STUDENTS' GROWING UNDERSTANDING IN SOLVING MATHEMATICS PROBLEMS BASED ON GENDER: ELABORATING FOLDING BACK

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Abstract

Students' previous knowledge at a superficial level is reviewed when they solve mathematical problems. This action is imperative to strengthen their knowledge and provide the right information needed to solve the problems. Furthermore, Pirie and Kieren's theory stated that the act of returning to a previous level of understanding is called folding back. Therefore, this descriptive-explorative study examines high school students' levels of knowledge in solving mathematics problems using the folding back method. The sample consists of 33 students classified into male and female groups, each interviewed to determine the results of solving arithmetic problems based on gender. The results showed differences in the level of students' understanding in solving problems. Male students carried out the folding back process at the level of image having, formalizing, and structuring. Their female counterparts conducted it at image-making, property noticing, formalizing, and observing. Subsequently, both participants were able to carry out understanding activities, including explaining information from a mathematical problem, defining the concept, having an overview of a particular topic, identifying similarities and differences, abstracting mathematical concepts, and understanding its ideas in accordance with a given problem. This study suggested that Pirie and Kieren's theory can help teachers detect the features of students' understanding in solving mathematical problems.

Keywords: Characteristics, Folding Back, Gender, Mathematical Problems, Understanding

Abstrak

Pengetahuan siswa sebelumnya pada tingkat dangkal ditinjau ketika mereka memecahkan masalah matematika. Tindakan ini sangat penting untuk memperkuat pengetahuan mereka dan memberikan informasi yang tepat yang dibutuhkan untuk memecahkan masalah. Teori Pirie dan Kieren menyatakan bahwa tindakan kembali ke tingkat pemahaman sebelumnya disebut *folding back*. Oleh karena itu, penelitian deskriptif-eksploratif ini mengkaji tingkat pengetahuan siswa SMA dalam menyelesaikan masalah matematika dengan menggunakan metode folding back. Sampel terdiri dari 33 siswa yang dikelompokkan menjadi kelompok laki-laki dan perempuan, masing-masing diwawancarai untuk mengetahui hasil pemecahan masalah aritmatika berdasarkan jenis kelamin. Hasil penelitian menunjukkan adanya perbedaan tingkat pemahaman siswa dalam menyelesaikan masalah. Siswa laki-laki melakukan proses *folding back* pada level *image having*, *formalising*, dan *structuring*. Rekan-rekan perempuan mereka melakukannya pada level *image-making*, *property noticing*, *formalizing*, dan *observing*. Selanjutnya kedua peserta mampu melakukan kegiatan pemahaman, antara lain menjelaskan informasi dari suatu masalah matematika, mendefinisikan konsep, memiliki gambaran umum tentang topik tertentu, mengidentifikasi persamaan dan perbedaan, mengabstraksikan konsep matematika, dan memahami ide-idenya sesuai dengan yang diberikan. masalah. Penelitian ini menyarankan bahwa teori Pirie dan Kieren dapat membantu guru mendeteksi ciri-ciri pemahaman siswa dalam memecahkan masalah matematika

Kata kunci: Karakteristik*, Folding Back*, Jenis kelamin, Masalah Matematika, Pemahaman

Problem-solving is an essential field in mathematics consisting of numerous requirements. In recent years, great attention has been paid to this field in education. The problem-solving process has always been the primary and fundamental area of study since the early 1980s (Schoenfeld, 2007; Bayat & Tarmizi, 2010). Its significance has been recognized at the international levels (NCTM, 2000). Problem-

solving is the most significant cognitive activity in everyday life (Jonassen, 2000; Verschaffel et al., 2020). In addition, it is a cognitive process that requires a solution to a certain problem (Sweller, 1988; Holyoak, 1990; Jonassen, 2003; Düşek & Ayhan, 2014). Furthermore, this process is closely related to understanding students' concepts to solve the problems at hand and the basis of the associated mathematical concepts (Pape & Tchoshanov, 2001; Stylianides & Stylianides, 2007). Conversely, students' inability to understand mathematical concepts makes it difficult to solve problems (An, Kulm, & Wu, 2004). Therefore, they must have adequate understanding to solve problems, particularly regarding the resolution of those requiring 'understanding.'

Based on classroom observations, some students experience inconsistencies with problem-solving activities. They have difficulty restating and presenting concepts in various forms of mathematical representation. Students' inability to solve mathematical problems indicates they do not have an adequate understanding of the subject. Their inability to solve problems indicates that the implication of problem-solving in mathematical learning is not well educated.

One of the prominent factors that support problem-solving in practice is understanding and NCTM (2000) emphasized its importance as a fundamental aspect of learning mathematics. The process of studying to understand has become an overwhelming priority among educators and psychologists, as well as one of the most critical targets for students in all subjects because it is physically more rewarding and practical (Stylianides & Stylianides, 2007; Skott, 2019). Theoretically, the understanding is defined as a growth process that is complete, dynamic, layered, continuous, and not linear (Pirie & Kieren, 1994; Pirie & Martin, 2000). It is also a passionate and organized process needed to abstract mathematical concepts based on the properties that emerge and build new knowledge from previous experiences.

This study utilized the Pirie-Kieren theory and the associated model, which are well-established and recognized theoretical perspectives on the nature of mathematical understanding to understand growth (Pirie & Kieren, 1994; Martin & Towers, 2016). According to Martin (2008), this theory emphasizes the integration of mathematical understanding in more localized ways, such as intuitive ideas, concrete representations, specific aspects of action, as well as acts of generalization, formalization, and the repetition of less complex understandings. The Pirie-Kieren theory provides an insight into how knowledge is organized and reorganized, as well as the strategies used by learners to reflect upon and build on their understanding accordingly. The growth of understanding in this theory is a dynamic, active process that involves development and action. This involves a constant move among different levels of thought without the involvement of a straight-like system (Pirie & Kieren, 1994). The act of re-examining the existing understandings and ideas of a mathematical concept is called "folding back" in the Pirie-Kieren theory, which is the focus of this study.

In the problem-solving process, folding back the way in which learners work with and build on existing knowledge offers a potentially powerful tool to follow and characterize the process by which mathematical comprehension emerges and develops. However, there was a lack of substantial evidence

showing how and why folding back occurred and its relationship with subsequent mathematical activities (Martin, 2008). Therefore, this study aims to closely explore the concept and nature of folding back, elaborate on the phenomenon, and understand more fully the part played by action in the development of mathematical understanding.

The Pirie-Kieren theory contains 8 potential levels of action to describe an individual's development of understanding and to describe a particular concept. Those levels are called Primitive Knowing, Image Making, Image Having, Property Noticing, Formalizing, Observing, Structuring, and Inventising (Pirie & Kieren, 1994; Pirie & Martin, 2000; Thom & Pirie, 2006; Martin, 2008; Martin & LaCroix, 2008). The eight levels provide a theoretical model or two-dimensional diagram, and each level includes all prior layers to emphasize the integrated nature of mathematical understanding. These levels are further elaborated as follows (1) primitive knowing, is the process of growing students' understanding of mathematical concept, (2) image-making, a level that enables students to have an understanding based on previous knowledge of mental and physical actions, (3) image having, a stage where students use mental images on a topic without taking specific actions that lead to the topic, (4) property noticing or the manipulation of a topic aspects to form related properties, (5) formalizing, enables abstract concept possession based on existing properties, (6) Observing, supports formal activities coordination to use them for the problem at hand, (7) structuring, a phase that facilitating students to relate the relationship between one theorem to another and prove it based on logical arguments, and (8) inventising, a period which signified by a structured and complete understanding, with the ability to create questions and grow into a whole new concept.

Folding back is the technique used by students to review their previous knowledge at a superficial level. This process helps them solve various mathematical problems (Gülkılık, Uğurlu, & Yürük, 2015; Yao & Manouchehri, 2020). Martin, Lacroix, and Fownes (2005) stated that folding back is an integral part of the learner's mathematical understanding, which helps students to develop the right knowledge that fits their task. It folds back the source, form, and outcome to expand students' mathematical understanding (Martin, 2008). According to Slaten (2010), students that fold back understand the development of mathematical concepts appropriately.

The description showed that folding back is essential in the growth of student understanding because it expands, sharpens, and strengthens their knowledge of the material while providing information that can be used to solve mathematical problems. Furthermore, this process allows students to renew their understanding and even replace their knowledge with new versions relevant to the math problem. Sagala (2017) stated that the structure of understanding the concept of derivative functions of student pre-service mathematics is based on gender and in accordance with Pirie & Kieren's (1994) theory. The result indicated that female and male subjects understood the basic knowledge layer in accordance with the folding back theory of Pirie-Kieren. Another essential aspect in the folding back theory is gender differences, which affect practical and theoretical issues in learning and solving math problems.

Over the last couple of decades, numerous studies have been conducted to solve mathematical problem resolution, considered an important factor in gender differences in education (Zhu, 2007). The meta-analyzes from 100 studies indicate that gender differences in mathematical performance of females in high school were minor (Royer et al., 1999; Gallagher et al., 2000). Multiple factors like cognitive ability, processing speed, styles of learning, and socialization contribute to gender differences in mathematical problem-solving. However, the contributions of some factors are still in doubt and are only applicable in some specific areas (Royer et al., 1999). Therefore, based on these findings, the authors can assume that female and male have various mathematical problem-solving patterns built on a multi-step approach. Furthermore, with standardized testing, students can come up with a correct solution by selecting and combining a set of appropriate strategies.

In problem-solving, boys are seen to return to performing more image-making and are confronted with problems while working with sophisticated mathematics (Pirie & Kieren, 1994; Martin, 2008). Therefore, they fold back to a lower level of activity to extend their overall and formal understanding. However, the procedures not offered by the analysis and the framework developed in this study provide is a detailed examination of why and how females fold back and how their actions in the lower level could facilitate their continuous work, thereby increasing understanding. Therefore, the variety of results is an essential reason for conducting gender-related study.

Some of the literature described above indicates the diversity of examining gender-related concepts in solving mathematical problems. However, this study was designed to explore the characteristics of the level of understanding used by senior high school students to solve mathematical problems based on gender. In particular, it focuses on the "folding back theory" originally developed by Pirie & Kieren (1994).

METHOD

This is a descriptive-explorative study designed to explore the characteristics of high school students' understanding of mathematical problems that focused on "folding back." The purposive sampling method was used to obtain data from 33 students at Public Senior High School, Bone, South Sulawesi, Indonesia. The students were grouped based on their gender and told to complete the Mathematical Ability Test question. Furthermore, to explore each group's characteristics, both participants were instructed to solve arithmetic sequences. Afterward, a task-based interview was conducted with a student from each group. They were both selected because (1) they both fulfill mathematical ability test results for the criteria based on the Minimum Exhaustiveness Criteria standard ≥ 75, (2) have good and qualified communication skills, and (3) are ready to participate in the study.

The questions used as a Mathematics Ability were adapted from the UN examination bank for the 2019/2020 high school year, which was modified into a description of 5 items by observing the process of students' understanding level (focus folding back) and recording interviews. This process also consists of 3 open-ended questions, which were used to explore students' understanding in solving

mathematical problems (arithmetic). The instrument was also tested for validity and reliability to validate the questions, and 2 mathematicians and an educational expert carried out interview sheets. The instrument's validity criteria included the feasibility of the test questions, content, language, and appropriate instructions, which were used to reveal the understanding level process of student high school. Furthermore, these results were used to instruct participants on mathematical problems, such as arithmetic sequences. The pattern is based on the addition or subtraction of operations using fixed patterns. Therefore, it is very suitable to be used to explore the growth of students' understanding, as shown in Figure 1.

Given the quadratic equation $x^2 - 7x + (r + 2) = 0$, which has roots p and q, with $r \in R$. If $p,\ q$ and r form an arithmetic sequence, then what is the arithmetic sequence!

Figure 1. Arithmetic Sequences Problems

To analyze data, each participant was thoroughly observed, based on their growing understanding of solving problems. Furthermore, the triangulation process was carried out to verify the data collected through interviews. This process was also used to confirm the findings of students' answers, which were coded as S (Students) and R (Researcher). Conclusively, the results of the folding back at each growth understanding of the 2 students' in solving mathematical problems were also summarized.

RESULTS AND DISCUSSION

Among the 33 students that carried out the Mathematics Ability Test, 7 comprising 2 male and 5 female had a score of ≥ 75. Among the 7 prospective potential participants that achieved these criteria, 1 male and female candidate with relatively similar ability were selected from the mathematical ability and gender. Furthermore, 26 students comprising 10 males and 16 females had a score of <75. The following are the interview results with 2 participants, namely Male Students (MS) and Female Students (FS), to obtain more information on the folding back at each level of student understanding growth.

Folding Back in Solving Mathematical Problems Process of MS
Primitive Knowing Level of MS

MS understood the given mathematical problem and provided detailed information, including known problems and commonly asked questions. Excerpts from interviews by MS on the level of primitive knowing are as follows.

R   : What did you do after given the mathematical problems?
MS   : I read the problem first to determine the information.
R   : What is the information you got from the mathematical problem?
MS   : Given that $x^2 - 7x + (r + 2) = 0$ is a quadratic equation with roots p and q,

where r∈R with p, q, and r are used to form an arithmetic sequence.

| R | : Why don't you write everything down on the answer sheet? |
| MS | : I will do that right away, it actually skipped by memory |

MS read the mathematical problem given in advance to determine the associated information without writing the answers to what is known and asked on the answer sheet. In addition, they understand the given mathematical problem due to its ability to explain the information. Therefore, by understanding, they can identify the information presented. However, without writing the information obtained during the problem-solving phase, the understanding activity carried out by male students is the ability to define concepts verbally based on the previous knowledge (Codes et al., 2013; Martin, 2008). At this level, there is no folding back activity because primitive knowing here does not imply low-level mathematics, rather it is the starting place for the growth of any particular mathematical understanding (Pirie & Kieren, 1994).

Image Making Level of MS

Quotations from the interview by MS regarding the level of image-making are as follows:

R	: What is your opinion on quadratic equations?
MS	: The general form of a quadratic equation is $ax^2 - bx + c = 0$ where $\alpha \neq 0$, where the highest power is 2, and the root is determined by factoring ABC.
R	: What was it like?
MS	: The form of factoring is $(x + x_1)(x + x_2)$ while the ABC formula is $x_{12} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
R	: What do you know about arithmetic sequences?
MS	: The form of the arithmetic sequence is $U_1, U_2, U_3, \dots, U_n$, each adjacent term has the same difference obtained using the following formula, $B = U_n - U_{n-1}$.

MS stated that the general form of quadratic equations and ways to determine the roots is by using the factoring method and the ABC formula. This is in addition to the general form of the arithmetic sequence and the conditions. They specified the arithmetic sequence difference formula and described a concept based on prior knowledge. Some of the procedures used in understanding activities by MS are developing specific ideas, making conceptual images, combining factoring methods, and using the ABC formula to solve arithmetic problems based on possessed knowledge. The understanding level shows that MS can make distinctions in the previous knowledge and use it in new ways. Its growth level in mathematical understanding strengthens Pirie & Kirien's theoretical model, especially at the image-making level (Gulkilik et al., 2020; Martin, 2008).

Image Having Level of MS

Figure 2 is a quotation of interview results carried out by MS on the level of an image having.

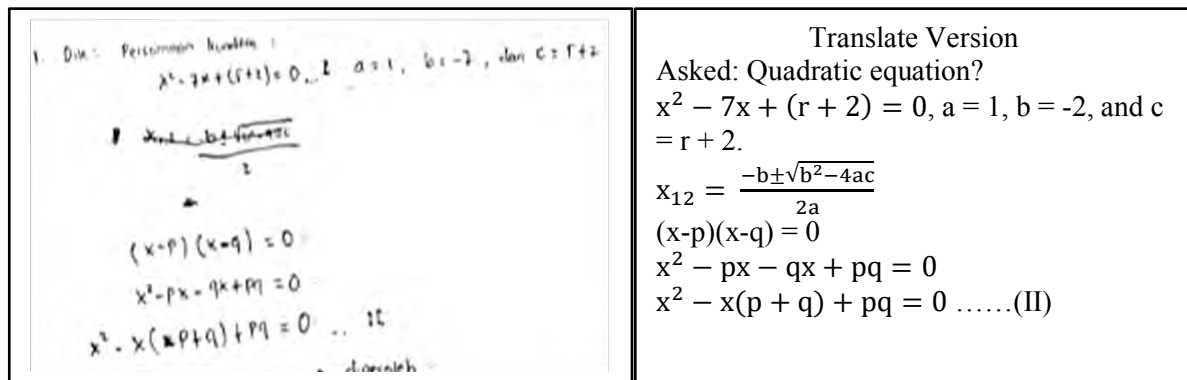|	Translate Version
	Asked: Quadratic equation?
	$x^2 - 7x + (r + 2) = 0$, a = 1, b = -2, and c = r + 2.
	$x_{12} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
	$(x-p)(x-q) = 0$
	$x^2 - px - qx + pq = 0$
	$x^2 - x(p + q) + pq = 0$ ......(II)

Figure 2. Problems Solving Activities Image Having Level by MS

R   : Apart from the ABC formula, is there any other way to determine the roots of a quadratic equation?

MS   : Yes, it can also be determined using the factoring method, $(x + x_1)(x + x_2) = 0$, although I utilized the ABC formula.

R   : Why?

MS   : At first, I calculated $b^2 - 4ac$ and got $(49 - 4r + 8)$, and due to my inability to obtain the root, I crossed the equation.

R   : After crossing out the ABC formula, what did you do and think?

MS   : I re-read the problem to determine what I know, and it turned out that the equation $x^2 - 7x + (r + 2) = 0$ already has roots p and q. I tried to think of another way to determine the roots and remembered solving a similar problem using quadratic equations.

MS explained the initial steps used to solve mathematical problems by determining the roots of the quadratic equation using the ABC formula. Furthermore, they utilized the folding back to the primitive knowing level to procedure and re-read mathematical problems and examine known procedures. The folding back result carried out by MS is to remember that the problem is like the given mathematical equation. Hence, they concluded that when a quadratic equation has roots such as p and q, the new formula can be searched using the sources, which shows that male students already had a mental picture of the topic. Therefore, by understanding, MS solves arithmetic sequence problems by folding back to primitive knowing based on Pirie & Kirien's theory (Martin, 2008). At the image having level, students use a mental image of a case without taking specific actions that lead to the topic. This means they have an idea of the concept through activities conducted at the previous level (Gokalp & Bulut, 2018; Gulkilik et al., 2020).

Property Noticing Level of MS

Quotations from the interview by MS regarding the level of property noticing are as follows.

R   : You earlier stated that one way to determine the roots of the quadratic equation is by using the factoring method. Meanwhile, you used the ABC method. Is it

the same?

MS   : Yes, both methods are similar with different answers. This is because when the quadratic equation is determined using the factoring method, the roots will be obtained. However, supposing the roots are known, then the quadratic equation is obtained as shown in this study.

MS described the difference in using the factoring method based on what is known and stated that when the quadratic equation is determined using the factoring method, the roots are easily obtained. However, when the seeds are known, the quadratic equation is obtained as shown in the above excerpt. These results indicate that they achieved the property noticing level by checking for the similarity and difference of these descriptions and related to specific mathematical sentences. Students are able to recognize the properties of the different concepts that have been learned at the noticing level by having images. The activities carried out by MS are in accordance with the theory of mathematical understanding (Martin, 2008; Yao & Manouchehri, 2020). At this level, students can also notice the distinctions, combinations, or connections between multiple mental images. However, they do not conduct the folding back activities at this level. In mathematical understanding, there are 2 phases of folding back, the first is from image Having to Making and the second from property Noticing to Image Making (Pirie & Kieren, 1994; Thom & Pirie, 2006).

Formalising Level of MS

Following interviews with MS.

R   : Educate me on the steps you used to determine the new quadratic equation!

MS   : I used the factoring method, namely $(x - x_1)(x - x_2) = 0$, because the roots, which comprise p and q, are known, whereby $x_1 = p$ and $x_2 = q$. After that, I replaced x_1 and x_2 with p and q, to get $(x - p)(x - q) = 0$, then substituted the equation to become $x^2 - px - qx + pq = 0$, before simplifying it to obtain $x^2 - x(p + q) + pq = 0$.

R   : I see that you did not immediately continue here (the data referred to is p, q, r forming an arithmetic sequence). Why?

MS   : I had no idea on the procedure to utilize ma'am.

R   : So, what did you do?

MS   : I re-read the problem and realized that $x^2 - 7x + (r + 2) = 0$ has roots, namely p and q. Where p, q, and r form an arithmetic sequence.

MS used the factoring method to substitute the process's assumptions and the multiplication operations to obtain a new quadratic equation. This activity showed that they abstracted a mathematical concept based on related problems (Pirie & Kieren, 1994). However, at this level, male students had a hard time continuing their work, which led to folding back to rediscover the quadratic equation with p, q, and r as its roots in an arithmetic sequence. At this level, MS's folding back action contradicts Pirie & Kieren's theory of mathematical understanding. At the formalising level, the students are able to think consciously on the generalised properties and work with the concept as a formal object, without specific

reference to a particular action or image (Pirie & Kieren, 1994; Pirie & Martin, 2000; Martin, 2008). When formalizing, students abstract the mathematical characteristics or properties of the image and create a concept written into a formal definition or algorithm. Therefore, at this level, students generalize statements on an idea and develop common concepts that are similar to the mathematical definition (Gulkilik et al., 2020).

Observing Level of MS

Following interviews with MS.

R	: Please explain the factoring method you used!
MS	: Here $q^2 - 4q - 5 = 0$, Therefore, I looked for numbers that tend to produce -5 when multiplied and -4 when added, namely $(q - 5)(q + 1) = 0$, q = 5, and q = -1.
R	: After getting a score of q, what did you do?
MS	: I substituted the value of q into equation 3 ($p = 7 - q$). For q = 5, it became $p = 7 - 5$, therefore, the p score equals 2. Whereas for q = -1 it became $p = 7 - (-1)$), hence the p score equals 8.
R	: OK, how did you get the r score?
MS	: I utilized the same procedure in determining p score, by substituting the p and q scores in equation 4, as follows $r = pq - 2$. For p = -2 and q = 5, $r = (2)(5) - 2$ equals r = 10-2, therefore r equals 8.

MS carried out the factoring process by thinking of numbers that correspond to the quadratic equation owned when multiplied and added together. The factoring method used shows that MS has a q score, therefore, it links the mathematical concepts with the problem at hand. MS substituted the q score with 3 to obtain a p score in the 4 equations to get an r score. Therefore, MS reached the level of observation where students are able to coordinate and use formal activities to solve problems (Figure 3).
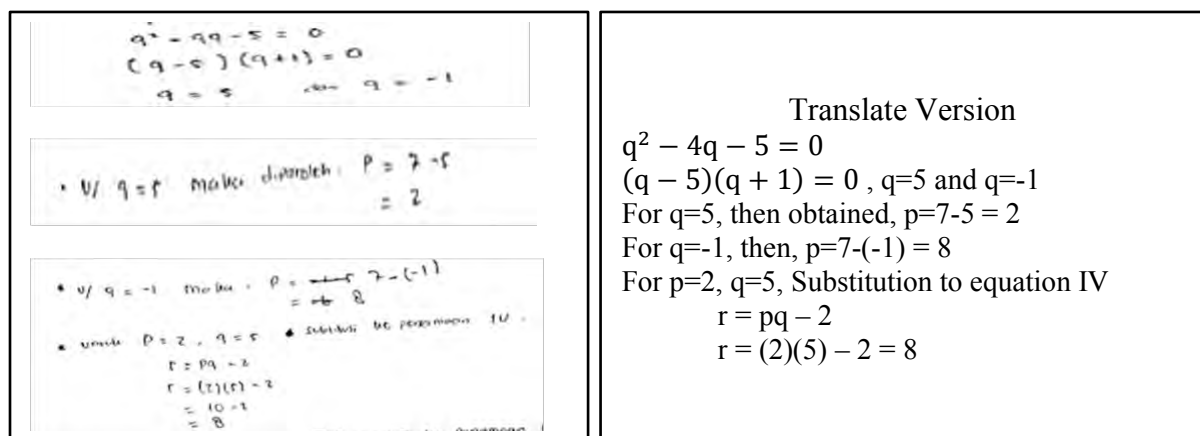

Translate Version
$q^2 - 4q - 5 = 0$
$(q - 5)(q + 1) = 0$, q=5 and q=-1
For q=5, then obtained, p=7-5 = 2
For q=-1, then, p=7-(-1) = 8
For p=2, q=5, Substitution to equation IV
$r = pq - 2$
$r = (2)(5) - 2 = 8$

Figure 3. Activities Observing Level by MS

Furthermore, at the observation level, students are able to make formal statements on mathematical concepts and determine algorithm or theorem patterns (Gulkilik et al., 2020). Moreover, students at this level are able to observe, structure, and organize personal thought processes and

recognize the ramifications of problem-solving. This activity is in accordance with the theory of mathematical understanding by Pirie & Kieren (Pirie & Martin, 2000).

Structuring Level of MS

The following interview was conducted with MS.

R   : Okay, why did you cross this out? ((r = $(-6)(-2) - 2 = 10$)
MS   : At the beginning, I got the second p-score which was equal to -6, then I substituted it for the equation $r = pq - 2$, to obtain an r-score of 10.
R   : So, how did you find r?
MS   : After, determining the various values of p, q, and r which equals 2, 5, and 8 in the first rows and -6, -1, and 10 in the second, I used the different arithmetic sequence formula to determine the difference in the first and second sequence. These are -1, -6, and -7, while the difference between the third and the second term is 10 - (- 1) = 11.
R   : After you got the difference, what did you do?
MS   : I used the factoring equation to determine the q and p scores. However, there was an error in the second p score, I wrote $p = -1 - 5$, which culminated in 6? Therefore, the difference between the second term and the first is not similar to the third and second.

MS made a mistake in creating a substitution; hence a write-off was found on the worksheet with folding back carried out on the observed level using the factoring method. This process was used to determine the error after checking the arithmetic sequence obtained using the difference formula. The activity further showed that male students make logical formal observations and verify previously developed ideas, presented in Figure 4.

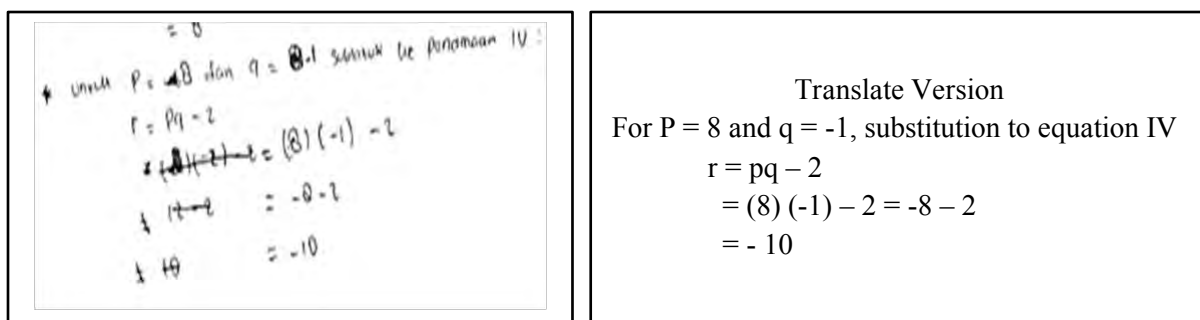| Translate Version |
|:---:|
| For P = 8 and q = -1, substitution to equation IV |
| r = pq – 2 |
| = (8) (-1) – 2 = -8 – 2 |
| = - 10 |

Figure 4. Activities Structuring Level by MS

They are also able to link the relationship between one theorem and another at the structuring level and prove it based on rational arguments (Thom & Pirie, 2006; Martin & Towers, 2016). MS checked the factoring and p-score obtained previously and determined an error in substituting the second p. This error causes the difference between the 2 adjacent terms in the second arithmetic sequence, thereby indicating that MS Races its thought process into an axiomatic structure (Gülkılık, Uğurlu, & Yürük, 2015; Gulkilik et al., 2020).

Based on the interview results, an understanding map was created, as shown in Figure 5, which confirms that MS conducted the folding back process 3 times at the level of mathematical understanding growth.
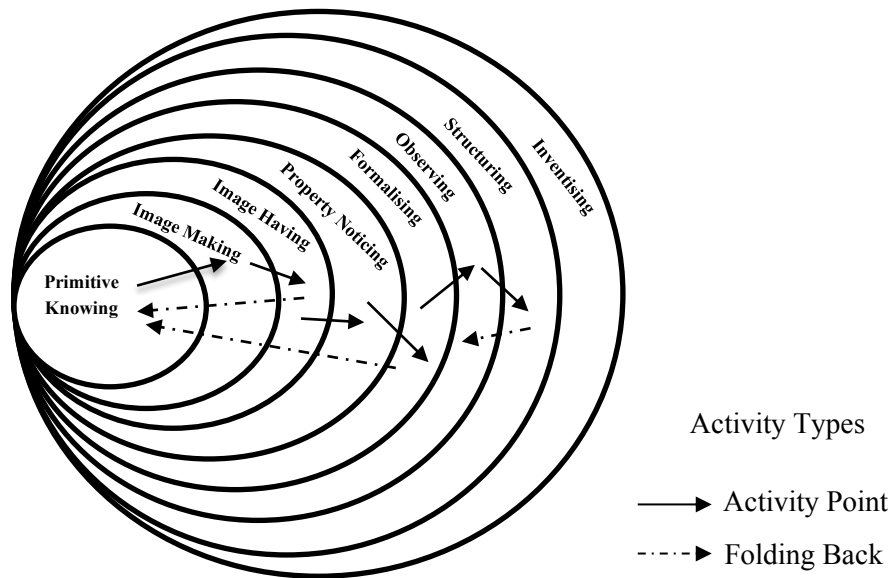

Figure 5. Folding Back in Solving Mathematical Problem of MS

Figure 5 showed the folding back activity of MS at the level of mathematical understanding. This activity suggests that students tend to return to a lower level of understanding when faced with problems that cannot be solved immediately. It also depicts 3 folding back phases, with the first from image Having to Primitive Knowing, followed by Formalising to Primitive Knowing, and the last is from Structuring to Observing. The outcome of folding back is that male students have the ability to expand their current inadequate and incomplete understanding by reflecting on and rearranging their former concepts. They can also achieve this by generating and creating new images, supposing the existing constructs are not sufficient to solve the problem.

Folding back in Solving Mathematical Problems Process of FS
Primitive Knowing Level of FS

The FS explained all the information on the mathematical problem, such as what is known and asked. The excerpt from interviews by the FS on the level of primitive knowing is as follows.

R   : Did you ever get a question like this previously?
FS   : No, I was never questioned on the model.
R   : Now look at the answer sheet! Why didn't you write down the information
             obtained from this question?
FS   : Sorry, ma'am. I was too excited.
R   : Alright, so what information did you get from the questions given?
FS   : Given the quadratic equation $x^2 - 7x + (r + 2) = 0$, which has the roots p and

q, where p, q, and r form an arithmetic sequence.

The FS failed to write down all the information obtained from the given mathematical problem despite having an adequate understanding of what was asked. This understanding activity shows that female students describe the initial thinking process and new concepts. At the primitive level, knowledge on concepts that students are assumed to have prior understanding was explored (Gülkılık, Uğurlu, & Yürük, 2015). At this level, students need to construct new ideas and information on the learning situation for further understanding (Yao, 2020a, 2020b; Yao & Manouchehri, 2020).

Image Making Level of FS

Quotations from the interview by FS regarding the image-making level are as follows.

R	: What material is related to this math problem?
FS	: Arithmetic sequences and quadratic equations.
R	: What do you know about quadratic equations and arithmetic sequences?
FS	: The general form of a quadratic equation is $ax^2 - bx + c = 0$ where $\alpha \neq 0$ and the highest power is 2. Meanwhile, the general form of an arithmetic sequence is $U_1, U_2, U_3, \ldots, U_n$ is significantly different from arithmetic sequence.
R	: Ok, tell me what you did!
FS	: Usually I am meant to factor $x^2 - 7x + (r + 2) = 0$, and determined the roots. However, I was not able to factor it due to the presence of $(r + 2)$. Therefore, I re-read the problem again, and I got $x^2 - 7x + (r + 2) = 0$, it turns out that it already has roots, namely p and q, which motivated me to use quadratic equations.

The FS explained a concept based on prior knowledge by stating that arithmetic sequences and quadratic equations are closely related to the problem presented (Pirie & Kieren, 1994). FS also explained the quadratic equation problem in the form of $x^2 - 7x + (r + 2) = 0$ and found it difficult to determine its roots due to the constant $(r + 2)$. Hence, the folding back to the primitive knowing level was used to re-read the problem by remembering the quadratic equations (Martin, 2008). Nonetheless, FS reached the image-making level where students try to picture the concept using prior knowledge with mental and physical actions (Gülkılık, Uğurlu, & Yürük, 2015).

Image Having Level of FS

Quotations of interview results by MS regarding the level image having are as follows.

R	: Apart from factoring, is there any other way to determine the new quadratic equation?
FS	: That's all I remember Ma'am.
R	: Okay, please explain what you did here?
FS	: I determined the roots, namely p and q, using the factoring method by substituting $x_1 = p$ and $x_2 = q$, in $(x - x_1)(x - x_2) = 0$, and wrote it to be $(x - p)(x - q) = 0$. Furthermore, I multiplied $(x - p)$ by $(x - q)$, to obtain

$x^2 - qx - px + pq = 0$, and simplified it to $x^2 - x(p + q) + pq = 0$.

R   : Why did you convert $x^2 - qx - px + pq = 0$ to $x^2 - x(p + q) + pq = 0$?

FS   : I carried out the conversion process because I was trying to obtain
a new quadratic equation, therefore, I changed the old form by collecting
the variable x.

From the interview descriptions, FS provided an overview of a concept used to solve mathematical problems by explaining the factoring step procedure to determine quadratic equations. FS explained that the process design generates new quadratic equations, which means the FS understood the image-making process (Figure 6). This level indicates students' first abstraction to adjust and manipulate images without working on examples (Martin & Towers, 2014).
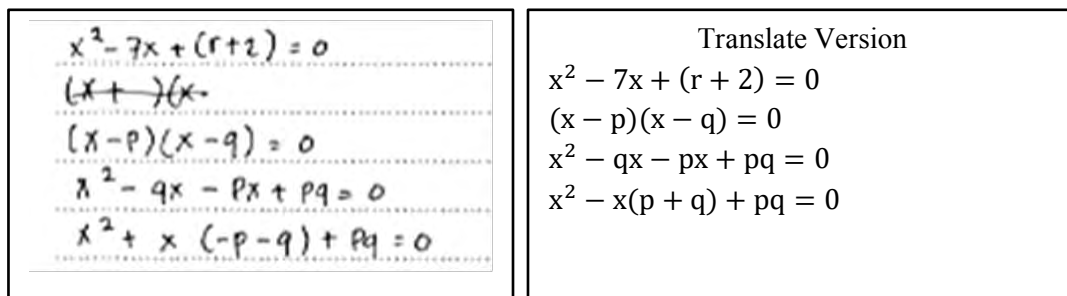

	Translate Version
$x^2 - 7x + (r+2) = 0$	$x^2 - 7x + (r + 2) = 0$
	$(x - p)(x - q) = 0$
	$x^2 - qx - px + pq = 0$
	$x^2 - x(p + q) + pq = 0$

Figure 6. Activities Image Having Level by FS

Property Noticing Level of FS

Quotations of the interview results by FS regarding the property noticing level are presented as follows.

R   : What is the difference between the factoring methods you want to apply here
$(x^2 - 7x + (r + 2) = 0)$ and $(x - p)(x - q) = 0$?

FS   : Both methods are similar, with differing results. For instance, in the question is
$x^2 - 7x + (r + 2) = 0$, I was able to obtain the roots using the factoring
method. Meanwhile, in $(x - p)(x - q) = 0$, I used the factoring method to
determine a new quadratic equation with known roots.

R   : I notice you were silent for a few minutes before continuing. What was the
problem?

FS   : I obtained a new quadratic equation while solving the problem, which left me
confused on how to operate the 2 equations.

R   : So, what did you do?

FS   : I related the 2 quadratic equations to the properties of the roots by substituting
x_1 and x_2 in $ax^2 - bx + c = 0$, hence the formula for the number of roots is
$x_1 + x_2 = \frac{-b}{a}$, and the product is $x_1 x_2 = \frac{c}{a}$.

Based on the interview excerpt, FS explained the difference in the use of the factoring method based on $x^2 - 7x + (r + 2) = 0$ and $(x - p)(x - q) = 0$. Conversely, FS has a slight difficulty with 2 equations of the previous operation, which led to the use of the folding back to connect the roots of the two new

quadratic equations obtained (Gokalp & Bulut, 2018). The understanding activities carried out show that the FS understands the existence of a relationship between the description of a topic and suggests the right strategy for its verification (Martin & Towers, 2014; Yao & Manouchehri, 2020).

Formalising Level of FS

Quotations of interview results by the FS on the level of formalising are as follows.

R   : Explain what you did with the 2 new equations created?
FS   : In the quadratic equation $x^2 - 7x + (r + 2) = 0$, I substituted a, b, c, x_1 and
          x_2 with 1, -7, r + 2, p and q. Here I use the formula for the number of roots
          to determine the equation $p = 7 - q$.
R   : Ok. How about $pq = r + 2$?
FS   : For $pq = r + 2$, I used the formula for the product of roots and further
          substituted $x_1 = p$, $x_2 = q$, $c = r + 2$, and a = 1, therefore $r + 2 = 7q - q^2$.
R   : Okay, why did you cross out $r + 2 = (7 - q)q$? I noticed that you stopped here
          for a while, why did you and what do you think?
FS   : I thought of simplifying the equation, but it turned out to be back to the previous
          form, thereby leaving me confused on the process to utilize in continuing
          the operation. Therefore, I re-read the problem and thought of the relationship
          between the arithmetic sequence and p, q, and r.

The FS abstracts a mathematical concept based on a problem by describing the steps used to determine a new quadratic equation. The efforts used to solve the problem were described by making an example, substituting, and multiplying operations. After obtaining a new quadratic equation, they had difficulty continuing its work. Therefore, this leads to the folding back of the primitive knowing level by repeatedly re-reading the questions and thinking about the relationship between arithmetic sequences and quadratic equations whose roots are p, q, and r, presented in Figure 7.

Figure 7. Problem Solving Activities Formalising Level by FS

The folding back activity by FS contrasts with Pirie & Kieren's theory of mathematical understanding (Pirie & Kieren, 1994; Martin, 2008). The activities carried out show that the FS abstracts

a mathematical concept based on the properties that emerge with the ability to formalize prior understanding (Martin, Lacroix, & Fownes, 2005; Güner & Uygun, 2019).

Observing Level of FS

Following interviews with FS.

R	: What do you know about arithmetic sequences?
FS	: The general form of arithmetic sequences is $U_1, U_2, U_3, …, U_n$ with a difference usually referred to as arithmetic sequence difference.
R	: Okay, please explain the steps you used here!
FS	: I determined the relationships p, q, and r that form an arithmetic sequence, by substituting $U_1 = p, U_2 = q$, and $U_3 = r$. Next, for $B = U_2 - U_1$, I substituted q to U_2 and p to U_1, to obtain $B = q - p$. Similarly, with $B = U_3 - U_2$, I substituted r to U_3 and q to U_2, therefore, it became $B = r - q$.
R	: Why didn't you continue with the work in this section?
FS	: I wondered about the right steps to use and thought of relating it to the arithmetic sequence difference formula. Furthermore, I remembered that the difference between the two adjacent terms was the same.

FS stated that the general form of an arithmetic sequence with an edge can be used to determine the roots of the quadratic equations p, q, and r. MS substituted the q score for the equality of 3 to get a p score, while FS substitutes p and q into the equation to obtain B. This activity shows that MS linked the mathematical concept understood with the problem using new knowledge structures (Gülkılık, Uğurlu, & Yürük, 2015). However, FS has difficulty continuing its work at this level of understanding; therefore, folding back to the image-making level was used to determine the previous knowledge. The folding back is that FS gets the difference between the 2 adjacent terms, which is similar to ($U_2 - U_1 = U_3 - U_2$). Based on this, it shows that the FS has reached the observing level.

Structuring Level of FS

The following interview was carried out with FS.

R	: Is the factoring method the same as the ABC formula?
FS	: The result is the same with varying steps and a limited factoring method.
R	: Fine, please explain the method!
FS	: From the quadratic equation $q^2 - 4q - 5 = 0$, I supposed that $a = 1, b = -4$, and $c = -5$, then I substituted the values for the formula ABC. I carried out further operation in order to obtain the value of q, namely $q_1 = 5$ and $q_2 = -1$. Next, I substituted the values for q_1 and q_2, into the first equation $p = 7 - q$, to obtain $p_1 = 7 - 5 = 2$. Similarly, with p_2, I wrote $p_2 = 7 - (-1) = 8$.
R	: Ok. what about r?
FS	: I utilized a similar method for an equation with variables r and p or q. I used Equation 5, which is $r = -7 + 3q$, and substituted the value for q_1, for $r_1 = -7 + 3(5) = 8$. Similarly, with r_2, I substituted the value for q_2, which led to $r_2 = -7 + 3(-1) = -10$. Therefore, this led to the formation of 2 arithmetic sequences, namely 2, 5, 8, and 8, -1, -10.

FS logically related one concept to another based on argument and stated that the results obtained by the factoring method with the ABC formula are the same. FS performed a substitution at each step of the ABC formula taken to get the roots of the quadratic equation in the form of arithmetic sequences with the same selection of 2, 5, 8, and 8, -1, -10 (Figure 8).
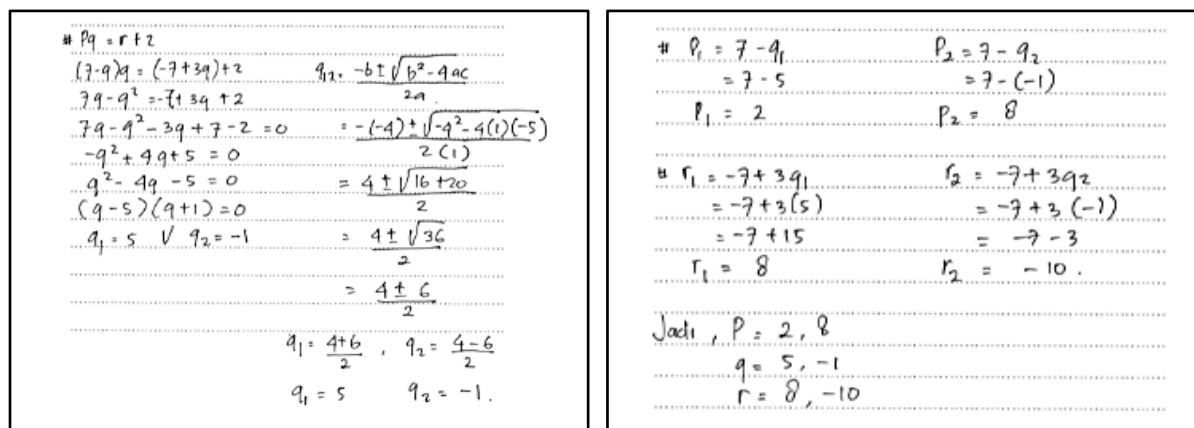

Figure 8. Activities Structuring Level by FS

The understanding activities carried out show that FS explained the formal observations logically and considered the observations as theories with relationships between theorems. At this level, students are aware of the interrelationship between a collection of theorems and demand that statements be justified or verified through logical or meta-mathematical argument (Pirie & Kieren, 1994; Codes et al., 2013; Yao, 2020b).

Based on the interview results, FS folding back is 4 times at the level of mathematical understanding growth, as shown in Figure 9.
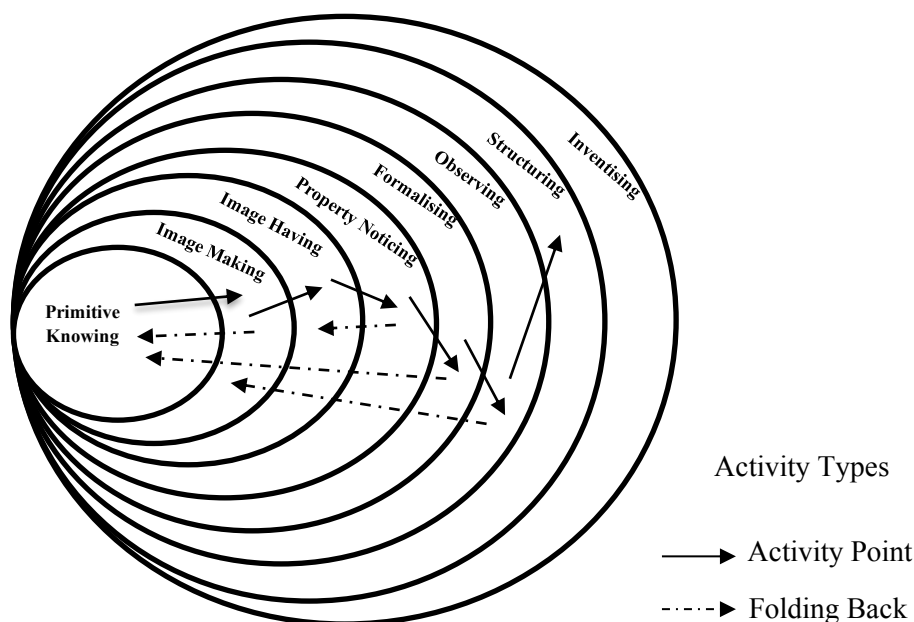

Figure 9. Folding Back in Solving Mathematical Problem of FS

Figure 9 demonstrated FS's folding back activity at the level of mathematical understanding, which suggests that students use lower-level understanding when confronted with any problem. This is shown in Figure 9, comprising of 4 folding back phases, namely Image-Making to Primitive Knowing, Property Noticing to Image Having, Formalising to Primitive Knowing, and Observing to Image Making. Folding back results fall into 3 categories, including returning to an outer level with/without an external prompt and effective folding back.

The first form is ensuring that female students are aware of the limitations of their existing understandings at the outer level and decide to shift to work at a lower level. The less sophisticated lower-level understanding activities are informed by what is already understood at an outer level. The second form is folding back to collect, which consists of female students' involvement in retrieving previous knowledge for a specific purpose and reviewing it considering the needs of current mathematical actions. Moving out of topic and working there is the third form of folding back, which enables them to develop the concept from a different mathematical area. The discussion has focused on the definitions of the levels and their embedded nature, which are necessary and structurally essential to the theory's mathematical understanding. However, a more vital issue is folding back, and according to Martin (2008), it is an important stage in the dynamical growth of mathematical understanding.

Data analysis in this study documents 7 levels of understanding growth based on mathematical problems by Pirie & Kieren theory, namely primitive knowing, image-making, image having, property noticing, formalizing, observing, and structuring, without describing the level of inventising understanding. An explanation of the characteristics of each type based on gender is shown in Table 1.

Table 1. Characteristics of Students (Male and Female) in Growth Understanding

Types of Growth Understanding based on Mathematics Problems	Description of Characteristic based on gender	
	Male Student (MS)	**Female Student (FS)**
Primitive Knowing	• Describing information obtained from mathematical problems. • Stating concepts related to mathematical problems.	• Describing information obtained from mathematical problems. • Stating concepts related to mathematical problems.
Image-Making	• Explaining the description of a concept based on previous knowledge develops a specific picture based on prior knowledge.	• Explaining the description of a concept based on previous knowledge develops a specific picture based on prior knowledge • Carrying out folding back to primitive knowing level

Types of Growth Understanding based on Mathematics Problems	Description of Characteristic based on gender	
	Male Student (MS)	**Female Student (FS)**
Image Having	• Having an overview of a concept used in solving mathematical problems. • Conducting folding back to primitive knowing level	• Having an overview of a concept used in solving mathematical problems.
Property Noticing	• Explaining the similarities/differences in the various descriptions of a topic.	• Explaining the similarities/differences in the various descriptions of a topic. • Conducting folding back to image having a level.
Formalising	• Making abstraction of a mathematical concept based on a mathematical problem • Carrying out folding back to Primitive knowing level	• Making abstraction of a mathematical concept based on a mathematical problem • Carrying out folding back to primitive knowing level
Observing	• Linking mathematical concepts understood with the problem at hand.	• Linking mathematical concepts understood with the problem at hand. • Conducting folding back to image-making level
Structuring	• Linking one concept to another based on logical arguments • Carrying out folding back to observing the level	• Linking one concept to another based on logical arguments

Table 1 shows the differences in the levels of understanding of male and female students in solving mathematical problems properly. This is in addition to the differences associated with the level of image-making, MS and FS understanding, and the ability to develop specific knowledge. However, FS folding back to the primitive knowing level due to the difficulty factor. Furthermore, at the image level, MS and FS have different levels of understanding, with an overview of the concepts used in solving mathematical problems. Similarly, there are differences in the level of understanding between male and female students in solving problems at the property, noticing, formalising, observing, and structuring levels. Male students fold back 3 times at the level of understanding growth, while their female counterparts carried out the process 4 times.

These results indicate differences in students' level of understanding in solving math problems based on gender, as shown in Figures 5 and 9. Male students engage in fewer folding back activities and understand problems than their female counterparts. This result in boys being more likely to correctly answer difficult, unfamiliar, and life-related math problems than girls, as supported by several

researchers (Hornburg, Rieber, & McNeil, 2017; Innabi & Dodeen, 2018; Reinhold et al., 2020). The ability of gender differences to affect the way students solve problems associated with learning was also acknowledged by Cvencek, Meltzoff, and Greenwald (2011).

Folding back is the primary key in the growth of Pirie-Kieren understanding of mathematics and essential activity in the building, strengthening, and expanding students' knowledge of mathematics in learning. Students' understanding of mathematics takes place with the help of folding back between levels (Pirie & Martin, 2000; Martin, 2008). Therefore, based on the results above, students do not always go through the problem-solving stages at every step. However, the dynamic growth of mathematical understanding varies between students in problem-solving, according to preliminary studies (Pirie & Kieren, 1994; Pirie & Martin, 2000; Martin, Lacroix, & Fownes, 2005; Martin, 2008; Martin & LaCroix, 2008; Martin & Towers, 2014; Martin & Towers, 2016). Furthermore, this study provides more insight provided by the maps than the original manufactured by Pirie & Kieren (1994).

The Pirie–Kieren theory was used in this study to unpack activities associated with students' mathematical understanding, known as folding back actions. Meanwhile, the use of this theory is not the focus of this study. However, it provided a framework for investigating the role of participants' folding back in the process of mathematical understanding. The prominence of folding back in mathematical problem-solving lends support to the notion that folding back is critical in the process of mathematical understanding, which is in accordance with the Pirie–Kieren theory. This discovery contributes significantly to the solution of mathematical problems by elaborating folding back and proposing a broader framework for its categorization based on its source, form, and outcome. The framework enables the identification of various sources and forms, as well as describe their impact on students' mathematical understanding practices, particularly mathematical problem-solving.

CONCLUSION

This study explored the characteristics of students' level of understanding in solving arithmetic problems, with a focus on folding back based on gender. The results showed differences at the level of image-making, image having, property noticing, observing, and structuring. The understanding activities performed by male students are image having level, students folding back to the level of primitive knowing. Students had a mental picture of the topic, and in formalising level, they utilize the folding back to the level of primitive knowing process. Furthermore, their abstract to mathematical characteristics or properties of the image, create a concept and then write it into a formal definition or algorithm. At the structuring level, folding back is accomplished to the level of observation. Students have the ability to link a theorem to another and demonstrate it based on rational argument. Meanwhile, the level of understanding of female students includes image-making level, students folding back to primitive knowing level. They could imagine the concept with mental and physical action utilizing preliminary information. At property noticing level, students folding back to image level and tend to link the description of a topic to others. In formalising level, they are folding back to primitive knowing

level is used to determine abstract mathematical concepts based on their properties. In observing level, students conduct folding back to the level of image-making to combine new knowledge structures with mathematical concepts.

Subsequently, the results showed that the 2 participants achieved understanding activities by explaining the information obtained from mathematical problems, describing the concept, making reports on a particular topic, identifying similarities and differences in various definitions of a topic, making mathematical abstraction concepts, and linking mathematical ideas to a problem. This study showed that students have not been able to acquire the inventising level. Therefore, further investigation needs to be carried out with qualitative studies at different grade levels, using various topics. The improvement of the understanding map needs to be tested using other issues. Current studies offer new techniques for describing growing students' understanding. The insights observed in this study suggested some implications for students' further development on a broader level of understanding. Some practical considerations were concluded from the results and impact when designing activities to solve arithmetic problems in the preparation program for mathematics teachers. However, this study was limited using data observation, which led to a small-scale investigation, including 2 different gender students, from 33 participants in one public school.

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REFERENCES

An, S., Kulm, G., & Wu, Z. (2004). The Pedagogical Content Knowledge of Middle School, Mathematics Teachers in China and the U.S. *Journal of Mathematics Teacher Education*, *7*(2), 145–172. https://doi.org/10.1023/B:JMTE.0000021943.35739.1c

Bayat, S., & Tarmizi, R. A. (2010). Assessing Cognitive and Metacognitive Strategies during Algebra Problem Solving Among University Students. *Procedia - Social and Behavioral Sciences*, *8*(5), 403–410. https://doi.org/10.1016/j.sbspro.2010.12.056

Codes, M., González Astudillo, M. T., Delgado Martín, M. L., & Monterrubio Pérez, M. C. (2013). Growth in the understanding of infinite numerical series: a glance through the Pirie and Kieren theory. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 652–662. https://doi.org/10.1080/0020739X.2013.781690

Cvencek, D., Meltzoff, A. N., & Greenwald, A. G. (2011). Math-Gender Stereotypes in Elementary School Children. *Child Development*, *82*(3), 766–779. https://doi.org/10.1111/j.1467-8624.2010.01529.x

Düşek, G., & Ayhan, A. B. (2014). A Study on Problem Solving Skills of the Children from Broken Family and Full Parents Family Attending Regional Primary Boarding School. *Procedia - Social and Behavioral Sciences*, *152*, 137–142. https://doi.org/10.1016/j.sbspro.2014.09.170

Gallagher, A. M., De Lisi, R., Holst, P. C., McGillicuddy-De Lisi, A. V., Morely, M., & Cahalan, C.

(2000). Gender Differences in Advanced Mathematical Problem Solving. *Journal of Experimental Child Psychology*, *75*(3), 165–190. https://doi.org/10.1006/jecp.1999.2532

Gokalp, N. D., & Bulut, S. (2018). A new form of understanding maps: Multiple representations with Pirie and Kieren model of understanding. *International Journal of Innovation in Science and Mathematics Education*, *26*(6), 1–21. https://openjournals.library.sydney.edu.au/index.php/CAL/article/view/12454

Gulkilik, H., Moyer-Packenham, P. S., Ugurlu, H. H., & Yuruk, N. (2020). Characterizing the growth of one student's mathematical understanding in a multi-representational learning environment. *The Journal of Mathematical Behavior*, *58*, 100756. https://doi.org/10.1016/j.jmathb.2020.100756

Gülkılık, H., Uğurlu, H., & Yürük, N. (2015). Examining Students' Mathematical Understanding of Geometric Transformations Using the Pirie-Kieren Model. *Educational Sciences: Theory & Practice*, *15*(6), 1531–1548. https://doi.org/10.12738/estp.2015.6.0056

Güner, P., & Uygun, T. (2019). Examining Students Mathematical Understanding of Patterns by Pirie-Kieren Model. *Hacettepe University Journal of Education*, *35*(3), 1–23. https://doi.org/10.16986/HUJE.2019056035

Holyoak, K. J. (1990). Problem solving. In *Thinking: An invitation to cognitive science* (Vol. 3, pp. 117–146). http://reasoninglab.psych.ucla.edu/wp-content/uploads/2010/09/Problem-Solving.pdf

Hornburg, C. B., Rieber, M. L., & McNeil, N. M. (2017). An integrative data analysis of gender differences in children's understanding of mathematical equivalence. *Journal of Experimental Child Psychology*, *163*, 140–150. https://doi.org/10.1016/j.jecp.2017.06.002

Innabi, H., & Dodeen, H. (2018). Gender differences in mathematics achievement in Jordan: A differential item functioning analysis of the 2015 TIMSS. *School Science and Mathematics*, *118*(3–4), 127–137. https://doi.org/10.1111/ssm.12269

Jonassen, D. (2003). Using Cognitive Tools to Represent Problems. *Journal of Research on Technology in Education*, *35*(3), 362–381. https://doi.org/10.1080/15391523.2003.10782391

Jonassen, D. H. (2000). Toward a design theory of problem solving. *Educational Technology Research and Development*, *48*(4), 63–85. https://doi.org/10.1007/BF02300500

Martin, L. C. (2008). Folding back and the dynamical growth of mathematical understanding: Elaborating the Pirie–Kieren Theory. *The Journal of Mathematical Behavior*, *27*(1), 64–85. https://doi.org/10.1016/j.jmathb.2008.04.001

Martin, L. C., & LaCroix, L. N. (2008). Images and the Growth of Understanding of Mathematics-for-Working. *Canadian Journal of Science, Mathematics and Technology Education*, *8*(2), 121–139. https://doi.org/10.1080/14926150802169263

Martin, L. C., & Towers, J. (2014). Growing mathematical understanding through Collective Image Making, Collective Image Having, and Collective Property Noticing. *Educational Studies in Mathematics*, *88*(1), 3–18. https://doi.org/10.1007/s10649-014-9552-4

Martin, L., Lacroix, L., & Fownes, L. (2005). Folding Back and the Growth of Mathematical Understanding in Workplace Training. *Adults Learning Mathematics*, *1*(1), 19–35. https://files.eric.ed.gov/fulltext/EJ1055423.pdf

Martin, L., & Towers, J. (2016). Folding back and growing mathematical understanding: a longitudinal study of learning. *International Journal for Lesson and Learning Studies*, *5*(4), 281–294. https://doi.org/10.1108/IJLLS-04-2016-0010

NCTM. (2000). *Priciples and Standards for School Mathematics*. Reston, VA: NCTM.

Pape, S. J., & Tchoshanov, M. A. (2001). The Role of Representation(s) in Developing Mathematical Understanding. *Theory Into Practice*, *40*(2), 118–127. https://doi.org/10.1207/s15430421tip4002_6

Pirie, S., & Kieren, T. (1994). Growth in Mathematical Understanding: How Can We Characterise It and How Can We Represent It? In *Learning Mathematics* (pp. 61–86). Springer Netherlands. https://doi.org/10.1007/978-94-017-2057-1_3

Pirie, S., & Martin, L. (2000). The role of collecting in the growth of mathematical understanding. *Mathematics Education Research Journal*, *12*(2), 127–146. https://doi.org/10.1007/BF03217080

Reinhold, F., Hofer, S., Berkowitz, M., Strohmaier, A., Scheuerer, S., Loch, F., Vogel-Heuser, B., & Reiss, K. (2020). The role of spatial, verbal, numerical, and general reasoning abilities in complex word problem solving for young female and male adults. *Mathematics Education Research Journal*, *32*(2), 189–211. https://doi.org/10.1007/s13394-020-00331-0

Royer, J. M., Tronsky, L. N., Chan, Y., Jackson, S. J., & Marchant, H. (1999). Math-Fact Retrieval as the Cognitive Mechanism Underlying Gender Differences in Math Test Performance. *Contemporary Educational Psychology*, *24*(3), 181–266. https://doi.org/10.1006/ceps.1999.1004

Sagala, V. (2017). Struktur Lapisan Pemahaman Konsep Turunan Fungsi Mahasiswa Calon Guru Matematika. *Jurnal Didaktik Matematika*, *4*(2), 125–135. https://doi.org/10.24815/jdm.v4i2.8384

Schoenfeld, A. H. (2007). Problem solving in the United States, 1970–2008: research and theory, practice and politics. *ZDM*, *39*(5–6), 537–551. https://doi.org/10.1007/s11858-007-0038-z

Skott, J. (2019). Understanding mathematics teaching and learning "in their full complexity." *Journal of Mathematics Teacher Education*, *22*(5), 427–431. https://doi.org/10.1007/s10857-019-09446-z

Slaten, M. (2010). Effective Folding back via Student Research of the History of Mathematics. *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education*, 1–10. http://sigmaa.maa.org/rume/crume2010/Archive/Slaten.pdf

Stylianides, A. J., & Stylianides, G. J. (2007). Learning Mathematics with Understanding: A Critical Consideration of the Learning Principle in the Principles and Standards for School Mathematics. *The Mathematics Enthusiast*, *4*(1), 103–114. https://scholarworks.umt.edu/tme/vol4/iss1/8

Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science*, *12*(2), 257–285. https://doi.org/10.1016/0364-0213(88)90023-7

Thom, J. S., & Pirie, S. E. B. (2006). Looking at the complexity of two young children's understanding of number. *The Journal of Mathematical Behavior*, *25*(3), 185–195. https://doi.org/10.1016/j.jmathb.2006.09.004

Verschaffel, L., Schukajlow, S., Star, J., & Van Dooren, W. (2020). Word problems in mathematics education: a survey. *ZDM*, *52*(1), 1–16. https://doi.org/10.1007/s11858-020-01130-4

Yao, X. (2020a). Unpacking learner's growth in geometric understanding when solving problems in a dynamic geometry environment: Coordinating two frames. *The Journal of Mathematical Behavior*, *60*(April), 100803. https://doi.org/10.1016/j.jmathb.2020.100803

Yao, X. (2020b). Characterizing Learners' Growth of Geometric Understanding in Dynamic Geometry Environments: a Perspective of the Pirie–Kieren Theory. *Digital Experiences in Mathematics*

Education, *6*(3), 293–319. https://doi.org/10.1007/s40751-020-00069-1

Yao, X., & Manouchehri, A. (2020). Folding back in students' construction of mathematical generalizations within a dynamic geometry environment. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-020-00343-w

Zhu, Z. (2007). Gender differences in mathematical problem solving patterns: A review of literature. *International Education Journal*, *8*(2), 187–203. https://files.eric.ed.gov/fulltext/EJ834219.pdf