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Elementary Students' Exploration of the Structure of a Word Problem Using Representations

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Abstract

Word problems are frequently used in school mathematics to offer students the opportunity to explore mathematical relationships and structure. However, previous studies have reported that word problems are misused or abused in ways that overlook the original intent of exploring mathematical structure and relationship. This study aims to share a series of a small group of third-grade students' explorations while debating the mathematical relationships in solving a word problem with representations over several days. Although the exploration took longer than planned, it was worthwhile. It offered students a space to express confusion, showcase their knowledge, test conjectures, and imagine alternative contexts. Ultimately, these explorations helped students recognize multiple relationships within the context of specific problems while bringing their attention to real-world related applications. The retrospective analysis of class episodes offers insight into learning opportunities to support students in exploring mathematical structure and relationships while discussing and debating the word problem context.

Keywords:

Additive and Multiplicative Relationships, Classroom Culture, Elementary Education, Schematic Representation, Structure, Word Problems

Introduction

Understanding and generalizing mathematical relationships and structures in learning mathematics are critical (Davydov, 1990; Mason, 2003; Sierpinska, 1994; Thompson, 2011). The ability to "look closely to discern a pattern or structure" is an essential skill that mathematics learners should develop (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 8). Also, "the detection and exploitation of structural relationships" is considered an essential mathematics component (Greer & Harel, 1998, p. 22).

Word problems are frequently used in school mathematics to offer students the opportunity to explore mathematical relationships and structure. However, studies have reported that students face varying challenges and difficulties in handling world problems (Verschaffel et al., 2020). Researchers discussed situations in which word problems are misused or abused, resulting in blocking the intended results of mathematical exploration (e.g., Mason, 2001; Verschaffel et al., 2000). In particular, studies reported that many students tend to dive into calculations by grabbing given numbers and using known procedures and operations or rely on keywords, rather than analyzing the structure of the problem as a means to solve the problem (Littlefield & Reiser, 1993; Savard & Polotskaia, 2017; Stigler et al., 1990; Verschaffel et al., 2000). Additionally, students generally produce one answer in the form of a numerical symbol and seem unwilling to bring anything further into the problem-solving process. Students also believe that there is only one correct answer or one correct process for finding a solution for a word problem. These two tendencies often keep students from paying attention to the context of the word problem, while these students generally have difficulty with problem-solving (Schwieger, 1999).

Educators employ various approaches to help students pay attention to and analyze the mathematical structure of a problem. For example, students are often encouraged to represent or model the relationships in ways that allow them to manipulate the quantities and reveal the structure, supporting their discovery of the required arithmetic operation. Some researchers supported schematic-based instruction, claiming that schematic diagrams better serve students (e.g., Terwel et al., 2009). Several studies highlighted a conceptual correlation between schemas and problem-solving (e.g., Jitendra & Star, 2011; Steele & Johanning, 2004; van Garderen et al., 2013). Other researchers offered activities that help students discern different word problem story grammar (e.g., Xin, 2012).

Previously, mathematics educators focused on teaching predefined schematic representations based on cognitive psychology (Fagnant & Vlassis, 2013). They asked students to memorize several predefined representations to solve certain types of problems, and students were expected to develop an ability to categorize problems based on the representations used (Schoenfeld, 1992). However, with increased attention to sociocultural perspective (Cobb & Hodge, 2011) and mathematical process (National Council of Teachers of Mathematics [NCTM], 2014), today's mathematics educators are encouraged to provide students with opportunities to examine mathematical structures of problems and represent them through student-oriented investigations (Fagnant & Vlassis, 2013). For example, NCTM (2014) suggests to "allow students to select and discuss their choices to represent the problem situations" (p. 28). When students gain authority in their mathematics investigation, they can make sense of connections between representations, understanding central mathematical ideas, and experiencing authentic mathematical problem-solving processes. Teachers should encourage students to engage in mathematical discussions about using and understanding schematic representation to improve students' problem-solving abilities in word problems.

In short, despite the many possible supporting tools and approaches, "what seems to matter most is not the apparatus itself, but how it is used" (Mason, 2018, p.332). Good tools and approaches to real-world problems, such as the aforementioned ones, can be (and often are) incorrectly presented through a teacher-led, top-down presentation rather than as an apparatus for student-centered exploration. When students have more opportunities to play with, be curious about, and explore word problems by changing the context and numerical parameters, it can be more enjoyable for them to explore structural relationships in the context of a word problem (Mason, 2018).

This study shares a series of explorations undertaken by a small group of third-grade students over several days. Using schematic representations and real-world examples, this group discussed the mathematical relationships involved in the following story problem: "A father is 32 years old, and his son is 4 times younger than him. How old will they be in 4 years?" The purpose of this study is to show the students' exploration process through three vignettes while providing interpretational space for readers. In this retrospective analysis of teaching episodes, this study focuses on the following questions: (a) What types of confusion and curiosity did the students exhibit while using schematic representations to identify the additive and multiplicative relationships? (b) How did the students make sense of the mathematical relationships underlying schematic representations? (c) What kind of classroom culture should be established to support student reasoning and justification?

Literature Review

Problem Structure

Although the term "structure" has been widely used in mathematics education without clear definitions, researchers consider knowledge about structure as an awareness of a network of local and general relationships (Venkat et al., 2019). Venkat et al. (2019) noted that emerging structures involving analyzing, forming, and seeing local relationships can be observed when young students analyze and distinguish local relationships, ultimately allowing them to identify mathematical structures with more general mathematical relationships and properties. For young mathematics students, exploring the different potential structures embedded in additive and multiplicative situations is a critical pathway for developing students' understanding and ability to operate within these structures flexibly (Mason, 2018).

Typically, curriculum using word problems includes multiple structures within additive and multiplicative situations. It is common that students make additive errors in multiplicative missing-value word problems and multiplicative errors in additive missing-value word problems.

Researchers highlighted that an important difference between additive and multiplicative relationships is the nature of invariance (Behr & Harel, 1990; Degrande et al., 2019). In other words, quantities are linked additively in the additive structure, and the actual difference between quantities remains invariant. In contrast, the ratio (e.g., relative difference) of quantities linked multiplicatively (e.g., linked through multiplication and division), what is invariant is the ratio between quantities.

> *When considering the word problem at hand, we can consider several structures.*
>
> *A father is 32 years old, and his son is 4 times younger than him. How old will they be in 4 years?*

First, the given relation ("4 times younger") supports students in multiplicative reasoning. Thus, the son's current age is 8 because 32 divided by 4 is 8. Second, the question turns students to additive reasoning. After finding the son's age, students can find missing values ("in 4 years") through different strategies. As the examples below show, 4 years are added to the current ages of father and son:

> *Father's age in 4 years: 32 + 4 = 36*
>
> *Son's age in 4 years: 8 + 4 = 12*

Alternatively, noting the actual difference between the current ages of father and son, 24, the final missing value is identified as follows:

> *The difference between the current ages of father and son: 32 – 8 = 24*
>
> *Father's age in 4 years: 32 + 4 = 36*
>
> *Son's age in 4 years: 36 – 24 = 12*

Thus, the situation can be explained differently depending on the relationships students recognize.

Representing the Problem Structure

Researchers noted that students have difficulties in understanding structures and analyzing quantitative relationships of word problems (Mason, 2018). Several researchers highlighted the importance of visualizing and representing the problem contexts to support students' attention and analyze the structure and relationships underlying a problem. Therefore, in mathematics curricula and programs, it is prominent to include various representations of a problem to elicit the structure and relationships within it. For example, materials used in the Math Recovery Program (e.g., Wright et al., 2006) frequently incorporate five or 10 frames to help students' early numeracy knowledge by illustrating the structure of numbers and the place value concept. In the Singapore curriculum, various models, such as bar models, support students' deeper understanding in solving word problems (Kaur, 2019; Ng & Lee, 2005, 2009).

More explicit use of models in elementary mathematics curriculum can be found in Davydov's curriculum (Davydov et al., 1999), where the critical role of symbols and models is emphasized. In the latter curriculum, students manipulate real objects and graphic models such as line segments and schematics to represent implicit and explicit structural relationships. As they progress, the use of concrete objects and graphic models decreases, and the use of symbolic formulas increases. For instance, physical objects or graphic models of a part-whole relationship help students initially see all involved quantities and their connection. Later, students can formulate algebraic equations for this mathematical relationship (Lee, 2002; Schmittau, 2005). Several studies reported the effectiveness of using various tools to represent and visualize relationships between quantities when solving word problems (Kaur, 2019; Ng & Lee, 2005, 2009; Schmittau, 2005).

Word Problem and Schematic Representation

Mathematics educators highlighted the importance of word problems in learning mathematics (NCTM, 2000; van Garderen et al., 2013; Vula et al., 2017). Word problems refer to problems that are "typically composed of a mathematics structure embedded in a more or less realistic context" (Depaepe et al., 2010, p. 154). Word problems help students construct mathematical representations and understand mathematical relationships and structures. They help them explore the relationship between reality and abstract mathematical concepts and operations (Jitendra, 2019). Studies showed that students usually go through problem-representation and problem-solution phases to solve word problems (Depaepe et al., 2010; Jitendra, 2019). In the problem-representation phase, students comprehend the problem and construct representations (or models) to illustrate the problem situation clearly. However, students work through the constructed representations in the problem-solution phase and interpret and evaluate the outcome.

In a well-known classification scheme for representation types, Lesh et al. (1987) emphasized flexibility and variability in meaningful use of representations among contextual, visual, verbal, physical, and schematic (or symbolic) representations. The visual representation retains most of the detailed information of the original contexts and clearly represents concrete visualization of objects to help students understand the problem contexts (Hegarty & Kozhevnikov, 1999; Viseu et al., 2021). However, schematic representations abstractly represent a structural relationship of mathematical elements in a problem. As schematic representations are "meaning-based representations" (Terwel et al., 2009, p. 27), they discard unimportant information and select mathematically important relationships and structures used in the problem-solving process. Therefore, students are expected to convert verbal information into symbolic expressions, such as line, diagram, and shapes, and use them to construct arithmetic operations during the problem representation phase.

Some studies reported that mathematics educators often introduced predefined schematic representations and asked students to memorize those representations to solve word problems (Fagnant & Vlassis, 2013). However, findings of some studies revealed that solving word problems with representations does not always increase students' performance (Diezmann & English, 2001; Terwel et al., 2009; Verschaffel et al., 2020). For example, Terwel et al. (2009) examined the effect of teacher-provided representations on solving word problems with fifth-grade students and reported minimal improvement in student problem-solving abilities. However, their counterpart group, which was asked to construct representations through collaboration, showed considerable improvement. As the reasons for these different outcomes, the researchers explained that the collaboration allowed students to improve their understanding of problem structures and enhance students' capabilities to generate new problem-solving strategies. Similarly, Lehrer et al. (2000) examined elementary school students and found that student-generated representations were more beneficial for developing their conceptual competence than using teacher-sanctioned representations. However, these findings did not reveal that teachers should not teach schematic representations to their students; instead, it means that teachers should first give students opportunities to learn and use predefined representations. Teachers should then allow students to construct their schematic representation based on their understanding paired with thoughtful discussion and analysis among classmates (Diezmann & English, 2001; Lehrer et al., 2000).

Previous studies have largely adopted quantitative research methods to examine the effect of employing schematic representations on students' word problem-solving abilities. Thus, limited qualitative information on what types of confusion and curiosity is exhibited by students when using schematic representations. We also lack understanding of how students make sense of mathematical relationships underlying schematic representations, and there is little guidance on what types or aspects of classroom culture should be established to best support students as they learn word problems. Therefore, further studies can be conducted to examine students' exploration of mathematical structures of word problems with representations.

Methods

Context and Participants

The class episodes were taken from a three-year teaching experiment conducted in a private school in the US (Lee, 2002). The first author taught a cohort of seven students using the first three years of elementary mathematics curriculum developed by Davydov and his colleagues (Davydov et al., 1999). There were two male students and five female students. For five students, this private school was their first formal education setting, and two students had some public school experience. There were three or four mathematics classes per week, and each class session lasted approximately 50–60 minutes. The curriculum consists of a series of problems. Students were accustomed to engaging in an in-depth discussion (or debate) on a small number of problems each session.

The class was in the third year of the experiment when discussing the word problem that this study discusses. Prior to this discussion, the students were accustomed to using literal variables, while they had the freedom to refer to known or unknown quantities using some tools such as question marks, blanks, underlines, or verbal descriptors. These students were also accustomed to problems that were impossible to solve due to insufficient or contradictory information. Such problems aimed to facilitate the students' justification and reasoning process. The students called them trap problems (Lee, 2007). The students were also familiar with using various representations such as line segments and schematic representations. Students used self-invented schematic representations at times, but they usually used mutually agreed-upon representations. Figure 1 shows some examples. As shown, students were encouraged to relate various relationships by analyzing the structure of the given schematic representations.

When the class episodes in the following section occurred, the students had already studied additive and multiplicative relationships and analyzed various contexts (word problems). In previous experiences, the problem contained only one relationship — either the additive or multiplicative relationship. Thus, the invariance of difference or ratio was maintained. The discussion presented in this study happened when students needed to consider both additive and multiplicative relationships in the same context.

Data Sources and Data Analysis

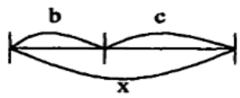
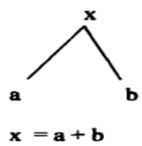
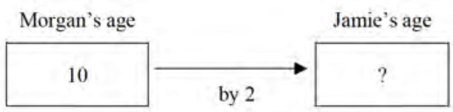
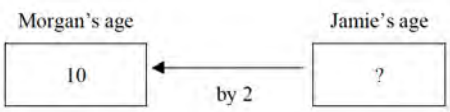
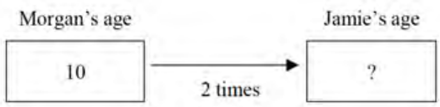
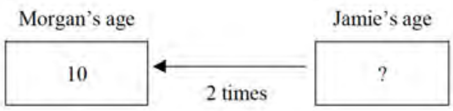
The primary data sources of the study were classroom discourse and field notes that the first author documented after each class session, describing interactions between the students, the teacher, and among students. For this study, the authors focused on the three days of class vignettes related to the discussion on the given word problem. A descriptive case study design (Yin, 2003) was used to examine student challenges during word-problem solving, and how students resolved those challenges through a series of small group discussions.

A case study examined a few cases of a phenomenon in a real context (Creswell & Poth, 2016). As individual cases are strongly connected in space and time, it is important to examine their context. A descriptive case study clearly describes a phenomenon and focuses on tracing "the sequence of interpersonal events over time (and describing) a subculture of it" (Yin, 2003, p. 4). For example, if researchers investigate the development of students' interactions over time, they can examine student participation and discourse, as well as their teachers' roles and discourse, to get a complete picture of the classroom environments.

The second author had an unbiased third-party role. As the first author was a teacher and thus directly participated in the classroom interactions, the second author also independently examined the raw data. Then, the two authors collaborated during several online meetings to compare and discuss the interpretation of the raw data at hand.

Figure 1

Examples of Schematic Representations Students Used

Schematic Representation	Example	Related symbolic representations
Line segment for a part-whole relationship		x = b + c b + c = x x - b = c x - c = b
Schematic for a part-whole relationship		x = a + b a + b = x x - a = b x - b = a
Schematic for the additive relationship	Morgan is 10 years old. Jamie is 2 years older than Morgan. How old is Jamie? 	10 + 2 = ? ? - 2 = 10 ? - 10 = 2
	Moran is 10 years old. Jamie is 2 years younger than Morgan. How old is Jamie? 	10 - 2 = ? 10 - ? = 2 ? + 2 = 10
Schematics for the multi-plicative relationship	Morgan is 10 years old. Jamie is 2 times older than Morgan. How old is Jamie? 	10 × 2 = ? ? ÷ 2 = 10 ? ÷10 = 2
	Morgan is 10 years old. Jamie is 2 times younger than Morgan. How old is Jamie? 	10 ÷ 2 = ? ? × 2 = 10 2 × ? = 10

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This process helped ensure the credibility of this study and provided the two authors the opportunity to retrospectively analyze key learning instances that occurred during the process (Yackel, 2001) from insider and outsider perspectives. More specifically, we took the following steps: (a) independently review classroom discourse and descriptive field notes and identify what important elements and moments, (b) compare each other's elements and moments, (c) jointly identify the most salient themes, and (d) juxtapose the data with interpretations based on extant literature. In short, we allowed the analysis to emerge from our understanding and interpretations of the events unfolding on the data, instead of approaching the data with a predetermined coding scheme. The reliability of the analysis is not obtained by the coincidence of interpretations among us. Instead, we directly presented classroom discourse to increase the external validity and transparency of the study (Creswell & Poth, 2016).

Descriptions of Class Episodes

The following three vignettes taken from the first author's field notes show the students' confusion regarding the use of schematic representations, differentiation between additive and multiplicative relationships and the critical moments in which the students chronologically shifted their attention. The class episodes also show how students make sense of the mathematical relationships underlying schematic representations. In the teaching episodes, the teacher's role was minimal and focused only on facilitating the discussion and recording (or helping record) students' discussions in visual forms. All students' names are pseudonyms.

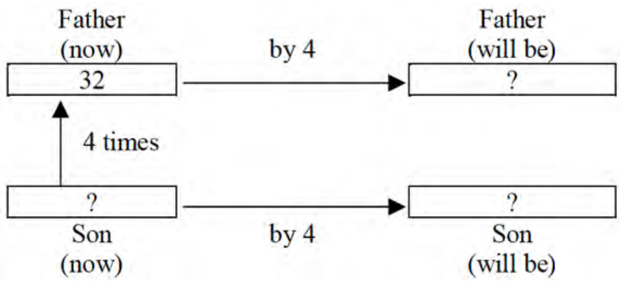
Vignette 1

"A father is 32 years old, and his son is 4 times younger than the father. How old will they be in 4 years?"

To solve this question, the student first determined the known (father's age now) and unknown quantities (son's age now, father's age in 4 years, and son's age in 4 years) and the relationships between given quantities. Based on that, the students drew the following schematic representation to help class discussion (see Figure 2). Students used descriptors for the quantities instead of literal variables in the schematic representations. In students' terms, the father's age in 4 years was noted as "father will be" and son's age in 4 years as "son will be." Unknown quantities were noted using question marks. Students noted the additive relationship between quantities "by" and multiplicative relationship "times."

Figure 2

Initial Schematic Representation



Using the information provided, students were able to find the values for all unknown quantities, as illustrated in Figure 2.

- [Father now]: 32 years old
- [Son now × 4]: 32, [Son now]: 32 ÷ 4 = 8
- [Father will be]: 32 + 4 = 36
- [Son will be]: 8 + 4 = 12

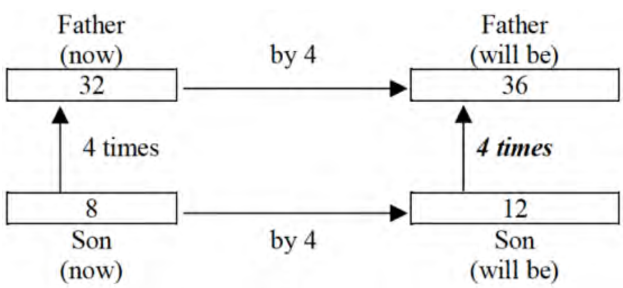
Meanwhile, the students found the values of all unknown quantities and completed the problem. However, one student, Jordan, started talking about the relationship between Father will be (father's age in 4 years) and Son will be (son's age in 4 years), noting that they did not show the relationship between these two quantities.

> Jordan: "Father (will be) is 4 times older than his son (will be)."
>
> Chris: "The problem did not ask for that relationship."
>
> Jordan: "However, we know that the relationship between the father's age and the son's age stays the same."

Although Chris resisted to do extra work (not due to any mathematical reason), students agreed that they could put "4 times" in the schematic representation between the father's age and son's age in 4 years (Figure 3).

Figure 3

Examining the Unasked Relationship



When completing the problem, students realized that something was wrong in the schematic representation because the father will be in 4 years (36) was not 4 times greater than the son will be in 4 years (12). At this point, Chris again suggested deleting "4 times" between the father's and son's ages in 4 years. Chris believed that we were not responsible for explaining the relationship between father will be and son will be by deleting the connection between them.

Other students disagreed with Chris, stated that the relationship would still exist even after it was deleted. Then, they concluded that this was a trap problem due to the contradictory information. Chris also agreed that it was a trap problem because the relationship between the father's age and the son's age should stay the same in 4 years. However, he continued to argue that there was no need to talk about this issue as the question did not ask about this relationship.

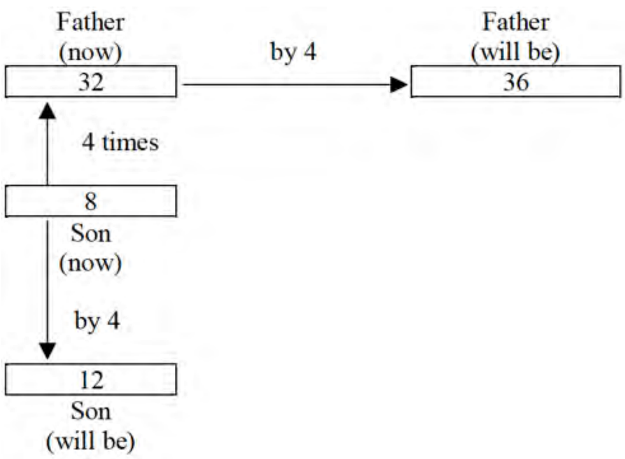
Vignette 2

The discussion began with a review of previous day's conclusions as to why this problem was a trap:

The relationship between the father's age and the son's age should remain the same even after 4 years. The father is always 4 times older than the son. However, here, 36 divided by 12 is 3, not 4. It does not make sense. So, it is a trap.

While all students agreed on this conclusion made in the session, one student, Morgan, changed her mind. Morgan stated, "I changed my mind. I think it is not a trap." When asked to explain, Morgan redrew the previous schematic representation as an attempt to disconnect father (will be) from son (will be), as depicted in Figure 4.

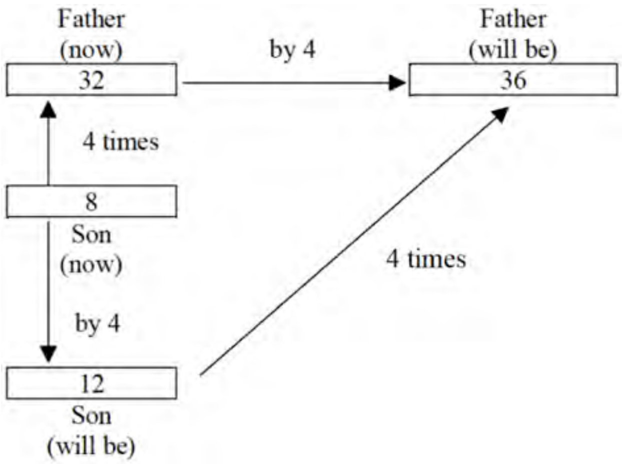
Figure 4
Morgan's First Attempt to Change the Shape of the Schematic Representation



Morgan tried to justify her argument by changing the shape of the schematic representation, believing that father will be and son will be cannot be connected in this situation. This attempt was more likely to support Chris' argument in Vignette 1. While Chris argued that we did not need to find that relationship in the problem, Morgan tried to show that it was not possible to find the relationship by changing the shape of the schematic.

> *Morgan: "We cannot connect father (will be) and son (will be) now, so we cannot say that it is a trap."*
>
> *Chris: "But we can still figure out the relationship between father will be and son will be, and it should be the same as the relationship 4 years ago."*
>
> *Morgan: "We don't need it."*
>
> *Chris: "We don't need it, but we can do it. (Chris connected father will be and son will be and noted the relationship as "4 times" as illustrated in Figure 5).*

Figure 5
Peers' Reaction to Morgan's First Attempt

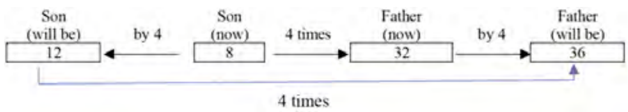


Morgan, then drew another schematic on the board, as illustrated in Figure 6.

Figure 6
Morgan's Second Attempt to Change the Shape of the Schematic Representation

> *Morgan: "Now, we cannot connect father (will be) and son (will be)."*
>
> *Jamie: "Still, we can connect them, and we can write '4 times' there" (she added a line between father will be and son will be, as shown in Figure 7)*

Figure 7
Peers' Reaction to Morgan's Second Attempt



The students rejected Morgan's conjecture that the shape of schematic representation would make a difference. However, they could not find why the

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relationship between father will be and son will be were not 4 times bigger or smaller. Thus, this continued to be a trap problem.

Vignette 3

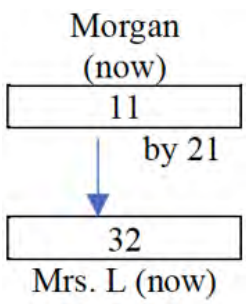
In this session, a student suggested taking a different example. Alex tried another example using Morgan's age and the age of an adult in the classroom. We drew a schematic on the board along with Alex's explanation.

> *Alex: "Morgan, how old are you?"*
>
> *Morgan: "11."*
>
> *Alex: "Mrs. L., how old are you?"*
>
> *Mrs. L.: "32."*

At this moment, the teacher asked the students how we could write this relationship in the schematic. The students answered easily (by 21), and it was recorded in the schematic (Figure 8).

Figure 8

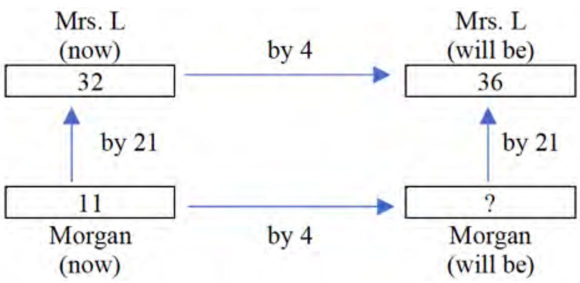
Alex's New Example



> *Alex: "Morgan, how old will you be in 4 years?"*
>
> *Morgan: "15."*
>
> *Alex: "How old will Mrs. L be in 4 years?"*
>
> *Morgan: "36."*
>
> *Alex: "OK. What is the age difference between you and Mrs. L in 4 years?"*
>
> *Morgan: "21."*
>
> *(Again, students wrote 'by 21' in the schematics.)*
>
> *Alex: "See, the age difference did not change." (Figure 9)*
>
> *Morgan: "Yes, but we don't need to figure that out."*

Figure 9

Alex's New Example: Expanded Version



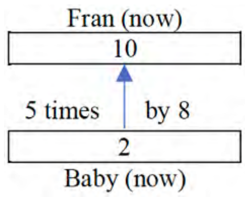
Although Alex's explanation was good and the students used the phrase "age difference," they were unable to connect the different explanations for this problem and the original problem regarding the ages of father and son in 4 years. At this point, the teacher encouraged the students to think about an additional problem in a similar context.

> *Teacher: "Alex's explanation was very interesting. Can we make another example? Can you use Fran's age and Mrs. L's baby's age this time?" (They all knew that Fran was 10 years old, and Mrs. L's daughter was two years old.) What is the relationship between Fran now and Mrs. L's daughter now?"*

Interestingly, this time, some students said, "by 8," and some of them said "5 times." The teacher wrote down both relations in the schematic (Figure 10).

Figure 10

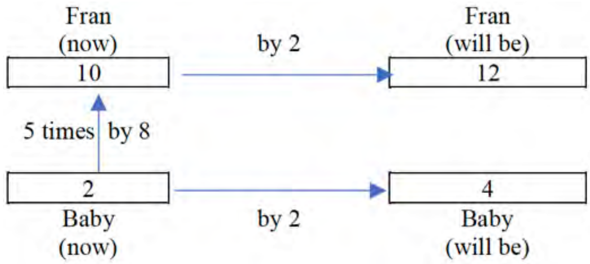
Teacher's Variation Problem



> *Teacher: "Let's think about their ages in two years."*
>
> *Students: "Fran will be 12, and Baby will be 4."*
>
> *(The teacher put the numbers in the blanks as illustrated in Figure 11).*

Figure 11

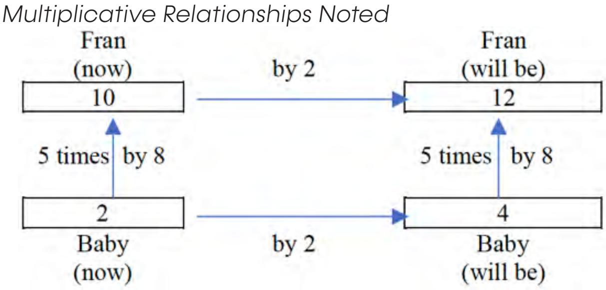
Teacher's Variation Problem: Expanded Version



> *Teacher: "Can you tell me the relationship between Fran will be, and Baby will be?*
>
> *Students: "It will be the same."*
>
> *(The teacher put both – "by 8" and "5 times" – in the schematic as illustrated in Figure 12).*

Figure 12

Teacher's Expanded Version with Additive and Multiplicative Relationships Noted



Jamie: "Wait a minute. 12 minus 4 is 8, but 12 divided by 4 is 3, not 5..."

Chris: "It is another trap!"

(But this time, nobody agreed with Chris).

Jordan: "No, it is not a trap."

Students talked about this situation for a while. Eventually, they concluded that the age difference was the same, but it was not true for the times relationship. They returned to the original problem. Students said, "The age difference of 24 is the same every year, but the "times" relationship is not like that.

Interpretations of Class Episodes

This section revisits each vignette and comments on several key points by juxtaposing the descriptive data from class episodes with our interpretations based on what we learned from other studies.

Comments on Vignette 1: Smooth Beginning, Doing Unasked Work, and a Trap!

Initially, the solution process was smooth. Examining the given multiplicative relationship between the father's age (now) and son's age (now), students identified the son's age (will be). Students then noted the additive relationships between current ages and ages in 4 years, which should have a 4-year difference. Students' recognition of these relationships resulted in incorrect answers. Jordan's proposal to further investigate the relationship between the father's and the son's ages in 4 years completely changed the way of the discussion. Two aspects were notable in terms of students' attitudes toward problem-solving. First, students had discussions on doing unasked work. Although there was some resistance to doing extra work, there was a consensus that they could do it. Rather than using the "keyword method" or adopting "number grabbing" approach (e.g., Littlefield & Reiser, 1993), being curious about unasked questions helped focus more on the structure of the problem context. Second, although there was no valid conclusion, the students acknowledged the possibility that the problem may have incomplete or contradictory information. In the end, it could not be solved (i.e., a trap problem). These two aspects show different attitudes toward problem-solving from what Schwieger (1999) points out as reasons for students' difficulties with problem-solving: (a) unwillingness to bring anything additional to the problem-solving process other than one numerical, symbolic answer and (b) belief of a singular solution and method for problem-solving.

While students showed desirable attitudes toward problem-solving, this session ended with an incorrect mathematical conclusion. The entire group agreed upon the erroneous conclusion that the multiplicative relationship ("times" relationship in students' words) between the father's and son's ages would remain the same in 4 years. Two aspects are noteworthy. First, the students were unable to identify the additive relationship between the father's age (now) and son's age (now) bound by the given multiplicative relationship between these two quantities.

Second, the students seemed to overgeneralize the invariance of relationships. In previous experiences, the students only examined problems involving the additive or multiplicative relationships; thus, the invariance of difference or ratio was preserved. In the problem context reported in this study, the students had to think about both additive and multiplicative relationships. Perhaps, their past learning experiences in techniques and language patterns triggered this overgeneralized conclusion (Mason, 2003). What happened in this vignette is evidence of students' unstable understanding of multiple relationships in one context.

Comments on Vignette 2: False Attention to the Shape of the Schematic Representations

In this vignette, Morgan attempted to prove that the problem was not a trap problem because there should be no relationship between the father's age and the son's age in 4 years. In Vignette 1, students' discussion was about "we don't need to do it" vs. "we still can figure it out." In this vignette, Morgan tried to explain that it was not possible to find the relationship between ages in 4 years by changing the shape of the schematic representations. Morgan attempted to separate these two quantities as much as possible, thinking that students could not "connect" them in the representations, and thus there should be no relationship. Other students' reactions to Morgan's idea (i.e., students were able to connect two quantities, regardless of the shapes Morgan created) eventually persuaded Morgan. Still, the idea that the problem itself was a trap remained.

Morgan's inaccurate attention to the shape of the schematic representations revealed the students' potential confusion, which had not been surfaced before. At the same time, the overall discussion in Vignette 2 showed the flexibility of the students in using schematic representations. Ng and Lee (2009) reported in their study that some students seem to treat the schematic models as an algorithm. However, it was not an algorithm learned by rote; rather, it was a problem-solving heuristic that required students to reflect on accurately depicting the information presented in word problems. Similarly, in this vignette, Morgan's false attention provided an opportunity to demonstrate students' cognitive flexibility in using schematic representations.

Comments on Vignette 3: Reconstructing the Problem

Noting the importance of distinguishing between additive and multiplicative situations, Mason (2018) suggested that encouraging students to make connections or develop problems with given structural relations would be an important area for further exploration. Students' discussion in Vignette 3 appeared to align with this suggestion. Alex's attempt to restructure the problem using different quantities helped students attend to the additive relationship. Alex focused on the difference between the ages because it was not immediately feasible to find the multiplicative relationship between the selected two quantities (32 and 11). At this point, the teacher intentionally suggested two quantities (10 and 2) so that the students could identify both additive and multiplicative relationships. Alex's and the teacher's attempts resonate with the notion of variation in structuring sense-making regarding tasks (Watson & Mason, 2006). Both aimed to expose the target mathematical structure by strategically varying some features of the problem while keeping other features. Reconstructing the schematic representations using these quantities (i.e., strategic variation) promoted students' focus of attention and encouraged them to notice what was invariant in this context. The students' willingness to reconstruct the context with the teacher's purposeful support was helpful.

Discussion and Implications

The story problem used in this article can be quickly solved using several steps of analysis and calculation. However, exploring this word problem with schematic representations took an unexpected path, resulting in a much longer exploration than expected. Some may say that this is a failure of lesson planning and its enactment. Others may question whether it was worth spending a long time discussing only one problem. While admitting that the presented class episodes in this study were atypical in terms of the duration of the discussion, we saw the value of allowing such an atypical learning process to occur.

Regarding the mathematical content, the students' lengthy investigation was fueled by their initial confusion and curiosity about the additive and multiplicative relationships and the related invariant and variant relationships. Considering the importance of constructing multiplicative reasoning for students' learning of mathematics throughout the middle grades and beyond (Zwanch & Wilkins, 2021), this was a timely opportunity for students to think about different relationships among quantities. Although it took longer than planned, it was worthwhile because it offered students a space to express their confusion, demonstrate their knowledge, test conjectures, construct a similar but different problem context, and eventually recognize multiple relationships within the particular problem context and general contexts. Additionally, the exploration revealed students' unexamined assumptions about the use of schematic representations.

We particularly noted that there were several instances where students themselves exhibited intellectual perturbations (Harel et al., 2014) or cognitive conflicts. Without such student-generated perturbations and conflicts, the proposed problem might have ended up as a computational problem. What if Jordan did not ask to find the unasked question in Vignette 1? What if Morgan did not pay her false attention to the shape of the schematic representations in Vignette 2? What if Alex did not suggest restructuring the problem using different examples? What if the teacher did not provide strategic variation in the quantities to shift students' attention? Such unexpected questions helped the students focus on the structures and relationships rather than just performing calculations.
Consistent with previous studies (e.g., Lehrer et al., 2000; Terwel et al., 2009), the findings of this study revealed that students could resolve their confusion through collaboration. While the teacher did not ask students to use a particular schematic representation, they constructed and reconstructed the representations through group discussion to reveal the difference between the father's and son's ages concerning additive and multiplicative relationships. Therefore, students could identify important mathematical elements in additive and multiplicative word problems and explain structural relationship of these problems using schematic representations. These findings revealed that teachers might provide students with mathematical tools to support their investigation, reasoning, and justification.

These findings also highlighted the teachers' roles in solving complex word problems with representations. Using mathematical tools, such as representations, alone could not guarantee students' mathematical learning (Lehrer et al., 2000). As Mason (2018) claimed "what seems to matter most is not the apparatus itself," (p. 332) but how teachers and students use them. If the apparatus is not used properly, its use might lead to rote learning. Therefore, teachers should be cautious when using schematic representations in mathematics classrooms. For example, as shown in this study, teachers could first teach their student types of schematic representations that they could use and explain the meanings of individual representations. Next, teachers could provide challenging problems and ask students to justify their reasoning by presenting additional questions. These processes might arouse students' curiosity and help them manipulate the quantities to reveal the mathematical structure of the problem.

As such, teachers should create a mathematics learning environment that allows students to investigate, reason, and justify (Depaepe et al., 2010; NCTM, 2000). As active investigators, teachers should believe in their students' mathematical abilities and refrain from transmitting mathematical knowledge and algorithms (NCTM, 2014). When teachers consider their students as passive listeners, students are unlikely to present unasked questions and investigate them. Moreover, teachers should respect students' authority in learning mathematics and create a classroom culture where all students' answers are respected (Cobb & Hodge, 2011). While some students gave incorrect answers in this study, most students did not criticize their ideas, and the teacher did not directly correct them. Instead, the students attempted to justify their arguments using representations and discussion, while the teacher supported their argumentation. Therefore, mathematics educators should be concerned with their classroom culture, particularly whether it facilitates or hinders students' understanding of mathematical relationships and structures in learning mathematics (Davydov, 1990; Mason, 2003; Zwanch & Wilkins, 2021).

The study has some limitations. Given that this study examined a small group of students in a single classroom, the findings of the study could not be generalized to other contexts. Therefore, studies with larger samples might yield more generalizable results. However, we hope that these classroom episodes give teachers and teacher educators ideas to explore more optimal learning environments for students to raise awareness about the mathematical structure and relationships in solving word problems.

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