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Representations in the Learning of Sequences and Regularities by Third-Grade Portuguese Students

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Abstract

Due to their abstract nature, representation of mathematical concepts through different registers favors their understanding. In the case of "sequences and regularities", it becomes propitious the exploration of different registers of representation in the institution of topics, such as term, order, formation law, and generating expression. Considering these assumptions, a teaching experiment was performed to understand the contribution of multiple representations in the learning of "sequences and regularities" by 3rd-grade students. The study adopted a qualitative methodology and the findings of the study reveal that students initially presented a smaller variety of representations, which increased during the teaching experience. Students showed a greater preference for pictorial representations and made explicit connections between different representations throughout their resolutions. Pictorial representations and tables allowed close and distant generalizations, the determination of the formation law, and the generating expression. The greatest difficulties of the students resulted from the interpretation of the statements of the proposed tasks, which were also evident in the representation (natural language), showing a greater number of incorrect answers. This result shows that some students still have difficulties in justifying their reasoning, either in writing or orally.

Keywords:

Third-Grade Students, Difficulties, Teaching, and Learning Sequences and Regularities, Representations

Introduction

Mathematics is characterized by the abstract nature of the objects that constitute it. To a certain extent, mathematics serves as a justification for the difficulties that some students have in its learning. Such difficulties constitute an obstacle to understanding certain mathematical concepts (Canavarro & Pinto, 2012), which impel students to manifest an attitude of rejection to this discipline. One way to alleviate such difficulties emerges from exploring different representations of mathematical concepts. Representations are a configuration that translates something, such as an object, an idea, or a mathematical content (Goldin, 2008). Duval (2012) argues that the lack of understanding of representations necessarily causes a misunderstanding of the mathematical content. This finding aroused our interest in exploring multiple representations in learning "Sequences and Regularities". The multiple representations of mathematical objects can help students give shape and visibility to their thoughts and communicate their ideas. An example of this can be seen in the content "Sequences and Regularities", which is favorable to the exploration of different registers of representation in the establishment of its specific topics, such as term, order, formation law, and generative expression. Considering such assumptions, this study aims to investigate the contribution of multiple representations in the learning of 'Sequences and Regularities' by students in the 3rd grade.

Sequences and Regularities in the First Years

The exploration of "sequences and regularities" is transversal to the different academic years, as suggested in the Curriculum Management Guidelines: "[the content 'sequences and regularities'] should be worked in every year of schooling to allow a progressive development of algebraic thinking in students, in particular the ability to generalize" (Ministry of Education and Science [MEC], 2016, p. 6). Consciously or unconsciously, the human being is constantly looking for new patterns (Baratta-Lorton, 2009). Several authors (Devlin, 2002; Orton, 1999; Sawyer, 1995; Steen, 1998), have defined mathematics as the "science of patterns", as when a pattern is identified, there will inevitably be the possibility of doing math. Vale and Pimentel (2010) explained that any interaction the mind makes with patterns establishes relationships, even in everyday activities such as reading and shopping. Establishing relationships is essential in the students' path to counteract the tendency of being "limited to remembering a set of facts, concepts and procedures in isolation" (Vale & Pimentel, 2010, p. 33).

According to Vale (2009), "sequences and regularities" enhance the ability to abstract and communicate using multiple representations. Moments when students are encouraged to communicate and share their ideas allow them to develop their reasoning (National Council of Teachers of Mathematics [NCTM], 2000). For Lannin et al. (2011), reasoning in mathematics is an evolutionary process that involves "conjecturing, generalizing, investigating why, and developing and evaluating arguments" (p. 13). According to some studies (Branco, 2008; Pimentel et al., 2010; Rivera & Becker, 2009) working with "sequences and regularities" enhances algebraic reasoning ability, an important aspect for learning Algebra, which "is very useful for the student in his/her everyday life and further studies" (Borralho et al., 2007, p. 193). Hence, all students should learn Algebra (NCTM, 2000; Kiziltoprak & Köse, 2017), as early algebraic thinking and working with patterns encourage students to identify relationships and make generalizations. Pimentel et al. (2010) and Ponte (2005) argued that it is necessary to formulate generalizations starting from sequences and regularities, from the first years of schooling, as generalizing sequences students develop their algebraic thinking (Radford, 2010). In the study of sequences, Ponte et al. (2009) reported that one of the biggest obstacles for students arises in studying repetitive pictorial sequences, which is due to a lack of understanding, as only after assimilating the sequence will they be able to generalize. The generalization difficulty in this type of sequence is due to the relationship between term and order. On the one hand, children at an early learning level do not understand that sequences can be extended in both directions (Warren, 2005). On the other hand, children with more advanced levels have difficulty in the symbolic writing of a generalization and the attribution of a meaning to the letters of a numerical expression framed in a functional context (Saraiva & Pereira, 2010). In exploring growing pictorial sequences, Lannin (2003) reported that several children only use the additive strategy in describing generalizations, revealing difficulty in using other strategies. This can be explained by the fact that students only focus on presented dataset and do not understand the relationship between the datasets (Warren, 2005). Rivera and Becker (2008) and Ponte and Velez (2011) advocated that students had difficulties in justifying their reasoning, either in writing or orally, and they failed to formulate a valid justification for the generalization. The exploration of "sequences and regularities" also enables students use of multiple representations, such as gestures, tables of values, letters, and natural language (Warren, 2009). They are the main instruments students use in the first years of schooling (Alvarenga & Vale, 2007; Pimentel et al., 2010).

Mathematical Representations

Mathematical representations translate mathematical objects as these "are not directly accessible to immediate intuitive perception or experience" (Duval, 2012, p. 268), unlike everyday palpable objects. As such, it is necessary to create representations that signify, produce, and resemble them so that it is possible to reason for them and "give visibility to what we think" (Canavarro & Pinto, 2012, p. 53). According to Duval (2012), there are several ways to represent mathematical objects: a number, a function, a vector, and figures. However, a mathematical representation only makes sense when observed in a certain context, with defined rules and meanings (Ponte & Velez, 2011). For example, thinking of the number 3, it could refer to the "three little pigs", or represent something immaterial like the cardinal of a set of three elements.

Goldin (2008) claimed that understanding a mathematical concept implies that the subject can distinguish the mathematical objects from the representation that makes it accessible. If students confuse mathematical objects with the representation made of these same objects. In that case, this can lead to "a loss of understanding and acquired knowledge quickly becomes unusable throughout their learning context" (Duval, 2012, p. 268). Goldin (2008) distinguished two types of representations: external and internal. External representation is palpable and observable and can be found on paper, screens, or other support. They include mathematical symbols (symbolic writing); algebraic writing; pictorial representations (figures, images, and icons); objects, and verbal language (written). Goldin differentiated some external representations observed in a classroom context: mathematical symbols, verbal language, figures, and objects. As for internal representation, Goldin and Kaput (1996) considered cognitive constructions formed in students' minds. These are "mental images built on reality, referring to cognitive models, concepts or mental objects, and therefore not observable" (Almeida & Viseu, 2002, p. 195). Duval (2012) presented a similar thesis, distinguishing two types of representations: mental representations, where the subject uses a set of personal images that help give meaning to the mathematical objects in question; and semiotic representations, which are productions constituted using symbols belonging to a system of representations. It can, thus, be concluded that semiotic representations result from an exteriorization of mental representations. For Vygotsky (2008) and Piaget (1990), mental representations are an interiorization of what the subject assimilates and that depends on the interiorization of semiotic representations.

Ponte and Velez (2011) showed that, for a long time, only symbolic representations were worked in schools and, as emphasized by Dufour-Janvier et al. (1987), external representations are introduced in the school, "there appearing to be few opportunities for students to explore their numerical representations" (p. 119). According to NCTM (2000), children should learn conventional forms of representations, but they should also be guided to develop and create their representations that will support their learning.

The "iceberg model" developed by researchers at the Freudenthal Institute (Webb et al., 2008) is a metaphor illustrating the students' experience of the wide range of representations. At the top of the iceberg, there is a formal representation and at the submerged and broader part of it appear pre-formal and formal representations. Webb et al. (2008) emphasized three phases of learning: the informal phase, the pre-formal phase, and the formal phase. In the informal phase, the concepts are approached in a familiar context in a concrete way (with informal representations, such as figures, drawings, etc.). In the pre-formal phase, complexity increases, and representations appear more abstractly (for example, number lines). Finally, the formal phase implies that students resort to formal representations. When the students reach the formal stage, it does not mean that they will never resort to informal or pre-formal representations. Goldin (2000) and Webb et al. (2008) considered that students can resort to these representations again in moments of insecurity or confusion.

According to Duval (2012), "the use of many registers seems to be a necessary condition so that mathematical objects are not confused with their representations" (p. 270). The primary objective of representations is to give access to the represented object, which, for Duval (2012), brings together a set of necessary conditions promoting conceptual apprehension, implying the coordination of multiple representations. First, students must form an identifiable representation through the composition of a text, a drawing, a scheme, geometric figures, formulas, etc. Afterward, students should process, that is, transform a given representation into the same record in which it was initially created. This can be conceived through calculus, a way of handling symbolic expressions. Finally, students can perform a conversion, which means transforming this representation into a new one, keeping all or part of the content of the initial representation. It is important to emphasize that cognitive activities between conversion and treatment are distinct and independent, being two radically different types of semiotic representation transformation (Duval, 2012). A 2nd-grade student, for example, might look at a drawing with four circles and add those four circles to another three in a new drawing. The student will be able to get the sum of four and three without converting the drawing to another representation, such as for the numerals 4 and 3 or even for the symbolic writing of "4 + 3". However, students capable of converting the mentioned representations will show that they have mastered the mathematical content as they can apply it through multiple representations. The hypothesis underlying Duval's theory (2012) is that the complete understanding of a concept occurs in the coordination of at least two representation registers, and this manifestation occurs through cognitive conversion activity. The recommendations in the Curriculum Management Guidelines (MEC, 2016) are in line with Duval's theory, suggesting that the teacher should allow students to appropriately, consistently, and gradually use the "symbolic representation of data, ideas, concepts, and mathematical situations in various forms" (p. 16) and emphasize the importance of students being able to "pass information from one representation to another, to obtain different perspectives of the same situation" (p. 16). Carraher and Schliemann (2007) highlighted that working with "sequences and regularities" allows exploring

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geometric and numerical representations, where the ordinal positions (order) are related to the number of elements in that position (term). Alvarenga and Vale (2007) reinforced that "students, from the early years of schooling, can and should be encouraged to observe patterns and represent both geometrically and numerically, starting the study of algebra in a strongly intuitive and informal way" (p. 2). Students must be aware that there are multiple representations of the same situation and must "be able to move from one (representation) to the other understanding that the rules are equivalent" (Vale & Pimentel, 2005, p. 15). In working with "sequences and regularities" this allows them to "see," that is, understand, the existing pattern or relationship (Orton, 1999), regardless of how it is presented to them.

Methodology

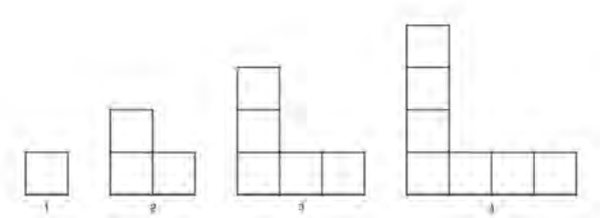
Considering the importance of representations in the learning of sequences and regularities, the study investigated the contribution of multiple representations in the learning of "sequences and regularities" in third-grade students. The study was conducted in the academic year of 2019–2020 in the context of the supervised teaching practice of the first author of this work. The study participants were 26 3rd-grade students, nine boys and 17 girls. All the participants are eight years old. More than half of the class (58.3%) indicated mathematics as the subject in which they had more difficulty; however, most students (66.7%) reported that they liked it. Regarding performance in mathematics, most students obtained "good" and "very good" levels and no student obtained insufficient level in their assessment at the end of each period of the school year. Given the nature of the objective outlined, this study adopted a qualitative and interpretive approach to understand the students' mathematical activities in solving the proposed tasks in the classroom context (Bogdan & Biklen, 1998). Twelve tasks were developed and explored in four classes to challenge students to develop their strategies, using their previous knowledge and multiple representations in the topic "sequences and regularities", with the students organized into 12 work pairs. Data were obtained from the students' resolutions and collected before their discussion in the class group. In this work, the study focused on the analysis of three tasks only. The analysis that results from the resolution of the tasks focuses on the following dimensions: (i) identification of numerical regularities; (ii) determination of the generating expression of a sequence; and (iii) transformation of representations. In each dimension, the types of representations used, and possible connections between the different representations are also analyzed. The strategies used by students in obtaining the following terms from the sequences are also analyzed. These strategies are classified using the typology of Ponte et al. (2009) as follows: (1) "representation and counting strategy", where students represent all the terms of the sequence until they determine the term that was asked of them; (2) "additive strategy", where students perceive the change that occurs from term to term and from there obtain the next term; (3) "whole object strategy", in which students, through one term, determine other multiple terms of that term; (4) "term decomposition strategy", where students decompose the term, thus realizing how it was constructed.

Presentation and Analysis of Results

Identification of Numerical Regularities

In determining the formation law of a sequence, the students started by solving Task 1, which asked them to identify the regularity that characterizes obtaining a given term from its predecessor and establish close generalizations.

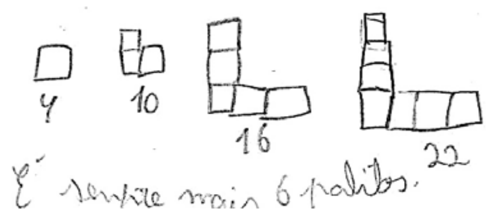
Task 1. Look at the following sequence of figures. In each figure, each square is formed by four equal toothpicks.



1. What happens to the number of toothpicks from one term to the next?
2. What is the formation law regarding the number of sticks in the sequence?
3. How many toothpicks are there in the order 6 figure? And in the order 8?

From the analysis of the answers given to each question, most of the students answered correctly in the first two questions. As for the first question, nine pairs correctly illustrated the formation law of the sequence, as exemplified by the answer given by pair P10, which represented each term and understood that six more toothpicks would be needed to form each subsequent figure (Figure 1).

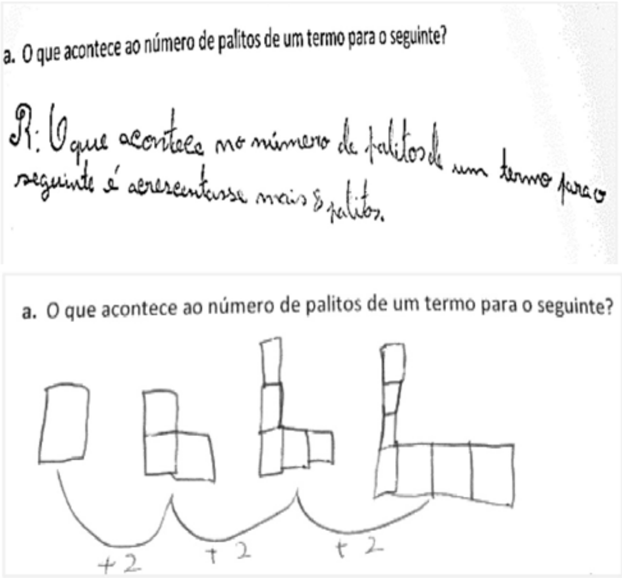
Figure 1
The Correct Answer of Pair P10 to Question 1 of Task 1 ["It's Always Six More Toothpicks"]



According to the resolutions of three pairs of students who answered incorrectly, two pairs miscounted the number of toothpicks to be added, and one pair considered the number of squares that increased from one figure to the next (Figure 2).

Figure 2

The Incorrect Response of Pairs P11 and P6 to Question 1 of Task 1

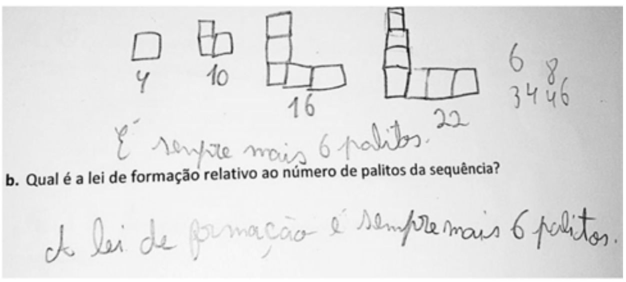


(Note: Pair P11's answer to Question 1 of Task 1: "What happens to the number of toothpicks from one term to the next is to add eight more toothpicks")

In the second question, nine out of 12 pairs of students answered correctly and reported that they related the formation law of a sequence to the phenomenon that occurs from one term to the next, successively, as illustrated by the response of pair P10 (Figure 3).

Figure 3

The Correct Answer of Pair P10 to Question 2 of Task 1

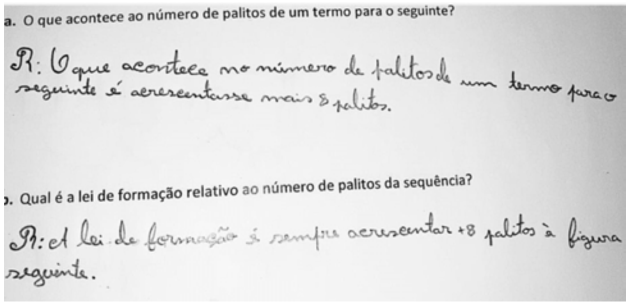


(Note: "The formation law is always more six toothpicks")

Two pairs gave partially correct answers as they made the correct association of the formation law to the phenomenon that happens next; however, as they incorrectly answered the previous question, the formation law presented does not correspond to the sequence under study, such as the response of the P11 pair suggests (Figure 4).

Figure 4

The Partially Correct Answer of Pair P11 to Question 2 of Task 1

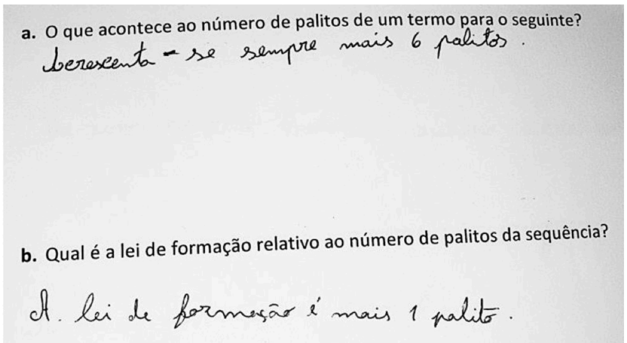


(Note: "The formation law always adds more eight toothpicks")

One of the pairs gave an incorrect answer. In the first question of the task, the pair understood what happens from one term to the next in the sequence; however, they could not relate this event as the formation law of the sequence, so they presented a different answer to the second question (Figure 5).

Figure 5

The Incorrect Response of Pair P2 to Question 2 of Task 1



(Note: Pair P2's answer to Question 1 of Task 1: "add six more toothpicks."] [Pair P2's answer to Question 2 of Task 1: "The formation law is more 1 toothpick")

Finally, in the third question, the students were asked to identify the terms of two distinct nonconsecutive orders. Three strategies used by the students to solve the problem stand out: the "term decomposition strategy," the "representation and counting strategy" and the "additive strategy." The "additive strategy" was presented by two pairs, who realized they had to successively add the formation law of the sequence (+6) to obtain the following terms. The P2 pair managed to achieve valid conclusions (Figure 6).

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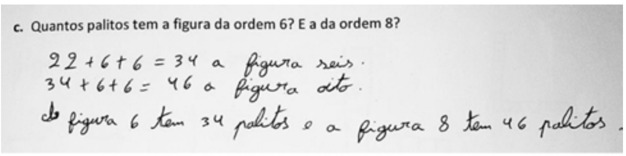
Figure 6

The Correct Answer of Pair P2 to Question 3 of Task 1

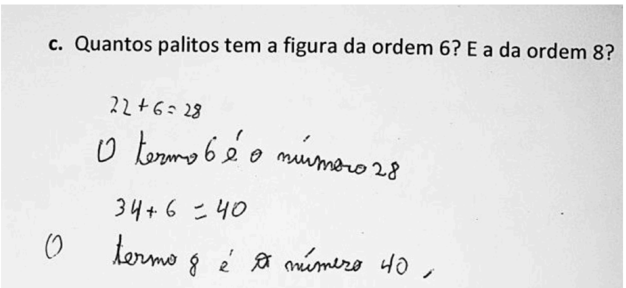


(Note: "Figure 6 has 34 toothpicks and Figure 8 has 46 toothpicks")

Pair P3 used this strategy and got a partially correct answer. This pair resorted to symbolic writing but could not visualize the sequence and realized that the orders requested were not consecutive nor immediately following those presented (Figure 7).

Figure 7

The Partially Correct Answer of Pair P3 to Question 3 of Task 1



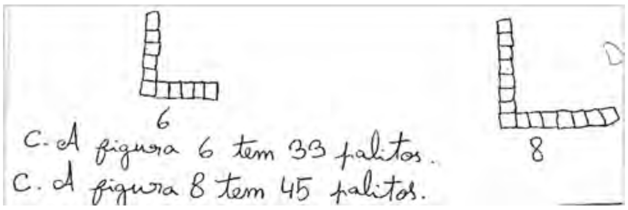
(Note: "The term 6 is the number 28. The term 8 is the number 40.")

As shown in Figure 7, the pair P3 added to the number of order fourth sticks (22) six units and assumed that the number of order sixth sticks would be 28. This shows that the representation used did not allow students to visualize the sequence and determine the following term (order 5 term) only. One can understand that in the second part of the task, in determining order 8, students started from the number of toothpicks they considered to be in figure 6 (28) and added six units, obtaining the order 7 term. To this result (34), they added another six units, reaching the order 8 term. Although the result is incorrect, students already showed a certain understanding of the formation law of the sequence. The choice of symbolic writing probably made it difficult to solve the task as it did not allow them to visualize the continuity of the sequence, as the degree of abstraction of students in this age group is still low.

As for solving strategies, it was observed that the "term decomposition strategy" was used by three pairs. Students represented, through pictorial representations, only the terms they intended to determine, which indicates that they understood how the figure was constructed. Students, through pictorial representation, were able to generalize a situation, realizing that the number of the figure is equal to the number of squares horizontally and the number of squares vertically; however, although pairs were able to represent the sequence correctly, they were not able to draw valid conclusions, probably due to little experience with the manipulation of pictorial representations. Students represented the terms correctly but were not able to accurately count the number of toothpicks in the figure. Figure 8, referring to the resolution of pair P9, depicts a counting error of one unit per figure, which translates into a partially correct answer.

Figure 8

The Partially Correct Answer of Pair P9 to Question 3 of Task 1

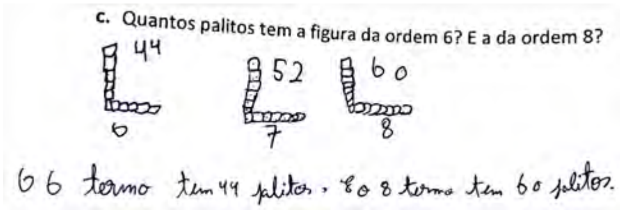


(Note: "Figure 6 has 33 toothpicks. Figure 8 has 45 toothpicks")

The use of the "representation and counting strategy" was also observed. It implies that students represent all the terms in the sequence until they obtain the desired term (Ponte et al., 2009). This strategy was used by a pair who incorrectly answer the question, and it may, once again, be a matter of lack of interpretation of the pictorial representation (Figure 9).

Figure 9

The Incorrect Response of Pair P4 to Question 3 of Task 1



(Note: "The term 6 has 44 toothpicks. And the term 8 has 60 toothpicks")

According to Figure 9, pair P4 correctly represented the terms, but the toothpick count is quite different from reality, with the order sixth term exceeding 10 units and the order eighth term exceeding 14 units.

Representations

In the analysis of the resolutions to the proposed questions, it is verified that the different pairs resorted to multiple representations (Table 1).

Table 1
Frequency of the Types of Records Used by Students in Task 1 (n=12)

	Response types														
	C					PC					I				
Question	PR	NL	S	SW	T	PR	NL	S	SW	T	PR	NL	S	SW	T
1	3	6	0	0	0	0	0	0	0	0	0	2	1	0	0
2	0	9	0	0	0	0	2	0	0	0	0	1	0	0	0
3	1	1	0	1	0	2	1	0	1	0	2	2	0	0	0
Total	4	16	0	1	0	2	3	0	1	0	2	5	1	0	0

Note: PR: pictorial representation; NL: natural language; S: schemes; SW: symbolic writing; T: tables. C: Correct answer; PC: Partially correct answer; I: Incorrect answer.

Table 1 shows that students mostly used natural language. The use of this type of representation allowed half of the pairs to obtaining a correct answer to the first question of the task. The predominance of this representation may be because the task is presented through a pictorial sequence, and the students do not feel the need to proceed with the treatment of this representation. Students may have understood the question only by analyzing the pictorial sequence, presenting their answer in natural language, without explaining the reasoning that resulted in the same answer.

The pictorial representation, being an informal representation, despite being less used, resulted only in correct answers. This may mean that, as it is a more visual representation, it facilitated understanding, which is corroborated by Vale (2009), who argues that exploring generalization through different representations of visual support is essential for understanding mathematical topics under study.

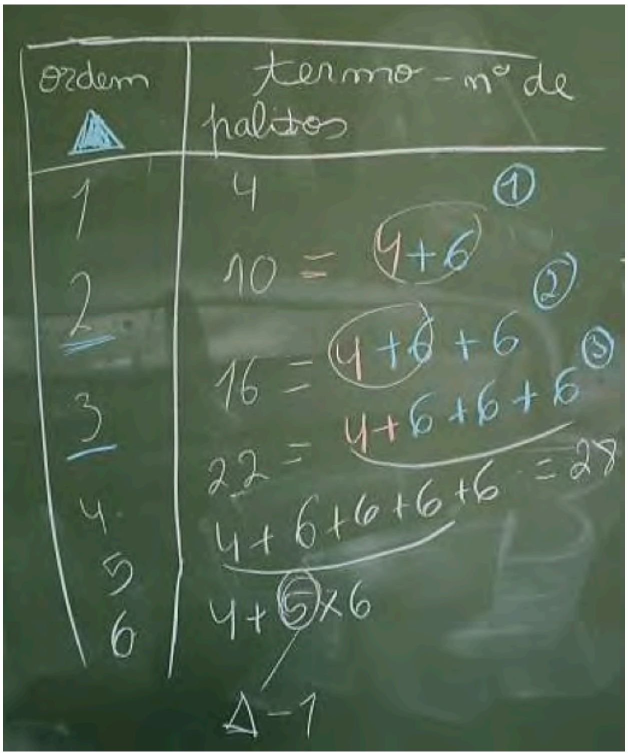
A pair of students chose to present their resolution with a pre-formal representation, the schema; however, this greater complexity in the representation may have led to confusion in the interpretation of the task, which led them to obtain an incorrect answer.

In the second question of the task, all pairs used natural language to present their answers, with nine pairs getting correct answers, in the third question, within the representations used, no representation stands out, as only three pairs managed to get a correct answer.

Connections between representations

A representation that was not used by any student was introduced in the discussion of resolutions in class. The aim was to offer new possibilities and develop the students' capacities in the treatment and conversion of representations. As the different representations previously used by the students were already registered on the board, (e.g., pictorial representations and symbolic writing), these same representations were used to build a table to organize the data (Figure 10).

Figure 10
Table Built in the Blackboard



This moment provided the class with the development of greater resourcefulness in the connections between multiple representations and allowed students to understand the mathematical content, even when represented in different ways.

A geometric figure (triangle) was assigned to the order to familiarize students with algebraic expressions. In determining the order sixth term, the numeral 5 of the symbolic expression in the table was converted using the following formula: "Δ-1." The conversion of representations and their respective treatment, in working with "sequences and regularities" allow students to understand the mathematical objects under study and develop the capacity for generalization and abstraction (Vale, 2009).

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Determining the Generating Expression of a Sequence

In determining the terms of a sequence defined by a formation law that allows obtaining each term from the previous ones, knowing the first terms, students solved Task 2.

Task 2. In a school canteen, four students can sit at a table. The tables are all the same. six students can seat, when two tables are combined.

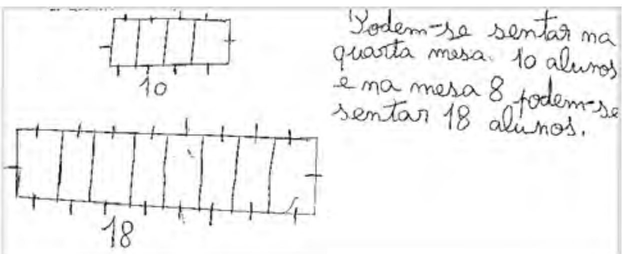
1. *How many students can sit at five and 10 tables? Justify your answer.*
2. *How many tables will be needed for 20 students to seat? Justify your answer.*
3. *Could there be a table with 31 students and all seats occupied?*

Only four of the 12 pairs transformed the presented representation into a new representation. Two pairs converted the statement to a pictorial representation, one pair to a schema and one pair to a table with symbolic writing, meaning that most pairs answered the questions in Task 2 correctly.

Ten pairs answered the first question correctly, while the other two gave a partially correct answer. However, different strategies emerged (Ponte et al., 2009). Eight of the 12 pairs developed the "term decomposition strategy." Although it is not explicit in their resolution, it is clear that students have decomposed the terms to understand that each one presents the same number of tables as the order to which it corresponds. In this case, when students draw the terms of the orders only, they aim to determine them with the number of tables corresponding to the desired order, as illustrated by the response of pair P5 (Figure 11).

Figure 11

The Correct Answer of Pair P5 to Question 1 of Task 2

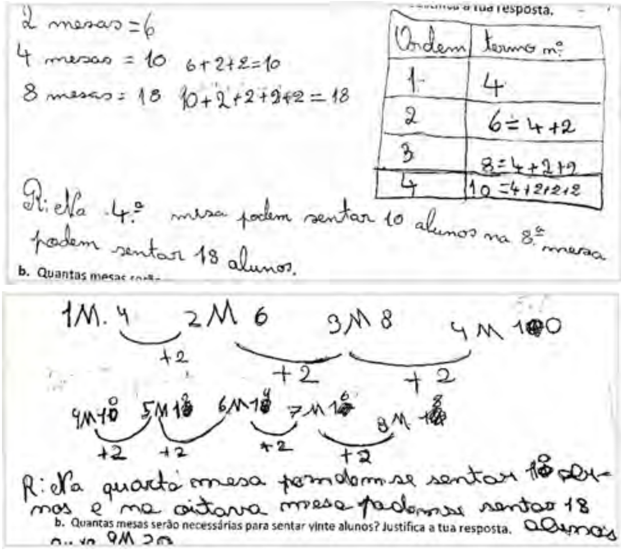


(Note: ["In the fourth table can seat 10 students, and in the table 8 can seat 18 students"])

The "additive strategy" was used by two pairs who obtained the correct answer by successively adding two to the previous term, as this is the formation law of this sequence, as illustrated by the answers of pairs P11 and P9 (Figure 12).

Figure 12

The Correct Answer of Pair P11 and Pair P9 to Question 1 of Task 2

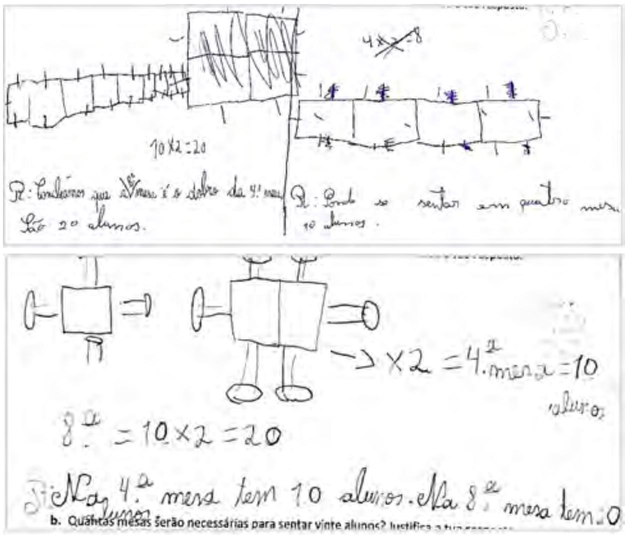


(Note: [Pair P11's answer to Question 1 of Task 2: "In the fourth table can seat 10 students, in the table 8 can seat 18 students."] [Pair P9's answer to Question 1 of Task 2: "In the fourth table can seat 10 students, and in eighth table can seat 18 students"]).

Finally, two pairs chose to resort to the "whole object strategy," which implies the determination of a term starting from a multiple term. However, this strategy is not feasible in this case, because, despite the number of tables doubling, the number of people seated does not correspond to the double, since two more places have to be considered at the head of the tables. Thus, students were able to answer the number of people who can sit at four tables correctly, but wrongly as to the number of people who can sit at eight tables, as illustrated by the responses of pairs P6 and P1 (Figure 13).

Figure 13

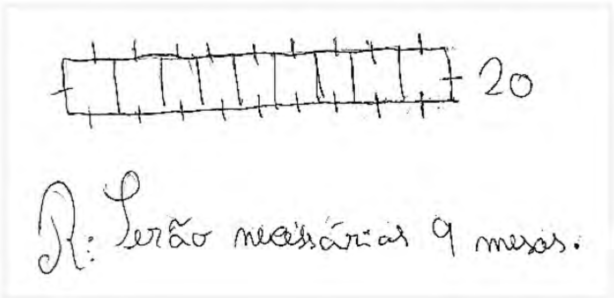
The Correct Answer of Pair P6 and Pair P1 to Question 1 of Task 2



(Note: [Pair P6's answer to Question 1 of Task 2: "We conclude that the eighth table is the double of the 4th. There are 20 students."] [Pair P6's answer to Question 1 of Task 2: "Can seat in four tables 10 students."] [Pair P1's answer to Question 1 of Task 2: "At the fourth table, there are 10 students. At the eighth table there are the double")

Nine pairs managed to determine the correct solution in the second question, and only three had an incorrect answer. Of the correct answers, seven pairs only drew the figure corresponding to the solution to the problem, so they may have obtained this result by trial and error, as illustrated in Figure 14.

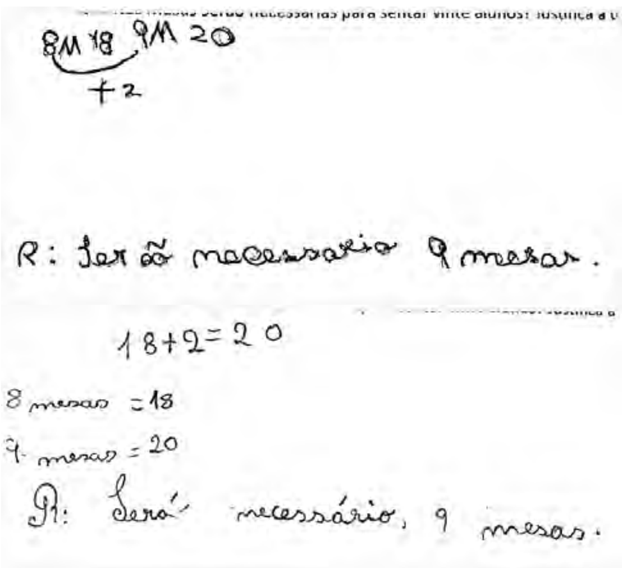
Figure 14
The Correct Answer of Pair P7 to Question 2 of Task 2



(Note: "Will be needed nine tables?")

Two other pairs obtained the correct answer through the "additive strategy", as to each term the formation law is added to obtain the next term (Figure 15).

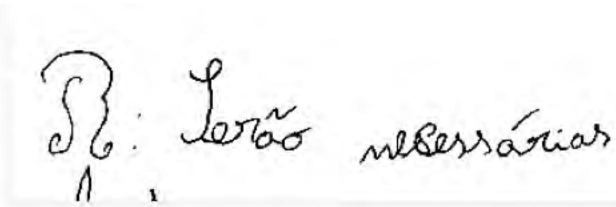
Figure 15
The Correct Answer of Pair P9 and Pair P11 to Question 2 of Task 2



(Note: Pair P9's answer to Question 2 of Task 2: "Will be needed nine tables"] [Pair P11's answer to Question 2 of Task 2: "Will be needed nine tables?")

Finally, two pairs had an incorrect answer, determining terms where the condition of sitting only 20 students is not verified, as shown in the answer of pair P7 (Figure 16).

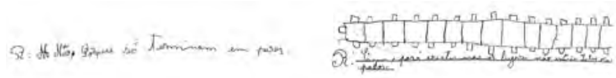
Figure 16
The Incorrect Answer of Pair P7 to Question 2 of Task 2



(Note: "Will be needed 10 tables to seat 20 students?")

In the third question, it is confirmed that 10 pairs answered the question correctly. Students found different conditions justifying it was not possible for 31 people to seat and occupy all seats. For example, pair P6 noticed that it was not possible because the terms are all equal, so 31 does not belong in the sequence, and pair P12 chose to design the tables, ending up verifying the impossibility of seating 31 people and having all seats occupied (Figure 17).

Figure 17
The correct Answer of Pair P6 and Pair P12 to Question 2 of Task 2



(Note: Pair P6's answer to Question 2 of Task 2: "No because they only end in even numbers"] [Pair P12's answer to Question 2 of Task 2: "Yes there may be but the seats are not occupied")

Representations

Multiple representations were used in solving the questions in Task 2, which prompts us to analyze which ones contributed the most for the students to solve the presented task (Table 2) effectively.

According to Table 2, it appears that most of the correct answers result from the pictorial representation.

Connection between representations

When discussing the resolutions of Task 2, the students were asked about the relationship between the term and the order of the sequence. This interaction resulted in a new representation: algebraic writing (Figure 18).

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Table 2

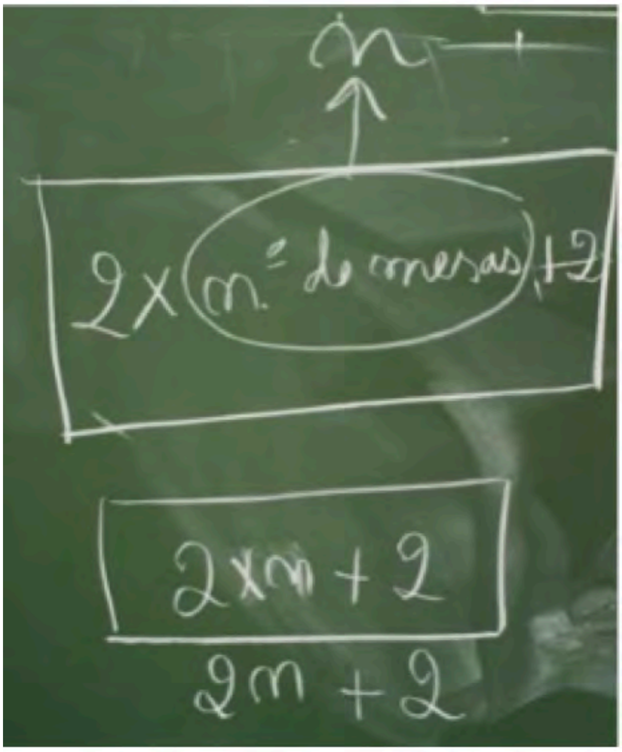
Frequency of the Types of Records Used by Students in Task 2 (n=12).

	Response types														
	C					PC					I				
Question	PR	NL	S	SW	T	PR	NL	S	SW	T	PR	NL	S	SW	T
1	8	0	1	1	1	1	0	0	1	0	0	0	0	0	0
2	7	0	1	1	0	0	0	0	0	0	2	0	0	1	0
3	5	6	1	0	0	0	0	0	0	0	0	0	0	0	0
Total	20	6	3	2	1	1	0	0	1	0	2	0	0	1	0

Note: PR: pictorial representation; NL: natural language; S: schemes; SW: symbolic writing; T: tables. C: Correct answer; PC: Partially correct answer; I: Incorrect answer.

Figure 18

Generating Expression of the Sequence That Translates the Number of Students Who can Sit at a Set of Tables

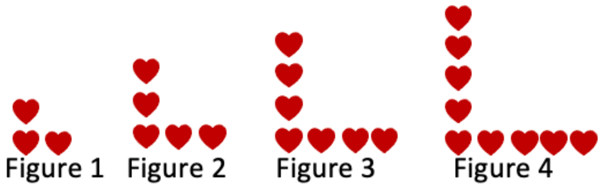


Although students are still at an embryonic stage in the development of their algebraic thinking, given the interest and resourcefulness shown by them in the proposed tasks, we think it is opportune to explore the representations that emerged to obtain the algebraic expression of the sequence, replacing the "number of tables" using the following formula: "n".

Transformation of Representations

To determine a formation law compatible with a partially known sequence and formulate it in symbolic language, students explored Task 3.

Task 3. Look at the following sequence of figures

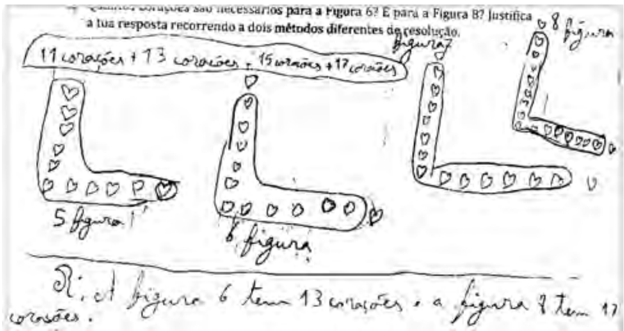


1. *How many hearts are needed for Figures 6 and8? Justify your answer using two different resolution methods.*
2. *Is there a picture with 100 hearts? Justify your answer.*
3. *Construct the generating expression of the terms of the sequence.*

Only a pair of students gave partially correct answers to the first question, while the rest answered correctly. Three pairs resorted to the "representation and counting strategy" to obtain the desired terms, as suggested by the resolution of pair P12 (Figure 19).

Figure 19

Correct Answer of Pair P12 to Question 1 of Task 3

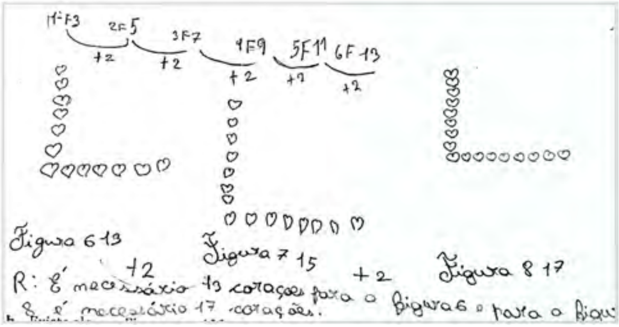


(Note: "Figure 6 has 13 hearts and Figure 8 has 17 hearts")

The "additive strategy" was applied by seven pairs, adding the formation law to each term to obtain the next term. Six pairs obtained the correct answer, as illustrated by the answer of pair P9 (Figure 20).

Figure 20

The correct Answer of Pair P9 to Question 1 of Task 3

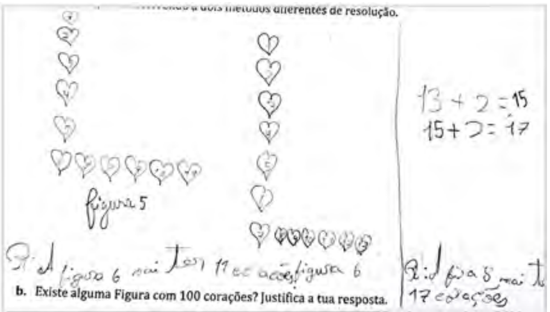


(Note: "It is necessary 13 for the figure 6 and for the figure 8 it is necessary 17")

However, pair P6, another pair that opted for this strategy, translated these representations into a partially correct answer, despite providing a correct pictorial representation and symbolic writing (Figure 21).

Figure 21

The Partially Correct Answer of Pair P6 to Question 1 of Task 3

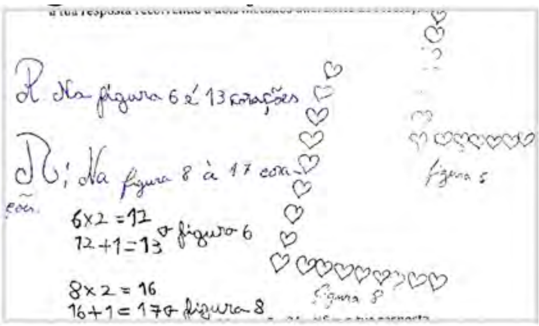


(Note: "Figure 6 will have 11 hearts. Figure 8 will have 17 hearts")

Finally, the correct answers include the "term decomposition strategy" used by three pairs. For example, P3 solved the question correctly by dividing the figure into three parts and realizing that the number of hearts is twice the number of the figure plus one (Figure 22).

Figure 22

The Correct Answer of Pair P3 to Question 1 of Task 3

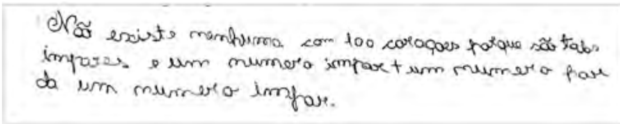


(Note: "In the figure 6 there are 13 hearts. In the figure 8, there are 17 hearts")

Nine of the 12 pairs answered the second question correctly, as suggested by the resolution of pair P5 (Figure 23).

Figure 23

The Correct Answer of Pair P5 to Question 2 of Task 3

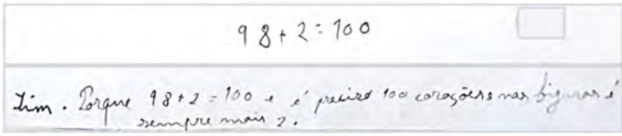


(Note: "There are none with 100 hearts because they are all odd, and an odd number + an even number gives an odd number")

Two pairs of students answered the question incorrectly, probably because they did not distinguish the characteristics of the sequence, as illustrated by the resolution of pair P12 (Figure 24).

Figure 24

The Incorrect Answer of Pair P12 to Question 2 of Task 3

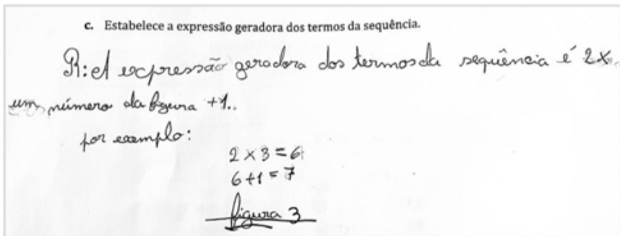


(Note: "Yes. Because 98+2=100 and we must have 100 hearts, and in the pictures, it is always +2")

In the last question, half of the pairs identified the generating expression obtaining a correct answer, as suggested by the resolution of pair P11 (Figure 25).

Figure 25

The Correct Answer of Pair P11 to Question 3 of Task 3



(Note: "The generating expression of the terms of the sequence is 2x a number in the figure +1. For example, 2×3=6;6+1=7")

Four of the remaining six couples did not give any response. Two pairs gave the wrong answer as they presented the formation law of the sequence instead of its generating expression, which were requested, as observed in the resolution of pair P8 (Figure 26).

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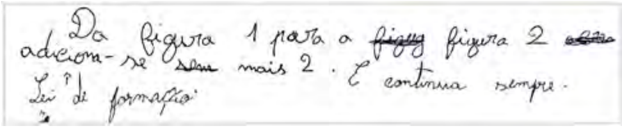
Figure 26

The Incorrect Answer of Pair P8 to Question 3 of Task 3

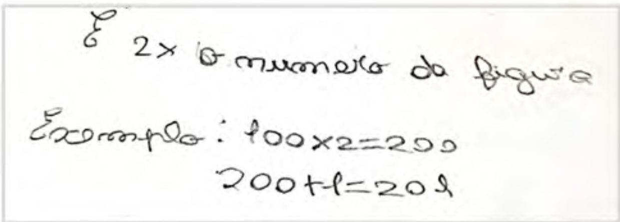


(Note: "From figure 1 to figure 2 add +2. And it always continues")

Also, in question 3, a pair of students gave partially correct answers. Although they presented an example in which they applied the generating expression, they were not able to identify its entirety (Figure 27).

Figure 27

The Partially Correct Answer of Pair P9 Pair to Question 3 of Task 3



(Note: It is 2x the number of the figure. Example: 100×2=200;200+1=201")

Representations

The analysis of students' responses to Task 3 identified which representations were used the most and which ones were used the most in correct answers (Table 3).

The first question asked students to present two different methods of resolution, which prompted them to use several representations in the same question. Of the pairs that with the correct answers, one pair only gave one representation, six pairs gave two representations, and four pairs gave three representations. As shown in Table 3, the students mainly used pictorial and table representations in solving the first question.

In the conversions between multiple representations, it was found that all pairs maintained the veracity of their solution, demonstrating a possible understanding of the mathematical objects, considering Duval's theory (2012) that a student able to coordinate two different representations would have achieved a complete understanding of the content under study.

In the second question, 11 pairs presented their answer in natural language, and nine of them answered correctly. As for the last question of the task, four correct answers, presented in symbolic writing, were obtained.

Connections between representations

In the first question, all pairs of students were able to convert representations without losing the "mathematical object" in question, revealing that they mastered in the mathematical contents. To understand the connections that students established between representations, a moment of discussion was created to discuss them.

As most of the students used pictorial representations, in the first moment, this representation was used to invite P8 to present their resolution on the board, which converted the pictorial representation into a table. A different reasoning was, then, explored using the pictorial representations used by pair P1 pair (Figure 28).

Figure 28

Exploitation of Item 1 by Pair P1

The discussion generated in the class group favored the determination of the generating expression (Figure 29).

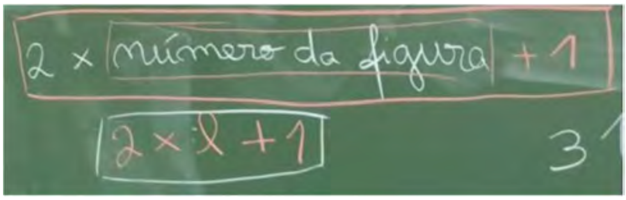
Table 3

Frequency of the Types of Records Used by Students in Task 3 (n = 12).

	Response types														
	C					PC					I				
Question	PR	NL	S	SW	T	PR	NL	S	SW	T	PR	NL	S	SW	T
1	11	0	3	3	10	1	0	0	1	0	0	0	0	0	0
2	0	9	0	0	0	0	1	0	0	0	0	1	0	1	0
3	0	2	0	4	0	0	0	0	1	0	0	1	0	1	0
Total	11	11	3	7	10	1	1	0	2	0	0	2	0	2	0

Note: PR: pictorial representation; NL: natural language; S: schemes; SW: symbolic writing; T: tables. C: Correct answer; PC: Partially correct answer; I: Incorrect answer.

Figure 29
Sequence Generating Expression



To continue with the familiarization of algebraic expressions, ''figure number'' is replaced by ''l''. Although the expression was not formally presented as an algebraic expression, students understood its meaning as they participated in its construction.

Conclusions

From the analysis of students' resolution of the sequence of tasks presented in the teaching, we conclude from experience that the students preferred the pictorial representations, with one of the most used representations being natural language.

Initially, students mainly resorted to this representation, and there was no variety of registers. It was also found that most students were not able determine near and far generalizations at first.

During the teaching experience and using a greater variety of representations, students were observed to respond more assertively to questions about near and distant generalizations and questions involving the determination of the formation law and the generating expression. This result leads to the conclusion that the exploration and connections between different representations stimulate the improvement of students' ability to argue and reason as they discuss important mathematical ideas (Lannin, 2003; Viseu et al., 2021).

Although pictorial representations stand out for facilitating understanding, students also encountered other representations such as tables during the teaching experience.

However, this representation almost always appeared to support other representations, and students solved the tasks with another register and, then, made connections between the previously used representations and the table.

Regarding the exploration of multiple representations, different pairs were able to make connections between different representations, which indicates that they helped determine generalizations. Despite their age range, the students showed improvement in their algebraic thinking, namely in generalization.

In this process, pictorial representations were the most helpful representations in constructing meanings of the mathematical objects under study, allowing them to express their reasoning more confidently and clearly, as they provided visual support that attenuated the abstract nature of the mathematical concepts and gave them meaning (Ozsoy, 2018).

The first difficulty arose in the interpretation of the statement of tasks. Although students resorted to strategies, such as the ''representation and counting strategy'', which implies a representation of all terms of the sequence up to the desired term, they obtained the wrong solution to the question presented.

This result is in line with the findings of the study by Duval (2012), where the greatest difficulties in mathematical thinking arise from the lack of proper understanding of a mathematical representation, originating the incomprehension of the mathematical objects. While natural language was one of the most used representations by students and gave rise to the most correct answers, it was also the most frequent representation in the incorrect answers. The students who used this register to develop their resolution had difficulties and often failed to obtain a valid answer to the questions.

Students perceived their greatest difficulties as ''understanding the questions'' and ''understanding the texts''. Also, most students reported natural language as the representation they found most difficult to learn. Tables and drawings were the most highlighted representations as the tables provided faster, easier, and more specific reasoning, and the drawings helped them understand the concepts. Regardless of the representation, it was concluded that involving students, from the early years of schooling, in the translation of their mathematical thinking promotes learning with understanding of concepts, communication, and reasoning skills.

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