimodal narrative (students of the group Exp.1 did not work the mathematical concept, task 8)

level, by task and by group (G.Exp.1 and G.Exp.2). We can observe that, globally, if the quality of the epistemic activity is high, the epistemic level (distance to the referent used – music) is also high. However, there can be high quality epistemic activity even if the epistemic level used to deal with the mathematical object is low. Furthermore, at a given epistemic level, epistemic work can be of greater or lesser epistemic quality.

There are two features in the experimental groups: variability of the epistemic level in the different tasks; in the tasks in which mathematics is worked from music as a mathematical object, higher epistemic levels can be found.

The epistemic path is the path from task 1 to task 8 characterized in each task by the epistemic level and the quality of the epistemic activity (**Figure 9**). There is a remarkable feature if we compare the paths of the two groups. The G.Exp.2 reached the highest epistemic level (EL4 the one with the greatest abstraction in relation to music). The difference between the two pathways is that the increase in the quality of epistemic work at a lower epistemic level in task 6 (greater proximity to music) in G.Exp.2 benefited from a slower progression of the epistemic level in the initial tasks. G.Exp.1 tries to work early at a high epistemic level (task 3 with EL3) and that might prevent it from working at a high epistemic level (task 8 with EL1).

Table 12. Quality of epistemic activity versus epistemic levels, by task, in both groups

		Epistemic level			
		EL1	EL2	EL3	EL4
Quality of epistemic activity	Low	Task 8(G.Exp1)			
	Poor	Task 1 (G.Exp1 e G.Exp.2)	Task 3 (G.Exp.2)	Task 5 (G.Exp1 e G.Exp.2)	
		Task 2 (G.Exp1 e G.Exp.2)	Task 4 (G.Exp1 e G.Exp.2)		
	High			Task 3 (G.Exp1)	Task 8 (G.Exp.2)
				Task 7 (G.Exp1 e G.Exp.2)	
	Very high		Task 6(G.Exp1 e G.Exp.2)		

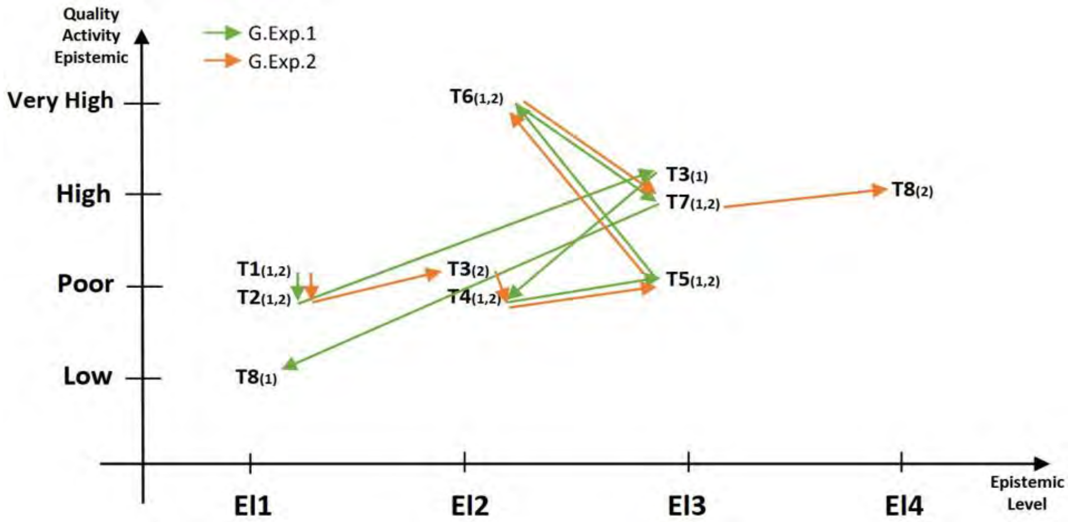


Figure 9. Epistemic pathways from task 1 to task 8 in experimental groups

Table 13. Difference between the medians of performance levels (pre- & post-tests), by question, in three groups of students

		Q1			Q2			Q3			Q4			Q5		
	N	Pre	Pos	Dif.	Pre	Pos	Dif.	Pre	Pos	Dif.	Pre	Pos	Dif.	Pre	Pos	Dif.
G.Co	18	2	2.5	0.5	3	3	0	4	4	0	0	2	2	0	1	1
G.Exp.1	17	0	1	1	3	3	0	4	6	2	0	7	7	0	8	8
G.Exp.2	19	1	3	2	3	3	0	4	6	2	0	7	7	0	2	2

Table 14. Significance level obtained by the Kruskal Wallis test in the three groups, by question

Q1	Q2	Q3	Q4	Q5
0.026	0.482	0.216	0.608	0.247

Table 15. Significance level obtained by the Wilcoxon test, by question, in each group (gray indicates those that are significant)

	G.Co	G.Exp1	G.Exp2
Q1	0.468	0.026	0.003
Q2	0.398	0.564	0.221
Q3	0.452	0.046	0.013
Q4	0.153	0.018	0.010
Q5	0.007	0.003	0.003

Learning Assessment and Statistical Comparison of Global Results of Learning Assessment Tests (Effects of Variables V1 and V2)

Considering the test performance scale used (qualitative ordinal) to compare the learning outcomes, medians of the data collected in the pre-tests and post-tests were used by question (Q1 to Q5) in the three groups of students corresponding to three curricula effective (**Table 13**).

In all questions, with the exception of Q2, the difference between the medians of the post-tests and pre-tests are higher in the experimental groups (G.Exp.1 and G.Exp.2) compared to the control group (G .Co). Thus, if we consider variable V1-class with and without intervention having music as a mathematical object using mediating artefacts-it is observed in **Table 13** that the experimental groups, that is, where the intervention took place within the scope of the adopted approach-doing mathematics with music–revealed better maths performance.

Considering variable V2-class with and without in-depth musical knowledge-it is observed in **Table 13** that the differences between the medians in the groups (G.Exp.1 and G.Exp.2) are very similar (except for Q5), in other words, having in-depth musical knowledge does not interfere in the mathematical performances achieved in groups where the approach of doing mathematics with music was adopted.

To understand the data obtained in the pre- and post-tests, non-parametric statistical tests were applied: Kruskal Wallis test and Wilcoxon test.

The Kruskal Wallis statistical test was used, in the pre-test (ensuring its application before performing any intervention), to compare the three groups (G.Co, G.Exp.1 and G.Exp.2), since are independent samples, in order to verify the existence of significant differences between the three groups at the beginning of the intervention (**Table 14**).

In the three groups, except for Q1, there are no significant differences between the groups, that is, the results obtained do not depend on the students' groups of origin.

In each of the three groups (between pre-test and post-test) the criterion of paired samples was applied, using the Wilcoxon test, to assess whether the evolution in each group is statistically significant (**Table 15**).

In the experimental groups, the evolution from pre to post-test is statistically significant in all questions, except for Q2. In the control group there is a significant evolution only in Q5. In other words, there is an effect of the intervention "doing maths with music" (from pre-test to post-test) on the mathematical performance of students in groups G.Exp.1 and G.Exp.2, since it is verified that the differences are significant in all questions, apart from question 2 (**Table 15**). The didactic approach "doing math with music" is more effective for students' math learning than conventional approaches (control group).

DISCUSSION

In response to the research questions, two contributions from this study are presented and discussed.

Contribution 1

1. The intervention "doing maths with music" is effective in terms of students' mathematical learning and the differences between students with and without in-depth musical knowledge does not affect this effectiveness.

The present study is in line with other studies that place music at the center of mathematical learning (An et al., 2008; Quinn et al., 2019; Thayer-Morel et al., 2018). This type of study is not very frequent. In addition, it deals with a teaching approach called "doing mathematics with music" that is distinguished from others, on the one hand, by its epistemic nature, both in the epistemic environment (Hershkowitz et al., 2001; Svahn & Bowden, 2021) underlying it, as in the study of the quality of epistemic work (Hudson, 2019) and the epistemic level at which it occurs; on the other, through the use of the concept of instrumental orchestration (Drijvers et al., 2010, 2020; Guin & Trouche, 2002; Lopes & Costa, 2019) to operationalize the referred epistemic environment.

"Doing maths with music" is a didactic intervention that created an epistemic environment in which students could assume a certain responsibility (Stroupe et al., 2019) in how they dealt with the challenges posed by the teacher (Asterhan & Schwarz, 2016) using several artefacts (digital piano, standard tasks, and script) properly orchestrated among themselves (Drijvers et al., 2010, 2020; Guin & Trouche, 2002; Lopes & Costa, 2019), some of which functioned as epistemic tools for students to learn mathematics (Svahn & Bowden, 2021). In short, the intervention consists of creating an epistemic environment in which music is the referent of real life and, as activities unfold, it becomes a mathematical object. It is with this mathematical object that it is possible for students to learn mathematical concepts new to them. The results show that it is not necessary for students to have musical knowledge for the experimental approach to be more effective than conventional approaches for the same mathematical subject. In fact, the "doing mathematics with music" approach, under the conditions mentioned above, is more effective than the conventional ones whether, or not, the students have deep musical knowledge. In this sense, our result contradicts some authors who argue that children who have musical training have a better performance in mathematics (Chao-Fernández et al., 2018; Gardiner et al., 1996), although, in fact, this study was not designed to contradict this assertion, but only to verify whether the intervention was sensitive to the students' musical knowledge.

Contribution 2

2. The educational artefacts used by students when they do mathematics with music, in an educational context, allowed them to deal with music (mathematical object) at various epistemic levels, with complex relationships between the quality of students' epistemic activity and the profile of epistemic levels they dealt with the music. Overall, if the quality of epistemic activity is high, the epistemic level (distance to the referent used–music) tends to be high. There are, however, specific aspects to be highlighted.
    a. The quality of epistemic activity in tasks with a lot of mathematical elaboration increases with the time devoted to the task.
    b. The group with less musical knowledge needs more time to transform music into a mathematical object.
    c. It is possible to appropriate music as a mathematical object by groups with and without in-depth musical knowledge. However, the group with the best musical preparation manages to work the mathematical object with greater abstraction.
    d. The epistemic path is most effective (reaching the highest epistemic level, the one with the greatest abstraction in relation to music) when one progresses (without leaps) from the lower epistemic level and when inflections are made along the path, increasing the quality of the epistemic work when read more closely with real life (at a lower epistemic level).

This study shows the importance of distinguishing quality of epistemic activity and epistemic level. Unlike the use of the epistemic level as a synonym for quality of epistemic activity by some authors (Mishiwo, 2021; Schwarz et al., 2018), our study shows that there is a relationship between both constructs, but this is not simple. It also shows that it is advantageous from the point of view of mathematical learning to start from lower epistemic levels (closer to real life) and to have a progression of the epistemic level at which the epistemic activity takes place. This result agrees with the results presented by Ndemo and Mtetwa (2021) and Švaříček (2019). We also found that there are advantages to there being inflections at the epistemic level with which one works epistemically, that is, there is a back and forth between mathematical knowledge and real life (Branchetti & Morselli, 2019).

In fact, students who used artefacts (experimental group) to deal with music as a mathematical object experience varied epistemic levels, but more than that, epistemic levels that increase throughout the intervention, obtained higher levels of mathematical performance than in the control group. This result agrees with Lopes (2019), who emphasizes that the quality of learning increases as artefacts become epistemic tools.

However, the novelty of the contribution we present is twofold: on the one hand, the progressive tendency of the epistemic level with which the mathematical object is dealt with may be concomitant with a certain variability of the epistemic level which, under certain conditions, may favor the achievement of higher levels of abstraction; on the other hand, that an epistemic path defined as the paths that students take, combining quality of epistemic work and epistemic level, is linked to better academic performance than those who do not.

It was not possible to study the relationship between the quality of the epistemic environment experienced and the levels of learning performance.

CONCLUSION

This study aimed, on the one hand, to study the effectiveness of the teaching approach to do mathematics with music in the performance of mathematics learning and whether this effectiveness depends on the musical knowledge of students and, on the other hand, to identify didactically relevant patterns in the relationships between the experienced epistemic environment and the epistemic levels at which students do mathematics.

The didactic intervention "doing maths with music" created an epistemic environment in which music is the referent of real life and as the activities unfold, it becomes a mathematical object, and which allows students to experience different epistemic levels when artefacts are used with the status of epistemic tools. The results show that whether, or not, the students have in-depth musical knowledge, the "doing math with music" approach is more effective than conventional approaches for the same mathematical subject.

This study shows the importance of distinguishing the concepts of "quality of epistemic activity" and "epistemic level". Effectively there is a relationship between them which, however, is not simple. It also shows that it is advantageous from the point of view of mathematical learning to start from lower epistemic levels (closer to real life) and to have a progression of the epistemic level at which the epistemic activity takes place. However, the progressive tendency of the epistemic level with which the mathematical object is dealt with may be concomitant with a certain variability of the epistemic level and that, under certain conditions, this variability may favor the attainment of higher levels of abstraction.

Author contributions: All authors have sufficiently contributed to the study, and agreed with the results and conclusions.

Funding: No funding source is reported for this study.

Declaration of interest: No conflict of interest is declared by authors.

REFERENCES

An, S. A., Kulm, G. O., & Ma, T. (2008). The effects of a music composition activity on Chinese students' attitudes and beliefs towards mathematics: An exploratory study. *Journal of Mathematics Education*, *1*(1), 91-108.

Asterhan, C. S., & Schwarz, B. B. (2016). Argumentation for learning: Well-trodden paths and unexplored territories. *Educational Psychologist*, *51*(2), 164-187. https://doi.org/10.1080/00461520.2016.1155458

Branchetti, L., & Morselli, F. (2019). The interplay of rationality and identity in a mathematical group work. In M. S. Hannula, G. C. Leder, F. Morselli, M. Vollstedt, & Q. Zhang (Eds.), *Affect and mathematics education* (pp. 323-344). Springer. https://doi.org/10.1007/978-3-030-13761-8_15

Cartiff, B. M., Duke, R. F., & Greene, J. A. (2021). The effect of epistemic cognition interventions on academic achievement: A meta-analysis. *Journal of Educational Psychology*, *113*(3), 477-498. https://doi.org/10.1037/edu0000490

Chao-Fernández, R., Román-García, S., & Chao-Fernández, A. (2017). Art, science and magic: Music and math the classroom. In *Proceedings of the 5th International Conference on Technological Ecosystems for Enhancing Multiculturality* (p. 77). ACM. https://doi.org/10.1145/3144826.3145427

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, *75*(2), 213-234.

Drijvers, P., Grauwin, S., & Trouche, L. (2020). When bibliometrics met mathematics education research: the case of instrumental orchestration. *ZDM Mathematics Education*, *52*, 1455-1469. https://doi.org/10.1007/s11858-020-01169-3

Edelson, R. J., & Johnson, G. (2003). Music makes math meaningful. *Childhood Education*, *80*(2), 65-70. https://doi.org/10.1080/00094056.2004.10521259

Elofsson, J., Englund Bohm, A., Jeppsson, C., & Samuelsson, J. (2018). Physical activity and music to support pre-school children's mathematics learning. *Education 3-13, 46*(5), 483-493. https://doi.org/10.1080/03004279.2016.1273250

Gardiner, M. F., Fox, A., Knowles, F., & Jeffrey, D. (1996). Learning improved by arts training. *Nature, 381*(6580), 284. https://doi.org/10.1038/381284a0

Goizueta, M. (2019). Epistemic issues in classroom mathematical activity: There is more to students' conversations than meets the teacher's ear. *The Journal of Mathematical Behavior*, *55*, 100691. https://doi.org/10.1016/j.jmathb.2019.01.007

Graziano, A. B., Peterson, M., & Shaw, G. L. (1999). Enhanced learning of proportional math through music training and spatial-temporal training. *Neurological Research, 21*(2), 139-152. https://doi.org/10.1080/01616412.1999.11740910

Guin, D., & Trouche, L. (2002). Mastering by the teacher of the instrumental genesis in CAS environments: Necessity of instrumental orchestrations. *Zentralblatt für Didaktik der Mathematik* [*Central Journal for Didactics of Mathematics*], *34*(5), 204-211. https://doi.org/10.1007/BF02655823

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. Dover Publications.

Hallam, S., & Price, J. (1998). Research section: Can the use of background music improve the behaviour and academic performance of children with emotional and behavioural difficulties? *British Journal of Special Education, 25*(2), 88-91. https://doi.org/10.1111/1467-8527.t01-1-00063

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, *32*(2), 195-222. https://doi.org/10.2307/749673

Hodges, D. A. (2005). Why study music? *International Journal of Music Education, 23*(2), 111-115. https://doi.org/10.1177/0255761405052403

Hudson, B. (2019). Epistemic quality for equitable access to quality education in school mathematics. *Journal of Curriculum Studies, 51*(4), 437-456. https://doi.org/10.1080/00220272.2019.1618917

Kämäräinen, A., Björn, P., Eronen, L., & Kärnä, E. (2019). Managing epistemic imbalances in peer interaction during mathematics lessons. *Discourse Studies*, *21*(3), 280-299. https://doi.org/10.1177/1461445619829236

Kelly, G. J., & Takao, A. (2002). Epistemic levels in argument: An analysis of university oceanography students' use of evidence in writing. *Science Education*, *86*(3), 314-342. https://doi.org/10.1002/sce.10024

Lee, S. W. Y., Wu, H. K., & Chang, H. Y. (2021). Examining secondary school students' views of model evaluation through an integrated framework of personal epistemology. *Instructional Science, 49*, 223-248. https://doi.org/10.1007/s11251-021-09534-9

Lerman, S. (2001). Cultural, discursive psychology: A sociocultural approach to studying the teaching and learning of mathematics. *Educational Studies in Mathematics*, *46*(1), 87-113. https://doi.org/10.1023/A:1014031004832

Lopes, J. B. (2019). Visual representation Artefacts used as epistemic tools to improve the quality of mathematics and science teaching practices. In B. Vogler (Ed.), *Teaching practices - Implementation, challenges and outcomes* (pp. 45-74). Nova Science Publishers.

Lopes, J. B., & Costa, C. (2019). Digital resources in science, mathematics and technology teaching how to convert them into tools to learn. In *International Conference on Technology and Innovation in Learning, Teaching and Education* (pp. 243-255). Springer, Cham.

Lopes, J. B., Branco, J., & Jimenez-Aleixandre, M. P. (2011). 'Learning experience' provided by science teaching practice in a classroom and the development of students' competences. *Research in Science Education, 41*(5), 787-809.

Lopes, J. B., Silva, A. A., Cravino, J. P., Santos, C. A., Cunha, A., Pinto, A., Silva, A., Viegas, C., Saraiva, E., & Branco, M. J. (2014). Constructing and using multimodal narratives to research in science education: contributions based on practical classroom. *Research in Science Education, 44*(3), 415-438.

Mannone, M. (2019). Have fun with math and music! In *Proceedings of the International Conference on Mathematics and Computation in Music* (pp. 379-382). Springer. https://doi.org/10.1007/978-3-030-21392-3_33

McLaughlan, R., & Lodge, J. M. (2019). Facilitating epistemic fluency through design thinking: A strategy for the broader application of studio pedagogy within higher education. *Teaching in Higher Education*, *24*(1), 81-97. https://doi.org/10.1080/13562517.2018.1461621

Mersenne, F. (1637). *Harmonie universelle, contenant la theorie et la pratique de musique* [*Universal harmony, containing the theory and practice of music*]. Editions du Centre National de la Recherche Scientifique [Editions of the National Center for Scientific Research].

Mishiwo, M. (2021). Effect of mathematics method course on pre-service teachers' knowledge of content and teaching fractions. *British Journal of Education*, *9*(7), 1-13.

Monaghan, J., Trouche, L., & Borwein, J. M. (2016). *Tools and mathematics*. Springer. https://doi.org/10.1007/978-3-319-02396-0

Muis, K. R., & Duffy, M. C. (2013). Epistemic climate and epistemic change: Instruction designed to change students' beliefs and learning strategies and improve achievement. *Journal of Educational Psychology*, *105*(1), 213-225. https://doi.org/10.1037/a0029690

Mun, R. U., Hertzog, N. B. (2018). Teaching and learning in STEM enrichment spaces: From doing math to thinking mathematically. *Roeper Review, 40*(2), 121-129. https://doi.org/10.1080/02783193.2018.1434713

Ndemo, Z., & Mtetwa, D. (2021). Mathematics education undergraduates' personal definitions of the notion of angle of contiguity in Kinematics. *JRAMathEdu (Journal of Research and Advances in Mathematics Education)*, *6*(2), 111-127. https://doi.org/10.23917/jramathedu.v6i2.11130

Peterson, I. (1985). Some labs are alive with...: The sound of data. *Science News*, *127*(22), 348-350. https://doi.org/10.2307/3969629

Quinn, C. M., Smith, D. K., Chappell, M. F., Carver, S. D., Duffy, S., Holcomb, J. P., Jackson, D. & Resnick, A. (2019). Music as math waves: Exploring trigonometry through sound. *Journal of Mathematics and the Arts*, *13*(1-2), 173-184, https://doi.org/10.1080/17513472.2018.1552822

Root-Bernstein, R. S. (2001). Music, creativity and scientific thinking. *Leonardo*, *34*(1), 63-68. https://doi.org/10.1162/002409401300052532

Schwarz, B. B., Prusak, N., Swidan, O., Livny, A., Gal, K., & Segal, A. (2018). Orchestrating the emergence of conceptual learning: A case study in a geometry class. *International Journal of Computer-Supported Collaborative Learning*, *13*(2), 189-211. https://doi.org/10.1007/s11412-018-9276-z

Sezen-Barrie, A., Stapleton, M. K., & Marbach-Ad, G. (2020). Science teachers' sensemaking of the use of epistemic tools to scaffold students' knowledge (re) construction in classrooms. *Journal of Research in Science Teaching*, *57*(7), 1058-1092. https://doi.org/10.1002/tea.21621

Silva A., Bernardino Lopes J., Costa C. (2021) Doing math with music - Instrumental orchestration. In A. Reis, J. Barroso, J. B. Lopes, T. Mikropoulos, & C. W. Fan (Eds.), *Technology and innovation in learning, teaching and education.* TECH-EDU 2020. Communications in Computer and Information Science, vol 1384. Springer, Cham. https://doi.org/10.1007/978-3-030-73988-1_8

Stroupe, D., Moon, J., & Michaels, S. (2019). Introduction to special issue: Epistemic tools in science education. *Science Education*, *103*, 948-951. https://doi.org/10.1002/sce.21512

Svahn, J., & Bowden, H. M. (2021). Interactional and epistemic challenges in students' help-seeking in sessions of mathematical homework support: presenting the problem. *Classroom Discourse*, *12*(3), 193-213. https://doi.org/10.1080/19463014.2019.1686998

Švaříček, R. (2019). The role of the teacher in supporting students' epistemic thinking in dialogic argumentation. A case study. *Studia Paedagogica* [*Studies in Pedagogy*], *24*(4), 143-171. https://doi.org/10.5817/SP2019-4-7

Sylvester, J. J. (1864). XVII algebraical researches, containing a disquisition on Newton's rule for the discovery of imaginary roots. *Philosophical Transactions of the Royal Society of London*, *154*, 579-666. https://doi.org/10.1098/rstl.1864.0017

Thayer-Morel, T., Venegas-Thayer, M. A., & Tejada-Giménez, J. (2018). Recursos informáticos para el aprendizaje de las matemáticas mediante metáforas musicales: El proceso de creación y evaluación de PicaLab [Computer resources for learning mathematics through musical metaphors: The PicaLab creation and evaluation process]. *Estudios Pedagógicos* [*Pedagogical Studies*], *44*(1), 351-376. https://doi.org/10.4067/S0718-07052018000100351

Turkka, J., Haatainen, O., Aksela, M. (2017). Integrating art into science education: A survey of science teachers' practices. *International Journal of Science Education*, *39*(10), 1403-1419. https://doi.org/10.1080/09500693.2017.1333656

Viladot, L., Hilton, C., Casals, A., Saunders, J., Carrillo, C., Henley, J., Gonzalez-Martin, C., Prat, M., & Welch, G. (2018). The integration of music and mathematics education in Catalonia and England: Perspectives on theory and practice. *Music Education Research*, *20*(1), 71-82. https://doi.org/10.1080/14613808.2017.1290595

Wilhelmi, M. R., & Montiel, M. (2019, June). Integrated music and math projects in secondary education. In *Proceedings of the International Conference on Mathematics and Computation in Music* (pp. 390-393). Springer. https://doi.org/10.1007/978-3-030-21392-3_35

APPENDIX A

Task Artefact (Artef-T1 and Artef-T2)

Algebraic Equations

Notion of equation

Solution of an equation

Solving linear equations

Part A–Completing the Score of a Song

Task 1: Happy Birthday to You

In the following score* are part of the notes of the song "Happy Birthday to You".

RULES to find the missing the note:

- Each space corresponds to a note;
- The five missing notes are all the same, although they may have different durations;
- When you discover one, the others are automatically revealed;
- You must play the music and it must be the same as the original music.

What is the missing note throughout the score?

A.: ____

* The score has the numbers corresponding to the notes, according to the following table, which corresponds to a digit from 0 to 9 for each note.

Number	Note
0	B
1	C
1*	C#
2	D
2*	D#
3	E
4	F
4*	F#
5	G
5*	G#
6	A
6*	A#
7	B
8	C
8*	C#
9	D

Task 2: National Anthem

In the score that follows are part of the notes of a song of the National Anthem.

RULES to find the missing note:

- Each space corresponds to a note;
- The five missing notes are all the same, although they may have different durations;
- When you discover one, the others are automatically revealed;
- You must play the music and it must be the same as the original music.

What is the missing note throughout the score?

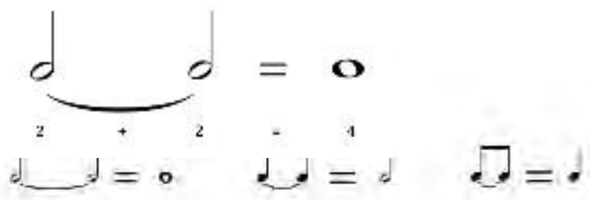
A.:____

Task 3: Musical Figures and Ligatures

Musical figures (or rhythmic figures) are symbols used to represent the beats of a song. The duration of each note in a score will be determined by the rhythmic figures.

Name	Value	Figure
Semibreve	4	
Minim	2	
Crotchet	1	
Quaver	1/2	
Semiquaver	1/4	
Demisemiquaver	1/8	
Hemidemisemiquaver	1/16	

The Ligature is a very common resource to increase the value of a figure; indicates the union of two or more figures of the same height, adding their values and, obviously, increasing their duration.



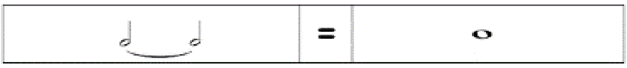
In the same way:

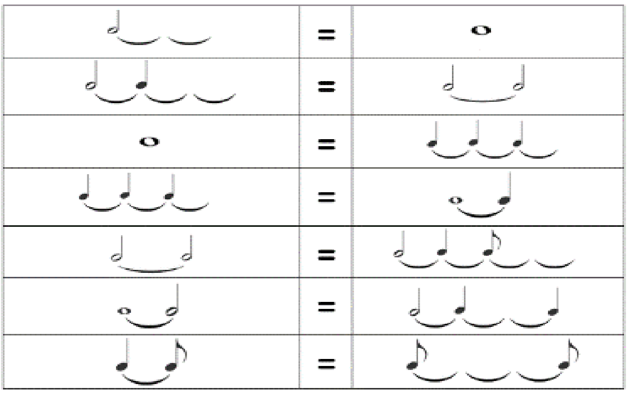
More:

RULES to complete the table:

- You can only use musical figures;
- There are no silences;
- The beats on the left side of the table equal the beats on the right side of the table;
- For the sound to be the same it is necessary to use ligatures.

Example:





Part B – Solving Equations

In Mathematics, the missing note is called the **unknown**.

When the note is found, we say that we have found the **solution** (that is, we find the value of the unknown that allows us to match the score to the desired music).

The equality, where there is at least one unknown, is called an **equation**.

The unknown is represented by a lowercase letter, the most common is letter x.

Task 4 - Solving an equation is finding the value of the unknown to obtain a true equality. Let's solve equations?

$x+3$	=	8	A.:
12	=	$2+x$	A.:
$2x$	=	82	A.:
22	=	$10x+2$	A.:
$x+1+x$	=	31	A.:
25	=	$5x+5$	A.:

Task 5 - To easily solve an equation, you just need to follow some rules that guarantee the equality given initially.

RULES to maintain equality:

R1-We can add or subtract the same amount on both "sides" of equality;

R2-We can multiply or divide by the same amount (except zero) on both "sides" of equality.

Example 1:

$x+3$	=	8	
$x+3-3$	=	$8-3$	R1
x	=	5	

Example 2:

10	=	$x-2$	
$10+2$	=	$x-2+2$	R1
12	=	x	

Example 3:

$2x + 5$	=	9	
$2x + 5 - 5$	=	$9 - 5$	R1
$2x$	=	4	
$2x - 2$	=	$4 \div 2$	R2
x	=	2	

Find the value of $\boldsymbol{x}$. You must use rules R1 and R2:

$3x + 4$	=	25
	=	
	=	
	=	
x	=	

Task 6–Of the following expressions, only a few are math equations. Which ones?

	Yes, it´s an equation	No, it's not an equation
$3 + 5 = 8$		
$x + 5 = 8$		
$1 + x + 5 + x = 10$		
$x + 2$		
$1 + 10$		
$25 = 2x + 5$		

Task 7–John thought of a number. He multiplied it by two and then added ten. He multiplied the result by five and got four hundred. What number did João think of?

(write the problem as an equation and solve it to get the solution)

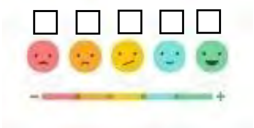
Task 8–Magic

1) Think of a number; 2) Double it; 3) Multiply by five; 4) Divide by the number you initially thought of; 5) Subtract seven; 6) Add twelve. I'll guess the result!

(it is at the end of the sheet, on the right side, in small letters, under the word "Congratulations!")

Try to understand how this trick is done. Discuss with colleagues.

Evaluation:



You did it! Congratulations!