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How a Measuring Perspective Influences Pre-service Teachers' Reasoning about Fractions with Discrete and Continuous Models

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Abstract

How teachers interpret and express fractions critically influences their teaching and their students' fraction knowledge. Internationally, the mathematics education community has been studying ways to enhance pre-service elementary teachers' rational number knowledge, particularly fractions. To address the challenge of augmenting pre-service teachers' fraction knowledge warrants theoretical and empirical revisions to standardized practices for teaching fractions. This study investigates how reexamining fractions from a distinctive measuring perspective influences pre-service teachers' reasoning about fractions. For four 75-minute sessions, 46 pre-service teachers enrolled in a teacher preparation program at a university in the United States revisited fractions from a measuring perspective. They engaged in tasks that focused on comparing continuous quantities and identifying relative magnitudes. The data for this study comprise their pre- and post-tests that assessed how they identify and represent fractions with discrete and continuous models. For each model, we analyzed participants' reasoning by attending to their written strategies. Findings revealed three main strategies: partition, construction, and symbolic manipulation. In general, participants expressed more strategies on the post-test for all fraction models. Partitioning was the most frequent strategy on the pre- and post-tests. However, the frequencies of strategies changed after the intervention. For example, with all models, there was an increase in partitioning strategy and a decrease in symbolic manipulation strategy. The results highlight affordances of a measuring perspective to support participants to shift from procedural strategies such as symbolic manipulation to more conceptual strategies to identify and represent fractions.

Keywords:

Fraction Models, Pre-service Teacher Knowledge, Mathematical Reasoning, Measuring Perspective

Introduction

For mathematics educators, how to support students' meaningful learning of fractions has been a significant challenge. Starting in students' initial schooling years,

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teachers struggle to help them conceptualize and operate on fractions (Zhou et al., 2006). Those who experience difficulties ordering and operating on fractions (Maher & Yankelewitz, 2017), underachieve mathematically, and are unsuccessful in learning higher subjects such as algebra (Fuchs et al., 2017; Siegler et al., 2012; Torbeyns et al., 2015). Evidence indicates that teachers' knowledge predicts students' achievement gains (Charalambous et al., 2020), and, therefore, teachers' understanding of fractions is crucial for students to learn rational numbers and operations on them. However, as Torbeyns et al. (2015) indicate, "systematic studies in Europe and North America point to deficits in (prospective) teachers' content and pedagogical content knowledge of mathematics in general and rational numbers in particular" (p. 7). Recent studies reveal challenges that pre-service teachers (PSTs) have with conceptual and procedural fraction knowledge (e.g., Bobos & Sierpinska, 2017; Busi et al., 2015; Depaepe et al., 2015; Harvey, 2012; Tobias, 2013; Toluk-Uçar, 2009; Utley & Reeder, 2011; Van Steenbrugge et al., 2014), particularly with fraction multiplication and division (Lo & Luo, 2012; Morano & Riccomini, 2019; Olanoff et al., 2014; Siegler & Lortie-Forgues, 2015; Young & Zientek, 2011).

Researchers have based their investigations into the learning of fractions and their operations on two ontological perspectives: partitioning and measuring (Powell, 2019a). The first perspective views a fraction as a relation between parts of a single whole or quantity subdivided into equal portions. This perspective emphasizes counting, leading to the commonly accepted part/whole conception of fractions (Schmittau, 2004). The Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and some mathematicians interested in pre-university and teacher mathematics education (Wu, 2014) suggest this conception for introducing fractions in elementary schools.

Nevertheless, research suggests that this partitioning perspective seriously limits the robustness of students' understanding of fractions (Kerslake, 1986). It encourages the erroneous idea that a fraction only represents two different discrete quantities and hinders understanding improper fractions (Tzur, 1999). While research on fraction knowledge has yielded information about fraction learning's cognitive issues (Behr et al., 1997; Dienes, 1967; Kieren, 1980; Lamon, 2007, 2012; Mack, 1990; Tzur, 1999), a critical limitation has been a focus on initial fraction learning from the partitioning perspective.

Few studies have investigated a known alternative source for fraction knowledge, a measuring perspective. Rather than basing fraction knowledge on counting discrete portions of a single quantity or collection, the measuring conception views fractions relationally. A fraction is a number or ordered pair of numbers that indicates a relation between two commeasurable quantities of the same kind (Davydov & Tsvetkovich, 1991; Gattegno, 1960/2009, 1974/2010). The relation is a multiplicative comparison, where one quantity measures a multiple of the other. In this perspective, learners conceptualize fractions through multiplicative reasoning (Vergnaud, 1983, 1988).

In both the partitioning and measuring perspectives, to help students conceptualize fractions, teachers use manipulatives and representational materials such as folded paper strips, area models, and collections of objects. Yet, few manipulatives allow for a tangible, flexible, and complete model of any fraction and the arithmetic operations on fractions. For instance, some researchers (Lee & Lee, 2019) note how circular models are inconvenient to illustrate fractions with large denominators and do not recommend set models for fraction comparisons. Contrastingly, length models can easily represent fractions of any denominator, fraction magnitudes (i.e., the numerical values fraction symbols represent such as ¾), and fraction operations (Carraher, 1993; Fazio & Siegler, 2011).

With or without manipulatives, learners use symbolic representations of fractions to resolve tasks. The strategies they use have been the focus of research. For example, in magnitude comparison tasks, researchers (Mack, 1990; Erol, 2021) found that students based their comparative judgments on the amount needed to reach one, so to compare 5/6 and 7/8, students used 1/6 and 1/8. Mack (1990) discovered that students judged 1/8 to be larger than 1/6 since eight is greater than six, thus misusing a whole number property for comparing fractions. Incorrectly applying properties of whole numbers on fractions is a common strategy known as whole number bias (Ni & Zhou, 2005). Another example concerns strategies related to locating fractions on a number. Siegler et al. (2011) found that middle school students used numerical transformation or segmentation strategies to locate fractions on a number line. With numerical transformation strategies, students transformed a fraction to an easier one, while segmentation strategies involve partitioning the number line into a certain number of segments. When comparing fractions, other researchers found that procedural manipulations are usually associated with incorrect comparisons and nonrelational thinking. In contrast, strategies based on number sense such as benchmarks and estimation support mathematical reasoning and evidence conceptual understanding (Sengul, 2013; Yang et al., 2009). Therefore, teaching fractions should consider approaches and practices that help learners develop conceptual strategies for representing and operating on fractions.

Our study investigates how a distinctive measuring perspective influences PSTs' reasoning about fractions represented in multiple models. Specifically, we ask this question: How does revisiting fraction knowledge using a measuring perspective influence PSTs' reasoning about fractions represented in rectangular, circular, and set models? To examine PSTs' reasoning about fractions, we attend to their strategies to solve fraction tasks within each of the three models.

In what follows, we present pertinent literature and our theoretical framework, methods, and findings. Finally, we discuss the strategies that PSTs used on pre- and post-tests in light of the current literature and suggest areas for further research.

Pertinent Literature and Theoretical Framework

Current influential perspectives on fraction knowledge have a common origin. Kieren (1980) introduces and analyzes a taxonomy of interrelated interpretations of rational numbers. Concerning fractions, researchers (Behr et al., 1993; Charalambous & Pitta-Pantazi, 2007; Kieren, 1993; Lamon, 2007) widely recognize five standard interpretations: part of a whole, quotient, operator, ratio, and measure. To illustrate, 3/4 as a part of a whole (an area or a collection) means three out of four equal parts; as a quotient means three divided by four; as an operator means a scalar or three-quarters of a quantity; as a ratio signifies three objects to four objects, where the objects are of different categories; and finally, as a measure represented by iterating the unit fraction, 1/4, three times on a number line. Kieren (1980) asserts that learners must understand and function with these interpretations as prerequisites for having complete, mature knowledge of fractions. Other researchers view that learners' difficulties stem exactly from what seems like perplexing, overlapping ideas about fractions (Ohlsson, 1988).

In the usual fraction taxonomy, the interpretations or "sub-constructs" share partitioning as their foundational cognitive action. As Kieren (1980) notes, "[p]artitioning is seen here as any general strategy for dividing a given quantity into a given number of "equal" parts. Thus, it can be seen as important in developing all of the five sub-constructs." (p. 138). Positing that partitioning is the cognitive basis for fraction knowledge implies that the part/whole interpretation is the initial entry to the concept.

After this introduction to fraction knowledge, current policy and curriculum documents suggest the measurement interpretation (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Siegler et al., 2010). Following Kieren's (1980) taxonomy, they mean positioning fractions on a number line and iterating a unit fraction (a fraction whose numerator is one) to locate a non-unit fraction. For instance, starting in the third grade, the Common Core State Standards recommends that the second interpretation of fractions to study is measurement: "[r]epresent a fraction a/b on a number line diagram by marking off a lengths 1/b from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line" (p. 28). An instructional imperative for this fraction interpretation is that number lines illustrate that fractions represent magnitudes (Fazio & Siegler, 2011) and transfer to tasks involving fraction magnitude comparisons (Hamdan & Gunderson, 2017).

In the following four subsections, we discuss our distinctive measuring perspectives, define unitizing, and relate it to fraction representational models. Finally, we discuss fraction strategies.

Measuring Perspective

Contrasting with the measurement interpretation of fractions within the partitioning perspective, which concerns the equal subdivision of a single entity such as an area or a length, another measurement standpoint can also yield fractional numbers. Distinctively, this standpoint posits that a fraction represents a particular relation between two quantities of the same kind. The relation is a multiplicative comparison. We call this standpoint a measuring perspective of fractions (Powell, 2019a, 2019b) since the quantitative comparison of continuous quantities (such as length, area, volume, and time) is the cultural practice of measuring one quantity by another of its kind considered as the unit. In this perspective, measuring is the material source of both whole numbers and fractions (Davydov & Tsvetkovich, 1991). In fact, rather than fractions being an extension of whole numbers, whole numbers arise as a special case of measuring. For example, to find the extent of a distance d, in comparison to a unit of measure u, there are two cases: Either d equals an exact multiple of u, or it does not, which historically occasioned the ideas of both rational and irrational numbers.

While describing the Elkonin-Davydov curriculum, Davydov and Tsvetkovich (1991) argue for a general concept of numbers based on measuring magnitudes as the support for learning about integers and real numbers. Numbers result from the count of the iteration of a unit when measuring a magnitude. Measuring occurs when objects have a common attribute that can be compared such as length, area, volume, or mass. Then, for example, a unit of length measuring the length of an object results in a count or number. The Elkonin-Davydov curriculum asks students first to consider the relationship between the unit's size and the measure obtained when comparing it to a quantity and then to notice that the measure of the quantity decreases as the unit of measure increases. This relationship is an essential idea for understanding inverse proportionality and fractions. Fractions are introduced by accepting partial units. Overall,

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as Schmittau (2005) notes, Davydov's approach intertwines three design elements: "initial development from the most generalized conceptual base, ascent from the abstract to the concrete, and appropriation of psychological tools" (p. 16). Our work incorporates these three curricular design elements. First, as psychological tools, we use Cuisenaire rods for mental models (see Figure 1) and engage relational thinking with them. Second, as the generalized conceptual basis for understanding units and fractions, we measure non-discretized quantified linear quantities. Finally, we structure a learning progression by starting with fractions of quantity and then moving to fractions as numbers. With these design elements within the measuring perspective, we provided examples of fraction tasks in the Methods section.

Figure 1

Cuisenaire rods, ten different sizes and colors, arranged in a "staircase" formation.

Our study aimed to instantiate an exclusive approach, comparing two distinct quantities of the same kind (Vergnaud, 1983), with a linear model employing Cuisenaire rods (Cuisenaire, 1952). Gattegno (1974/2010) references these manipulatives (see Figure 1) as he summarizes the role of measurement for elementary mathematics:

> *Measure, in the work with the rods, is borrowed from physics and introduces counting by the back door, since it is necessary to know how many times the unit has been used to associate a number with a given length. But measure is also the source of fractions and mixed numbers, and serves later to introduce real numbers. Thus measure is a more powerful tool than counting, which it uses as a generator of mathematics. Counting ... can be interpreted again as being a measure with white rods. Measure is naturally also an interpretation of iteration (p. 196, original emphasis)*

Using Cuisenaire rods to model fractions, Gattegno views length as their attribute of interest and to be measured. His approach is exclusive in that it ventures to find "how many times the unit has been used to associate a number with a given length," so the unit and the given rod are two distinct entities with length as their common attribute. The approach is consistent with the measuring perspective. The measurable characteristic of Cuisenaire rods is one reason we chose it to engage the PSTs in reexamining how they understand fraction magnitude, order, equivalence, and operations.

Unitizing

The concept of unitizing transcends the borders of the partitioning and measuring perspectives. Nevertheless, within each view, unitizing involves a different number of quantities. Fundamentally, unitizing concerns assigning a given quantity as a unit of measure (Lamon, 1996, 2007). For example, in the partitioning perspective, unitizing is a process alongside dividing and distributing equally:

> *Partitioning is an operation that generates quantity; it is an experience-based, intuitive activity that anchors the process of constructing rational numbers to a child's informal knowledge about fair sharing. Unitizing is a cognitive process for conceptualizing the amount of a given commodity or share before, during, and after the sharing process. (Lamon, 1996, p. 171)*

Moreover, Lamon (2012) emphasizes that unitizing is both natural and subjective. For instance, given a chocolate bar segmented into eight pieces, if a child wishes to share it fairly among herself and three other children, she must decide how to divide it into sizes or unitize the bar. The child has several choices for the unit. One possibility is that she selects the unit as two segments of the chocolate bar. In this case, each child receives one whole unit of chocolate or one-fourth of the bar. Instead, she might choose each segment as the unit, and, therefore, each child will receive two units of chocolate or two-eighths of the bar. In both distribution scenarios, as two-eighths and one-fourth describe equal portions of the chocolate bar, they are equivalent fractions. The child's sharing is seen as natural, and how to size the shared pieces or unitize them is subjective. Furthermore, unitizing in different ways can yield equivalent fractions. It is worth underscoring that unitizing in the partitioning perspective is a cognitive action on a single quantity.

In contrast, in our measuring perspective, two related but distinct material quantities, physical or mental, are necessary. Further, the idea of unitizing depends on the specific meanings of the concepts of measurement and measuring. Measuring requires two quantities, the one whose extent needs to be quantified and the quantity whose size is the measuring unit. A measurement quantifies a quantity's extent, a value representing how much it is of a given unit of measure. Measuring is the action to determine the size of a quantity. Unitizing, the choice of unit of measure, is contingent. For example, a person can choose to measure the distance between two cities, using a person's natural stride as the unit of measure or a more extended quantity. The choice is subjective. For instance, in Figure 2, we present the measuring action with Cuisenaire rods, where the tan rod's length measures eight white rods, four red rods, or two purple rods. Its length depends on whether we chose the white, red, or purple rod as the unit of measure. If the orange rod (equal to 10 white rods and five red rods) is the unit of measure, then the tan rod is either eight-tenths or four-fifths of the orange rod, contingent on whether the white or red rod is the subunit. The choice is an instance of unitizing, assigning quantities as the unit and subunit of measure.

Figure 2a
The length of the tan rod measured by white, red, and purple rods.

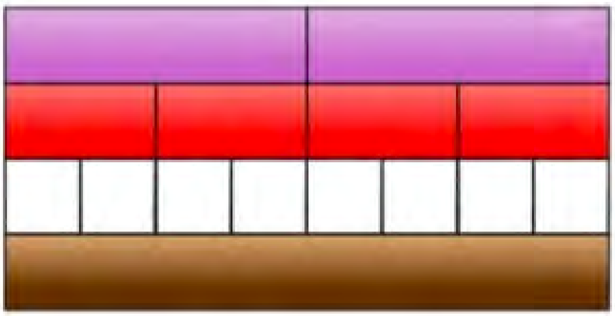
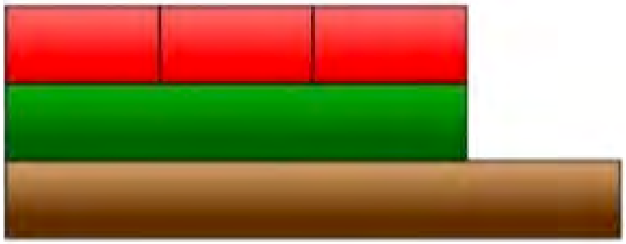


Figure 2b
The length of the dark green rod measured by red rods and compared to the tan rod.



Unitizing also pertains to determining the unit to measure a given quantity. For example, in Figure 2b, if we consider the length of a dark green rod to be three-fourths, then to unitize means to find what length is the unit of measure. In this case, since the dark green rod equals three red rods, and each red rod is one-fourth of the tan rod, then the measuring unit is the tan rod. The length of the dark green rod is three-fourths of the length of the tan rod. Since unitizing is subjective, three-fourths can also be measured by a different unit of measure. For example, the purple rod can be the unit of measure. It measures four white rods, and the light green rod measures three white rods, which means that the light green rod is three-fourths of the purple rod. That is, three measured by four is three-fourths. The need to unitize occurs when comparing two or more quantities, each measured by a different measuring unit.

Overall, unitizing is a critical operation for working adeptly with fractions. It is fundamental to fraction comparisons and operations (Van Ness & Alston, 2017a, 2017b, 2017c). Fraction comparisons, addition, subtraction, and division require that the involved quantities have the same unit of measure. For multiplication, the unit of measure of one fraction needs to equal the number of units of the other fraction's unit of measure. From the measuring perspective, in the Methods section, we illustrate comparing, adding, and subtracting fractions.

Fraction Representational Models

Mathematical representations are considered an essential element of mathematical knowledge. National standards call for supporting students to engage "in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving" (National Council of Teachers of Mathematics, 2014, p. 10). Representations are "processes and products that are observable externally as well as to those that occur 'internally,' in the minds of people doing mathematics" (National Council of Teachers of Mathematics, 2000, p. 67). They play a critical role in mathematics instruction (Watanabe, 2002). Policy and curriculum documents expect teachers to employ models that support students understanding and communicating mathematical ideas (Ball et al., 2008).

When teaching fractions, teachers rely on various models to illustrate the fraction concepts. Models include area, length, and sets. The area model uses geometric shapes such as circles and rectangles divided into equal parts. This model is widely used in textbooks and corresponds to fractions' part/whole interpretation (Hodges et al., 2008). The circular area model (or pies) is also commonly used to introduce fractions. However, research indicates that this model causes students difficulties partitioning circles into equal parts, especially with many equal parts (Cramer et al., 2002). The length models such as fraction strips, number lines, or Cuisenaire rods involve comparing or partitioning lengths. As mentioned above, the length model, specifically the use of number lines, is recommended by U.S.-based national organizations. Number lines support students to develop an understanding of fractions magnitudes. The set model involves comparing discrete collections of objects such as colored counters. Students consider a specific number of objects as the unit and use that number to name another group of objects' fractional relation to the unit.

Manipulatives and visual models also correspond to actions performed to identify or represent fractions. For example, Vergnaud (1983) and Watanabe (2002)

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identify two methods of representing fractions: inclusive or part/whole and exclusive or comparison. The inclusive approach uses one quantity (area, length, or a set of objects) to represent the whole and its parts. The exclusive approach involves two quantities; one quantity is the whole or unit used to compare the second. In an inclusive approach, the whole and the parts of interest are of the same object—for example, an area model represented with pizza results in comparing pizza slices to the whole pizza. The exclusive approach involves comparing two different entities of the same kind. For instance, a student compares a ruler's length to the length of a table's side. The two objects are comparable since they share a common attribute, length. To compare, the student unitizes and measures, which are both actions that correspond to the fraction concept's origin (Davydov & Tsvetkovich, 1991).

Choosing an inclusive or exclusive approach depends on the desired reasoning to be exercised. For example, when presented with partitioned parts, students use additive reasoning to count the highlighted parts and the total number of parts in the whole to identify fractions. However, this additive reasoning impedes proportional reasoning (Mack, 1995; Ni & Zhou, 2005). To develop proportional reasoning, "students must move from an additive method of comparing to a multiplicative one" (Vergnaud, 1983, p. 162). Multiplicative reasoning engages students with language that focuses on the scaler relation between quantities such as "how many times more or less."

In contrast, additive language reflects counting procedures and uses statements like "two or three objects more or fewer." This difference between additive and multiplicative reasoning highlights the importance of engaging students with continuous quantities when dealing with fractions. Continuous quantities are quantities for which there is another measure between any two measures (e.g., length, area, volume, or time). With any two non-discretized continuous quantities, students engage in unitizing, measuring, and comparing the two quantities to identify the fractional relationship between them. For example, Maher and Yankelewitz (2017) found engaging students with measuring and comparing lengths (Cuisenaire rods) to identify and compare fractions supports them to reason successfully about fractions (indirect, using cases, counterargument, and recursive reasoning) and to transition from additive to multiplicative reasoning.

Fraction Strategies

How learners resolve tasks involving fractions has been the focus of numerous studies. Several studies looked at how students compare fractions. Among these studies, a common finding relates to what has been called the whole number bias, strategies in which students employ properties of whole numbers to fractions. An example is Mack (1990), which we discussed in the introduction. Similarly, Erol (2021) asked fifth-grade students to compare fractions and then interviewed them to understand their reasoning. Relying on whole numbers properties to determine the greater of two fractions, students stated that a fraction is greater (1) when the numerator is larger among fractions with the same denominator, and (2) the fraction with the larger denominator is the greatest among fractions with the same numerator. Gabriel et al. (2013) observed this latter strategy for comparing fractions with fourth, fifth, and sixth-grade students. In two experiments, Meert et al. (2013) tested college students' componential and holistic processing of fraction comparisons. They asked the students to compare fractions with and without common components. When comparing fractions with the same denominators, college students only compared the magnitude of the numerators. With fractions having the same numerators, they used holistic processing to compare them.

Researchers observed other fraction strategies when learners located fractions on number lines. Siegler et al. (2011) identified two strategies, numerical transformation and segmentation. Zhang et al. (2017) subsequently investigated these strategies with middle school students who located fractions on 0-1 and 0-5 number lines. They found that only a few students used numerical transformation strategies by converting the fraction to a decimal or rounding it and comparing it with 0,1/2, and 1 as benchmarks and converting an improper fraction into a mixed number. Students used segmentation strategies accurately by dividing the number line into equal parts corresponding to the value of the fraction's denominator. However, they used segmentation strategies inaccurately by segmenting the unit into unequal intervals. When locating fractions on a 0-5 number line, some students treated the 0-5 number line as if it were a 0-1 number line, for example, by locating 7/8 close to 5. This action suggests that they think fractions are always less than one. Similarly, Bright et al. (1988) found that fourth and fifth-grade students had difficulties identifying the unit when locating fractions on a number line. Students counted the marks to locate fractions, which did not correspond to the equal interval related to the fraction.

Research about fraction strategies related to representational models such as set, rectangular, or circular models is limited. Additionally, there is scarce research about instructional interventions from a measuring perspective. Therefore, our study examines the strategies that PSTs used with these three models before and after revisiting fractions from a measuring perspective.

Methods

The intervention engaged pre-service elementary teachers in reexamining fractions from a measuring perspective to investigate changes in how they reason about fractions using different representational models of fractions. We use "reexamining" to indicate a re-consideration of PSTs' understanding of fractions, including a part/whole conception, from a measuring perspective. The intervention took place in an elementary mathematics methods course in a 15-week semester. During the first week of the semester, the PSTs completed a pre-test in which they expressed fractions using discrete and continuous models. In the last week of the semester, PSTs completed a similar assessment as a post-test. For approximately 75 minutes every two weeks, the PSTs used Cuisenaire rods (see Figure 1) to collaboratively solve fraction tasks.

The tasks initially engaged them with whole numbers and operations problems to familiarize the PSTs with Cuisenaire rods. Afterward, a set of tasks introduced them to fractions by comparing the lengths of different rods multiplicatively. The tasks did not include situations involving two-dimensional rectangular, circular, or set fraction models. Instead, PSTs interacted with three-dimensional physical objects formed by six parallelograms or parallelepipeds (Cuisenaire rods), focusing on one of their dimensions, length. An outline of the intervention tasks follows a description of the study participants.

Participants

The participants were 46 pre-service elementary teachers (43 females) enrolled in an elementary mathematics methods course at a medium-sized state university in the northeast of the United States. This study's participants consist of PSTs from two sections (n=22 and 24) of the course during the second semester of 2017. The participants were in their last year of a four-year early childhood baccalaureate degree program and one semester away from student teaching. In the program's first year, they completed two mathematics content courses specifically designed for pre-service elementary teachers and covered topics from elementary school mathematics, including rational numbers, ratio, and proportion.

The elementary mathematics methods course focused on problem solving and mathematical reasoning. It discussed the design and implementation of mathematical tasks that support elementary students to develop a conceptual understanding of various mathematical topics. The topics included the development of whole-number sense, operations on whole numbers, geometry, probability and statistics, early algebraic ideas, and fractions, including adding and subtracting fractions. For lack of time, the course did not include the multiplication and division of fractions. The course met for two 75-minute sessions a week during 14 instructional weeks. Starting the third week of the semester, the intervention for this study consisted of four sessions. Participants solved fraction tasks collaboratively, using Cuisenaire rods, every two weeks for an entire session. These fraction sessions represented approximately 14% of the semester. The sessions were staggered to lessen the cognitive load and give the participants extended time to reflect on their learning.

Reexamining Fractions from a Measuring Perspective

What follows is an outline of how we invited the PSTs to use Cuisenaire rods to rethink their fraction knowledge from a measuring perspective.

1. Familiarization with Properties and Operations: PSTs play with rods and note their properties. We define a train of rods as one or more rods placed end-to-end and have PSTs create trains. They construct trains and compare absolute lengths that go beyond the available rod sizes. For instance, the teachers build trains such as an orange and a purple train and compare its length to a train consisting of three rods: yellow, purple, and green. Next, they add and find the difference between pairs of lengths. Finally, they multiply lengths by iterating a chosen unit length a desired number of times and dividing lengths by seeing how many units create a length congruent to a larger length. These experiences enact a measuring perspective with whole numbers.

2.Introducing Fractions: From defining and iterating a unit to obtain a certain length, PSTs compare the length of the unit and the resulting length. For example, they repeat the red rod four times to create a train equivalent to the brown rod (see Figure 2a) and say that the length of the brown rod is four red rods. The inverse relation between the original unit and the resulting length yields a unit fraction. In the above example, the brown rod becomes the unit, and the red rod becomes the rod to be measured, whose length is one-fourth of a brown rod. Then, PSTs compare two red rods, five red rods, or 10 red rods to the brown rod and say that it is two-fourths, five-fourths, or ten-fourths of the brown rod, respectively. These comparisons lead participants to name non-unit and improper fractions. They also measure rod lengths by iterating a unit length where a whole number of unit rods does not create a length congruent to a larger length. For example, they measure the black rod (7 cm) using a light green rod (3 cm) as the measuring unit. They need two light green rods and a white rod to create a length equal to the black rod (see Figure 3). The name of a white rod's length emerges from the inverse relationship between its length and the length of a light green rod (one light green rod equals three white rods, and one white rod equals one-third of a light green rod). As shown in Figure 3, the measure of the length of the black rod is two and one-third of the light green rod.

Figure 3

Measuring a black rod using a light green rod, which measures 2 and 1/3 light green rods.

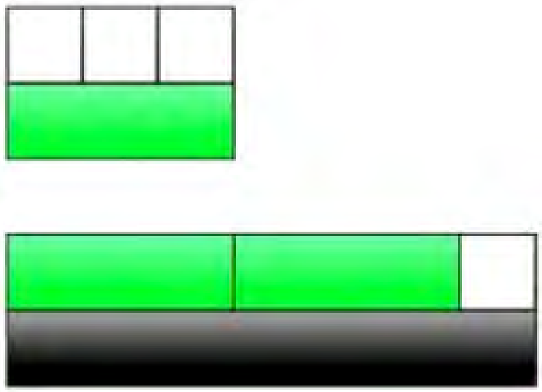
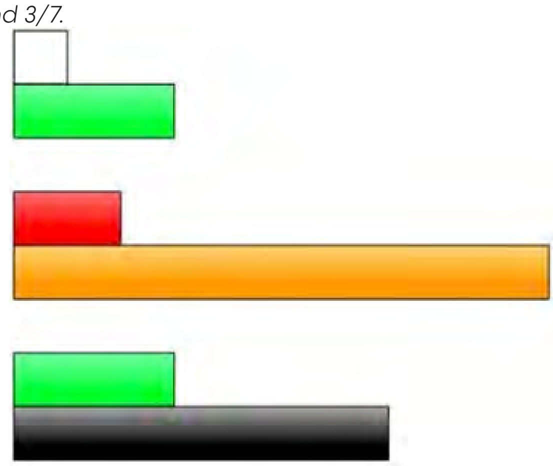


Figure 4

Using Cuisenaire rods to illustrate the fractions 1/3, 1/5, and 3/7.



Afterward, PSTs compare pairs of rods and name the fractions that represent the comparisons. Examples are that the white rod is one-third of the light green rod, the red rod is one-fifth of the orange rod, and the light green rod is three-seventh of the black rod (see Figure 4). They also compare the lengths of different rods to the lengths of trains of rods. For example, they compare the brown rod's length to a train's length composed of an orange rod and a yellow rod. PSTs then express the multiplicative comparison as eight-fifteenths since the brown rod is 8 cm and the train consisting of an orange rod and a yellow rod is 15 cm. We do not explicitly use a standard unit of measurement (centimeters or inches) to compare lengths; we use other rods such as the white rod (1 cm) or the red rod (2 cm) as an intermediary or subunit to assist with comparing lengths.

3. Comparing fractions: PSTs think of fraction magnitudes such as one-half and one-third and decide which is greater. They then demonstrate their choice using the rods. Some of them create those pairs of rods to model the fractions without regard to a standard unit of measure and then compare them. They create any of these three situations:

a) one-half is larger than one-third, using a red rod as one-half of a purple rod and a white rod as one-third of a light green rod; a red rod is larger than a white rod (Figure 5a);

b) one-half is equal to one-third, using a white rod as one-half of a red rod and also one-third of a light green rod (Figure 5b); and

c) one-half is smaller than one-third, using a white rod as one-half of a red rod, and a light green rod is one-third of a blue rod; a white rod is smaller than a light green rod (Figure 5c).

Figure 5a

Illustration of 1/2 and 1/3 where the rod representing 1/2 is larger than the rod that represents 1/3.

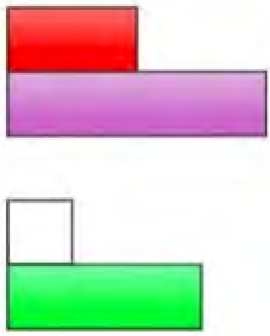


Figure 5b

Illustration of 1/2 and 1/3 where the rod representing 1/2 is equal to the rod that represents 1/3.

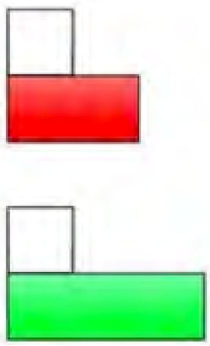
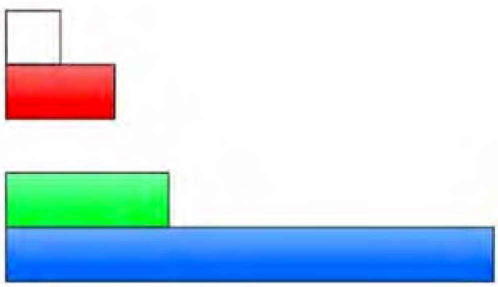


Figure 5c

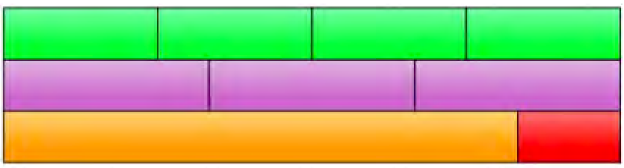
Illustration of 1/2 and 1/3 where the rod representing 1/2 is smaller than the rod that represents 1/3.



These discrepant results instigate a discussion about the role and importance of a unit length of measure to represent fractions such as one-half and one-third. Afterward, PSTs use one length as the unit of measure and find the corresponding lengths that represent one-half and one-third so that they can compare their relative magnitudes. Next, they learn the Train Race game to identify commensurable unit length for a set of fractions. The game results in the length that represents the least common multiple of the lengths of two rods by placing them next to each other and creating a single-color train using the two rods until they are equal in length. That length becomes the standard unit of measure for the fraction comparisons. For example, to compare one-third and one-fourth, PSTs take the smallest rod with which they can represent thirds (light green) and the smallest rod with which they can represent fourths (purple rod) and place them side by side (see Figure 6). From those two rods, they create two single-color trains. The light green rod is shorter than the purple rod, so they add another light green rod, then compare again to see which train is shorter and add another rod to it to make it longer. In this case, they add another purple rod and continue in turn until the two trains are equal in length. This Train Race ends with four light green rods and three purple rods since the length of the two trains is now equal. The last rod added to the trains is the rod color that wins the game. The length created from using a light green rod and a purple rod allows PSTs to represent one-third (one purple rod) and one-fourth (one light green rod) and compare their magnitudes.

Figure 6

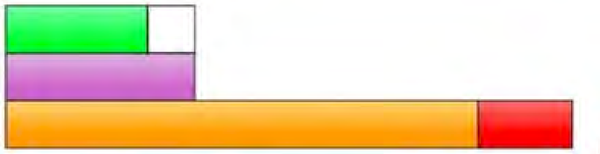
The result of a Train Race Game involving green and purple rods.



4. Adding and Subtracting Fractions: PSTs use the Train Race game to find a unit length to represent different fractions. They create a train composed of one-third and one-fourth and identify this new train's multiplicative comparison to the unit, which equals a train of an orange and a red rod. In this example, one-third is a purple rod, one-fourth is a light green rod, and the train composed of those two rods is seven-twelfths of the unit. To demonstrate subtracting fractions, PSTs identify the difference between the two lengths that represent the two fractions. For example, using the length that resulted from the Train Race above, teachers find the difference between one-third (a purple rod) and one-fourth (a light green rod). They identify the rod that fills the rod's gap when placing them side-by-side (see Figure 7).

Figure 7

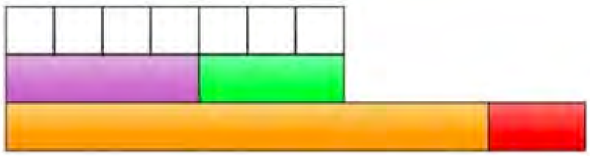
Using Cuisenaire rods to illustrate the difference between 1/3 and 1/4.



In this case, it is the white rod. They express the rod (or length) that fills the gap with the unit. Now, a white rod is one-twelfth (1/3-1/4=1/12). The PSTs map this physical experience of comparing, adding, and subtracting fractions to the symbolic manipulation procedure and finding other fractions' names. In the example above, they express the fraction one-third differently, and it would be four-twelfths as a purple rod is equal to the length of a train composed of four white rods. Each white rod is one-twelfth of the unit. Similarly, they express one-fourth as three-twelfths and write 1/3+1/4=4/12+3/12=7/12 (see Figure 8). A final interrogation concerns writing this statement without representing the fractions using the rods.

Figure 8

Using Cuisenaire rods to illustrate adding 1/3 and 1/4.



Most of the tasks above involve less-than-one fractions only to illustrate the measuring approach. In fact, each task engages participants with fractions greater than one immediately after working with fractions less than one. For example, participants measure the red rod (2 cm) using a dark green rod (6 cm) as the measuring rod and notice that one red rod is one-third of the dark green rod. Again, in relation to the dark green rod, they also measure two red rods and four red rods, respectively two-thirds and four-thirds (see Figures 9a, 9b, and 9c). The participants are encouraged to refrain from using the language of mixed numbers to name the fractions so that their language corresponds closely to what they see. After participants develop fluency with naming fractions, the language of mixed numbers is visited.

Figure 9a

Comparing one red rod to a dark green rod.

Figure 9b

Comparing two red rods to a dark green rod.

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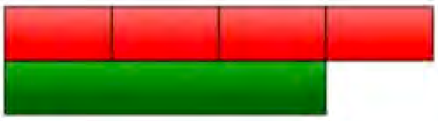
Figure 9c

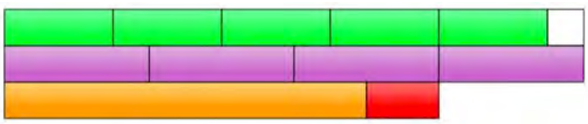
Comparing four red rods to a dark green rod.



Similarly, when comparing fractions or adding and subtracting fractions, participants are asked to compare, add, and subtract greater-than-one fractions immediately after working with fractions less than one. For example, when comparing fractions, participants are asked to determine which of these two fractions is greatest: four-thirds and five-fourths. They use the Train Race to find a unit length representing the two fractions (orange and red rods; see Figure 10). The unit length is equivalent to the train in Figure 4 since the comparisons involve thirds and fourths. Participants can observe that the train representing four-thirds (four purple rods) is longer than the train representing five-fourths (five light-green rods), which means four-thirds are greater than five-fourths. PSTs discuss the difference between the two lengths that represent the two fractions. They can see that the difference between the two trains (four-thirds and five-fourths) is one white rod, filling the gap between the trains. They express the white rod's name in relation to the unit as one-twelfth (see Figure 10), demonstrating the subtraction of four-thirds and five-fourths. When asked about the sum of four-thirds and five-fourths, participants use the two trains that represent the two fractions to create one train composed of those two trains, then compare this new train to the unit. They can identify that the new train is equivalent to 31 white rods, thirty-one-twelfths. Participants can also notice that the new train equals two trains of the unit and seven white rods (two and seven-twelfths).

Figure 10

Using Cuisenaire rods to illustrate comparing and subtracting 4/3 and 5/4.



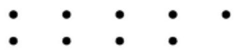
With this measuring perspective for reexamining fractions, participants compared any two quantities and choose an appropriate unit of measure. For example, the orange rod can be the unit of measure, making the yellow rod one-half, the purple rod four-tenths or two-fifths, and the white rod one-tenth. When the yellow rod is the unit of measure, the orange rod is ten-fifths, the purple rod is four-fifths, and the white rod is one-fifth. The PSTs continually consider the quantity to be measured to determine a unit and often a subunit of measure.

Data Collection and Analysis

This study's data come from pre- and post-tests that participants completed in the first and last weeks of the semester about aspects of their fraction knowledge. We adopted Norton and Wilkins's (2010) fraction assessment to examine PSTs' facility with unitizing and representing fractions less than one and greater than one, using two different continuous models (rectangular and circular) and a discrete model (set of dots; see Table 1). Each test includes 10 items that involve only two of the fractions' models, namely, dots and circles, dots and rectangles, or circles and rectangles. The fraction questions are parallel among the three representations. We randomly assigned participants to one of the three pre-assessment versions. For the post-test, we ensured that each participant received a different version of the assessment and answered items that used each of the three formats.

Table 1

Pre- and post-tests item samples. Adapted from (Norton and Wilkins 2010)

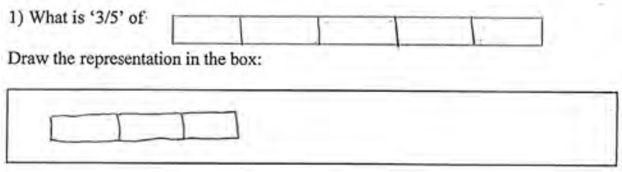
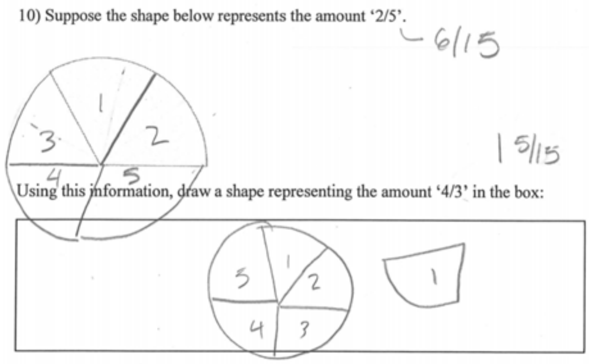
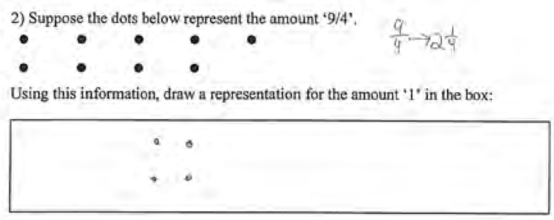
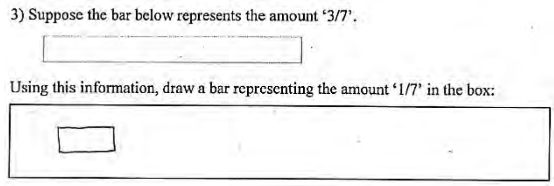
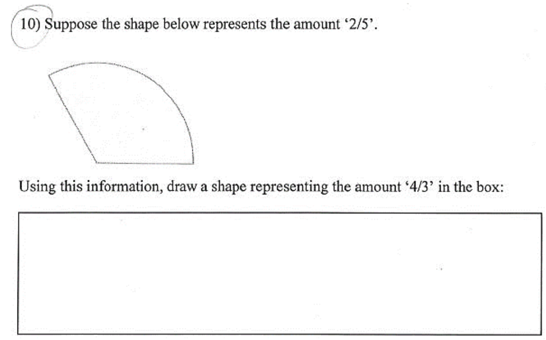
Circular model	7) Suppose the shape below represent the amount '9/4'.  Using this information, draw a shape representing the amount '1' in the box.
Rectangular model	7) Suppose the bar below represents the amount '9/4'.  Using this information, draw a bar representing the amount '1' in the box.
Set model	7) Suppose the dots below represent the amount '9/4'.  Using this information, draw a representation for the amount '1' in the box.

We conducted a conventional content analysis (Hsieh & Shannon, 2005) to identify PSTs' strategies for solving fraction problems. For multiple iterations, two researchers coded teachers' responses and discussed the codes until they agreed on a set of codes (see Table 2). After that, each researcher coded the same 240 responses separately and agreed on 257 codes out of 279, or 92.11% agreement.

Results

Our coding of PSTs' responses to the pre- and post-tests revealed five strategies for solving the fraction tasks described above, involving the set, rectangular, and circular fraction models. The tasks invited PSTs to identify (a) a fractional relation between two quantities or (b) the portion of a quantity that

Table 2
Definitions and examples of fraction strategies from pre- and post-tests

Partitioning: PSTs partition a quantity (set, length, or area).	
Constructing: PSTs draw an extension to a quantity (set, length, or area).	
Symbolic manipulation: PSTs create equivalent fractions or change improper fractions to mixed numbers.	
No visible strategy: PSTs provide an answer without showing any work.	
No answer: PSTs do not provide any written response.	

represents a certain fraction. Table 2 presents the five strategies and an example of each.

Generally, the frequencies and percentages of the strategies that PSTs expressed in their responses on the pre- and post-tests shifted. Our analysis revealed that the number of codes for strategies increased on the post-test from 527 to 578 codes. The percentages for partitioning strategy compared to all strategies increased from 35% in the pre-test to 52% in the post-test. In addition, fewer responses on the post-test included no visible strategy; its percentage decreased to 24% in the post-test from 40% in the pre-test. These findings indicate that compared to the pre-test participants employed more strategies on the post-test. In the following, we present PSTs' strategies for tasks involving circular, set, and rectangular models and fractions less than and greater than one.

Strategies Related to the Circular Model

In another study (Alqahtani & Powell, submitted), we scored the accuracy of PSTs' responses. For each response, the score was either 0, .5, or 1. For tasks that involved the circular model, Table 3 shows the percentages of each strategy associated with PSTs' responses and the mean score for the accuracy of their responses.

Table 3
Percentages of codes and mean score for the circle representational model.

Strategy	Pre-Test		Post-Test	
	%	Mean score	%	Mean score
Partitioning	55%	.74	66%	.80
Construction	6%	.89	6%	.69
Symbolic Manipulation	17%	.37	14%	.63
No Visible Strategy	15%	.67	14%	.42
No Answer	7%	0	1%	0
Total	100%	.63	100%	.69

As shown in Table 3, the most used strategy is partitioning in both the pre- and post-tests (55% and 66%, respectively) with a slight increase in the accuracy of the responses. The construction strategy did not differ between the two test times, but the accuracy decreased from a mean score of .89 to .69. The symbolic manipulation strategy decreased usage from the pre-test (17% to 14%) and increased the mean score (.37 to .63). Almost the same number of responses had no visible strategies but decreased the mean score, going from .67 in the pre-test to .42 in the post-test. There was a noticeable change in the number of responses that had no answer in the post-test, coming down to only 1% in the post-test from an initial 7% in the pre-test.

The change in the accuracy with the responses that had construction strategy might indicate that this strategy dose not lead to accurate estimations. When PSTs draw extensions to circular sectors, they cannot compare areas accurately and identify fractional relationships. The change with partitioning strategy might indicate that this strategy is more effective when dealing with circular sectors.

Strategies Related to the Rectangular Model

Similarly, partitioning strategy was the most common with PSTs' responses to questions that involved rectangular models in both pre- and post-tests (with 50% and 78% respectively; see Table 4). A more noticeable change was the decrease in the absence of strategies. In the pre-test, about 29% of the codes were for "No Visible Strategy." That percentage decreased to 9% in the post-test, and the mean score for accuracy of responses also decreased, from .66 to .5. The PSTs answered more questions on the post-test in comparison with the pre-test. Table 4 also shows that few PSTs used constructing strategy with rectangular shapes on post-test, while none of the PSTs used this strategy on the pre-test.

Table 4
Percentages of codes and mean score for the rectangular representation model.

Strategy	Pre-Test		Post-Test	
	%	Mean score	%	Mean Grade
Partitioning	50%	.69	78%	.71
Construction	0%	-	3%	1
Symbolic Manipulation	4%	.3	9%	.5
No Visible Strategy	29%	.66	9%	.5
No Answer	16%	0	2%	0
Total	100%	.52	100%	.65

The data in Table 4 indicate that PSTs reasoned more effectively on the post-test (more answers and more strategies). This change might be related to the similarity between the materials used in the intervention and the rectangular model. That is, like interacting with rods, the PSTs may have focused on the length of the rectangles. Findings show that working with a measuring approach can improve how PSTs partition rectangular shapes to compare them and identify fractional relationships among them.

Strategies Related to the Set Model

In our analysis, the most common code for PSTs' responses to questions that involved a set model was "No Visible Strategy." The percentages of no visible strategy were 61% on the pre-test and 70% on the post-test without any change in PSTs' accuracy. They used fewer symbolic manipulation strategy on the post-test than the pre-test, while their accuracy increased. In addition, PSTs provided more answers on the post-test than the pre-test.

Table 5
Percentages of codes and mean score for the set model.

Strategy	Pre-Test		Post-Test	
	%	Mean score	%	Mean score
Partitioning	15%	.78	16%	.8
Construction	2%	.6	2%	.7
Symbolic Manipulation	16%	.61	10%	.8
No Visible Strategy	61%	.75	70%	.75
No Answer	7%	0	2%	0
Total	100%	.68	100%	.73

With all strategies, the mean scores for the accuracy of PSTs' responses increased on the post-test. The change in the frequency of symbolic manipulation strategy (decreasing from pre-test to post-test) and the response accuracy (increasing from pre-test to post-test) might indicate that this strategy is ineffective with the set model. Data did not clearly show which type of strategy is more appropriate for set model questions. Nevertheless, the improvement in accuracy shows that revisiting fractions using a measuring approach can influence how PSTs solve set model questions.

Strategies Related to Fractions Less than and Greater than One

When examining the strategies that PSTs implemented with questions that involved less than and great than one fraction, we found that the largest increase occurs with partitioning strategy (see Table 6). Partitioning strategy comprised 43% of strategies used on the pre-test and 65% on the post-test for less-than-one fractions. The response accuracy did not have a notable change. Similarly, percentages for partitioning strategy with greater-than-one fractions increased from 24% to 43%, with a sizeable mean score increase from .46 to .57. In addition, the number of responses with no visible strategy decreased from 49% to 28% for less-than-one fraction questions and from 29% to 22% for greater-than-one fractions questions. The change for symbolic manipulation strategy was marginal for responses to both types of fraction questions. However, the accuracy of responses increased mean from the pre-test to the post-test. A more noticeable finding is the change in the number of questions that received no response on greater-than-one fraction questions. On the pre-test, 18% of codes were for questions that received no answers compared to only 3% on the post-test.

Table 6 above presents a few interesting findings. Strategies that PSTs implement vary depending on the type of questions. Understandably, PSTs used more symbolic manipulation strategy with fractions greater than one, including changing improper fractions to mixed numbers. In addition, the increase in the partitioning strategy for both types of questions, along with the increase in accuracy, indicate that working with fractions from a measuring perspective can support PSTs to reason visually through partitioning. Findings also show no notable difference between the construction strategy that PSTs used for both types of fraction questions.

Discussion

This study engaged 46 PSTs in reexamining fractions from a measuring perspective and investigated their strategies to compare quantities and identify fractional relations among them. On pre- and post-tests, PSTs worked with discrete and continuous quantities, presented in three models: set, rectangular, and circular. The set model involved a collection of dots, the rectangular model involved rectangles with a fixed width, and the circular model involved circular sections. The pre- and post-tests invited PSTs either to identify the fractional relation between two quantities of the same kind or to draw a set, rectangle, or circular section representing a certain fraction of a given set, rectangle, or circular area. The intervention employed Cuisenaire rods and engaged PSTs to compare the lengths of different rods to identify fractional relations between pairs of them and add and subtract fractions. Qualitative analyses show that PSTs implemented three main problem-solving strategies with the representational models: partitioning, constructing, and symbolic manipulation. Interestingly, two strategies were similar to findings from (Siegler et al., 2011), where participants used segmentation and numerical transformation strategies.

Our analyses also revealed changes in the frequencies of strategies from the pre- to the post-tests. On the post-test, PSTs used more strategies and answered more questions. Specifically, they used more partitioning strategy with the continuous models (rectangles and circular sectors). Furthermore, when analyzing PSTs' responses based on the type of fractions involved, findings also revealed that, in the post-test, partitioning strategy increased with fractions less than and greater than one.

Our findings show pronounced changes in strategies with the rectangular model. We believe this occurs

Table 6
Percentages and mean score of the pre- and post-tests for questions less than one and greater than one.

Strategy	Less than one				Greater than one			
	Pre		Post		Pre		Post	
	%	Mean score	%	Mean score	%	Mean score	%	Mean score
Partitioning	43%	.85	65%	.84	24%	.46	43%	.57
Construction	3%	.89	4%	.78	2%	.6	4%	.67
Symbolic Manipulation	3%	.89	2%	1	28%	.44	29%	.57
No Visible Strategy	49%	.90	28%	.85	29%	.35	22%	.31
No Answer	2%	0	<1%	0	18%	0	3%	0
Total	100%	.86	100%	.84	100%	.34	100%	.5

because of the close relationship between the rectangular model and the manipulative materials used in the intervention. The measuring approach with Cuisenaire rods asks learners to measure the length of one rod using another and express that measurement using rational number. Learners investigate and decide on an appropriate unit of measure and a subunit if needed. The rectangular model questions on the assessment ask PSTs to compare rectangular shapes with a relatively small width. PSTs only attended to the lengths of the rectangle and kept the width constant. In a related study, researchers first observed this close relationship between the rectangular model and the measuring approach (Alqahtani & Powell, submitted). In that study, the authors investigated the changes in PSTs' fraction knowledge after reexamining fractions from a measuring perspective. The participants from the current study comprised about half of the previous one. In general, the scores of 96 PSTs (including the 46 participants from the present study) on the post-test show a statistically significant increase (at $p < 0.01$) compared to the pre-test. With questions that involved greater-than-one fractions and for each of the three representational models, the scores also increased significantly (at $p < 0.05$ for the rectangular and circular models and $p < 0.01$ for the set model). With questions that involved fractions less than one, the authors found that participants' scores show statistically significant increase (at $p < 0.05$) only with the rectangular model. Again, the similarity between the rectangular model and the intervention's manipulatives might explain this change.

Even though PSTs did not work with the part/whole perspective or the partitioning action during the intervention, results show an increase in partitioning strategy with the rectangular and circular models. We believe that operating on continuous quantities such as length and area to compare and identify fractional relations is a conceptual process. The partitioning strategy that PSTs employed involves measuring or estimating the magnitude of lengths or areas. PSTs used partitioning to measure or estimate the size of the unit or the unit fraction. Working with Cuisenaire rods may have supported the PSTs to unitize, compare absolute magnitudes, and identify relative magnitudes between two continuous quantities. In alignment with Sengul (2013) and Yang et al. (2009), we contend that partitioning strategy based on measuring and estimating quantities reflects a conceptual understanding of fractions.

This study contributes to the literature by analyzing the implementation of a measuring perspective for fraction learning using Cuisenaire rods. This perspective aligns with the theoretical position and empirical studies that measuring is the material source of both whole numbers and fractions (Davydov & Tsvetkovich, 1991; Gattegno, 1974/2010). Another contribution of this study is the discussion of strategies that PSTs employ when working with fractions represented in three different models (set, rectangular, and circular) before and after revisiting fractions from a measuring perspective. The three models allowed PSTs to engage with counts, lengths, and areas. Conceptual strategies, such as measuring-based partitioning instead of counting-based partitioning, may support PSTs' fraction knowledge. The other two strategies, symbolic manipulations and construction, did not seem adequate for comparing two quantities and identifying fractional relations between them.

Future research may examine the influence of learning fractions using a measuring approach with both elementary and middle school students. Research is also needed to study how individuals use the three fraction models with fraction arithmetic and investigate how that compares to strategies used by those who worked with Cuisenaire rods within a measuring perspective.

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