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Factors Affecting Success in Solving a Stand-Alone Geometrical Problem by Students aged 14 to 15

Branka Antunović-Piton*[1] and Nives Baranović[2]

This paper investigates and considers factors that affect success in solving a stand-alone geometrical problem by 182 students of the 7th and 8th grades of elementary school. The starting point for consideration is a geometrical task from the National Secondary School Leaving Exam in Croatia (State Matura), utilising elementary-level geometry concepts. The task was presented as a textual problem with an appropriate drawing and a task within a given context. After data processing, the key factors affecting the process of problem solving were singled out: visualisation skills, detection and connection of concepts, symbolic notations, and problem-solving culture. The obtained results are the basis of suggestions for changes in the geometry teaching-learning process.

Keywords: geometrical problem, mathematic language, problem solving, visualisation

1 *Corresponding Author. Faculty of Educational Sciences, Juraj Dobrila University of Pula, Croatia.

2 Faculty of Humanities and Social Sciences, University of Split, Croatia.

Dejavniki vplivanja na uspešnost reševanja strogo geometrijskega problema pri učencih med 14. in 15. letom starosti

Branka Antunović-Piton in Nives Baranović

Prispevek obravnava in razišče dejavnike, ki vplivajo na uspešnost reševanja geometrijskega problema 182 učencev 7. in 8. razreda osnovne šole. Izhodišče za razmislek je geometrijska naloga z nacionalnega srednješolskega zaključnega ocenjevanja znanja na Hrvaškem (državna matura), ki zahteva uporabo osnovnih geometrijskih pojmov. Naloga je bila predstavljena kot besedilni problem z ustrezno grafično reprezentacijo in kot naloga v danem kontekstu. Po obdelavi podatkov so bili izbrani ključni dejavniki, ki vplivajo na postopek reševanja problemov pri učencih: spretnost vizualizacije, odkrivanje in povezovanje pojmov, simbolni zapisi in kultura reševanja problemov. Pridobljeni rezultati so osnova za predloge sprememb v procesu poučevanja in učenja geometrije.

Ključne besede: geometrijski problem, matematični jezik, reševanje problemov, vizualizacija

Introduction

Throughout all stages of mathematical education, students are involved in solving various types of mathematical tasks. Such tasks are used as motivational and thematic introductions, learning foundations for novel procedures, algorithms and formulas, and tools for the revision and application of obtained knowledge in new situations and contexts, as well as a discernment of new contents (Kurnik, 2000).

By solving different types of tasks, including tasks of different complexity, students have the opportunity to develop and improve various procedural skills, acquire adequate conceptual knowledge and experience the application of such knowledge and skills in more complex and new mathematics environments (Hsu, 2013). The acquisition and development of problem-solving skills allow students to acquire and apply various mathematical concepts and processes, as well as to develop and cherish adequate mathematical competencies.

In the teaching process, the teacher is a designer and executant of all teaching activities (Cavanagh, 2008; Odluka, 2019). Hence, teachers are tasked with choosing from among various types of mathematical tasks while ensuring an adequate learning environment characterised by equal opportunities to acquire and develop the knowledge, abilities and skills.

The current methods of teaching mathematics are often implemented using several topics/subjects focused on a particular subject area (e.g., triangles, quadrilaterals, linear function, vectors, etc.) (MZOS, 2006). Additionally, tasks are chosen to accommodate the learning, practice, and application of area-specific concepts and processes. Upon completing a specific topic, an assessment of students' skills and competences is carried out using relevant tasks. In such a learning context, students often complete the course without difficulties before moving to the context of the next subject area. However, when students are assessed outside of area-specific contexts, for example, during the PISA international benchmarking tests, TIMSS, and similar, and national level tests (State Matura in Croatia), they often underperform (PISA, 2012; Priručnik DM, 2017; Priručnik TIMSS, 2017).

The most significant underperformance occurs in geometrical tasks during external assessment. Numerous studies (e.g., Baranović, 2019; de Villiers, 2009; Fujita & Jones, 2007) indicate that students have difficulties working with geometrical concepts and processes (visualisation, classification, proving, etc.). Specifically, students perform poorly when given a mathematical task, for example, a mid-level task that can be solved using different paths and strategies outside of the context of the subject area. Thus, this task becomes an isolated problem (defined as a stand-alone (SA) problem). To solve an SA task, one has to implement a network of mathematical knowledge and skills. If this is not the case, meaning if knowledge and skills are insufficiently developed or have remained at the level of disjoined subject areas, students are unable to solve the given problem successfully.

This paper examines the possible factors affecting the students' success in solving stand-alone geometrical task.

Problem Tasks

In this paper, the concept of a 'problem' or 'problem task' implies any task for which a solution or solving method cannot immediately be found. Such a task can indeed represent a challenge for some students, whereas it is a matter of routine for others, highlighting individuals' knowledge and experience in solving problem tasks. These differences in perspective to the problem task can present themselves as an obstacle to learning and teaching, as well as in evaluating students' knowledge; a teacher who knows a certain subject matter well and visualises it skilfully tends to underestimate the difficulties faced by students who encounter the subject matter for the first time (van Hiele, 1986, p. 17).

Current research indicates that solving problem tasks provides multiple benefits for students, such as developing mathematical thinking (Foong, 2002; Leikin & Lev, 2007), stimulating and developing creativity (Klavir & Gorodetsky, 2011), ensuring the engagement of the majority of students during classes (Klavir & Herskovitz, 2014) and at appropriate learning levels (Sullivan, 2009), enabling the identification of mathematical giftedness (Leikin & Lev, 2007), developing communication and a positive atmosphere in math classes (Schukajlow et al., 2012) and similar. Also, the approach that uses problem tasks can transform the teacher's role 'from a transmitter of knowledge to a facilitator of learning' (Cavanagh, 2008, p. 123). Utilising problem tasks, the teacher has the opportunity to guide the flow of students' thoughts through discussion, encourage conclusion-making and creativity, and link the knowledge from disjoined subject areas into one functional unit.

The evaluation of the task-solving process is highly complex, as tasks can be solved in various ways, yet very advantageous as it provides teachers with a better insight into the students' thinking process (Bingolbali, 2011). Furthermore, the inclusion of students into the process of evaluation by their teachers provides students with the possibility of visualising and solving challenges while finding various approaches to the same problem. Discussing possible problem-solving approaches, students have the opportunity to develop and appreciate different solving strategies, such as analysis, synthesis, specialisation, generalisation, visualisation, analogy, among others (Kurnik, 2000). More precisely, solving problem tasks and the analysis of their respective solving processes enable students to develop their learning culture and various problem-solving strategies. Consequently, a practical intertwinement of mathematical concepts allows for a continuity of knowledge, a higher level of thinking, and a deeper understanding of mathematical concepts and processes.

This paper analyses a stand-alone problem (hereinafter abbreviated as 'SA problem') as a distinct type of problem task. The SA problem is defined as a 'problem to find', a problem task that requires greater cognitive effort to discover and link basic mathematical ideas and concepts, facilitating the acquisition of conceptual understanding. As such, the SA problem requires the application of prior knowledge using different approaches and problem-solving strategies, yet simultaneously, it serves as a learning platform for the discovery of new knowledge and concepts. The SA problem can be presented visually, symbolically, and contextually when it is necessary to establish connections between different representations. Furthermore, the SA problem is given outside area-specific confinements, allowing for various didactic purposes. To solve an SA problem, one must become familiar with the basic problem-solving principles of 'problem to find', as proposed by Polya.

The 'problem to find' consists of three main parts: given data, unknown elements, and the conditions linking them (Polya, 1966, p. 92). Therefore, to solve the 'problem to find' means recognising all possible mathematical objects that meet the conditions and their relationship to the given data. As problem tasks require greater cognitive effort, Polya (1966, p. 5) proposes four phases for an effective problem-solving process and finding the solution: understanding the problem, devising a plan, carrying out the plan, and looking back.

In the understanding phase, to develop an intuition about the possible solutions, it is necessary to observe and connect the main parts of the task by combining various task representation methods (text, visual presentation, symbolic notation). To understand the problem, the mathematical language and the students' reading literacy are essential (Gal & Linchevski, 2010; Polya, 1966; Yang & Li, 2016). According to Polya, a partial understanding of the task, which occurs either because of insufficient concentration or interest to solve the task independently, is the most common drawback in the task-solving process (Polya, 1966, p. 58).

In the devising-a-plan phase, it is very important to visualise the path towards the solution by gradually analysing the given situation, remembering a similar task, varying the task, examining the sketch or given figure, setting a particular expression, equation, formula, etc. During sketching or observation of the figure, the visualisation skill and spatial and geometrical thinking become prominent (Duval, 1999; Boonen et al., 2014). Polya emphasises that many students often become lost along the way to determining the solution as they start computing without a plan or idea, which results in a lack of control over their process (Polya, 1966, p. 58).

Once the problem has been well understood and the solving plan created, the carrying out of the plan phase is simple since it only remains to carry out the chosen procedures and computations. However, this phase requires patience and control over the process, as a mistake can easily pass unnoticed, and the goal can be missed. Suppose one becomes stuck or a mistake is noticed. In that case, one can always go back and try again because a clear plan facilitates the control of the solving process (Polya, 1966).

The looking-back phase has numerous advantages: it allows for the additional development and consolidation of knowledge by verifying whether a solution is in line with the given elements and contextual conditions. Teaching praxis has shown that students often pass over the opportunity to check their solutions as they are satisfied by reaching any, especially if their solution coincides with one of multiple choices. The review of the problem-solving process, which includes a discussion and a comparison of different problem-solving approaches to a single task, creates preconditions for the detection and development of adequate solving strategies while allowing for new knowledge acquisition. It is far more useful to solve a particular task in many different ways and then to compare chosen strategies mutually than to solve similar tasks using the same method (Yanhui, 2018).

In line with the phases given by Polya, the task-solving process often occurs as a nonlinear, phase-interchangeable process that constantly evolves by looking back at understanding, revising the plan, and repeating the process until a clear path to the required solution is achieved (Hodnik Čadež & Manfreda Kolar, 2018). Many current mathematics curricula emphasise the importance of developing mathematical processes, which are especially pronounced when solving problem tasks (Odluka, 2019).

Visualisation in Mathematics Teaching

Each person has a biological tendency to visualise his/her thoughts and conclusions. Although the tendency for visual thinking varies among individuals, educational research confirms that the frequency of these tendencies in the general population follows a normal distribution. Hence, some people will never turn to visual representation, whereas others always will, regardless of the opportunity to choose otherwise. However, in a proper context, the majority of people will turn to visual solutions (Presmeg, 2014, p. 152).

The status and the role of visualisation in mathematics learning and teaching is changing. Visualisation has been present and influential since the very beginning of mathematics, especially in geometry. However, the development of mathematical formalism downgraded geometry, and visualisation became secondary. While strict formalism was prevailing, the mathematics community thought of visualisation as a second-rate activity, resulting in its poor application throughout learning, teaching and knowledge evaluation processes. Although most mathematicians use visualisation in their work as an efficient help and support for learning various mathematical topics, they commonly describe visualisation as only an aid along the path to 'true' mathematics. In such an environment, students also develop an attitude that those utilising visualisation are not successful enough, resulting in the neglect of visual explanations and arguments (Dreyfus, 1991).

The development of modern technologies has increased the value of geometry and visualisation. The previous two decades of educational research, specifically research focused on the role of visualisation in learning and teaching mathematics, has uncovered anew the potential of visual reasoning in discovering, describing, debating, and evaluating mathematical results (Duval, 2014; Presmeg, 2006). Therefore, a new idea has gradually developed, suggesting that an overemphasis of abstract and analytical thoughts can have an adverse effect on mathematics teaching and indicating the importance of developing students' visualisation skills (Gal & Linchevski, 2010; Rellensmann et al., 2017; Sinclair et al., 2018).

In literature about learning and teaching mathematics, visualisation, as a notion, is described in different ways, usually implying both the product (visual representation) and process, and often followed by a definition that integrates various other definitions, by Arcavi (2003):

> Visualisation is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (p. 217)

As such, visualisation is crucial for the learning and teaching of geometrical concepts and for solving geometrical problems. However, to achieve a deeper understanding of geometrical concepts, a flexible transition between the spoken language, visual representation, and symbolic notations within the problem is required. This problem-solving process that utilises multiple representations is neither linear nor simple but can be mastered by learning and teaching (Duval, 1999). Nevertheless, the visual representation of complex conceptual structures requires high cognitive effort to observe and establish connections between adequate elements of these structures. As such, this process is not a routine, nor is there a procedure for students to rely upon as there is when working with formal symbolic notations (for instance, linear equations solving), which is also one of the reasons why students easily give up visualisation (Dreyfus, 1991), meaning that they have difficulties in geometry (too).

Moreover, as Duval (2014) states:

> For a mathematician and a teacher, there is no real difference between visual representations and visualisation. But for students, there is a considerable gap that most are not always able to overcome even throughout their mathematics education. They do not see what the teacher sees or believes that they will see. (p. 160)

Aims and Research Problem

This research aimed to examine the approaches by which students solved a chosen geometrical task, specifically, to analyse the entire solving process, not only the final solution. We investigated how students established connections among the task text, the visual representation of the described situation, and the symbolic notation. Namely, we asked to what extent students developed problem-solving skills with respect to Polya's phases. The chosen task encompassed several basic geometry concepts of elementary school mathematics, mainly focusing on the concept of the inscribed angle. The task was given to 7th- and 8th-grade students, since the inscribed angle topic is taught and applied in the 7th and the 8th grade, respectively, using various geometrical problems. To examine whether the posing of the problem impacted the problem-solving success, three variations of the same task were offered.

Our research questions were as follows:

1. What factors are contributors to the SA problem-solving process, whether as an asset or an obstacle?
2. How does the posing of the problem affect the success of its solving?

Method

To answer the research questions, students' assignments were processed using the descriptive method with qualitative analysis of the problem-solving process regarding the four phases by Polya.

Participants

The research involved 182 randomly selected 14- to 15-year-old students attending the 7th and 8th grades in different Croatian urban elementary schools. Personal information about the participants was not requested. The participation was voluntary and anonymous; each student was assigned a unique ID code in accordance with ethical research practice (Cohen et al., 2007, p. 61).

Instrument

For research purposes, a geometrical task from the National Secondary School Leaving Exam in Mathematics (DM, 2012) was selected. This task was a multiple-choice textual task with one correct out of the four offered answers (Figure 1). The compiling criteria of the State Matura tests classified the task as an intermediate-level task (40–59% of correct resolutions) for the assessment of the student's ability to apply the properties of inscribed angles intercepting the same circular arc or chord (Priručnik DM, 2017, p.20). The State Matura results showed the task was an advanced level task (20–39% of correct resolutions) and mostly solved by the guessing method. The selected task was offered to research participants in three versions.

The first version (Task 1) of the task was identical to the State Matura task (Figure 1). The second version (Task 2) contained the text of the original task without multiple-choice answers, but with the addition of incomplete drawing (Figure 2); all the given elements were pointed out in the figure, but not the unknown angle.

Figure 1

Task from the State Matura exam – Task 1

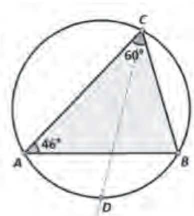
Consider triangle ABC. The measure of angle at vertex A is 46° and the measure of angle at vertex C is 60°. Angle bisector at vertex C intersects the circumscribed circle of the triangle at points C and D. What is the measure of angle ∠CBD?

Answers: (a) 104° (b) 120° (c) 134° (d) 150°

Figure 2

Task with incomplete drawing – Task 2

Consider triangle ABC. The measure of angle at vertex A is 46° and the measure of angle at vertex C is 60°. Angle bisector at vertex C intersects the circumscribed circle of the triangle at points C and D. What is the measure of angle ∠CBD?



In the third version (Task 3), the text of the selected task was put in the context of an astronomy problem, without multiple-choice answers, and the complete drawing was added to the context (Figure 3). Also, the new terminology was used in the text: the descriptive term 'measure of the angle' was replaced with 'at what angle...is seen'. This version represented a 'dressed up' problem, a problem in which the focus of interest was not on the modelling but on the manner of establishing connections between the text, drawing and computations (Schukajlow et al., 2012).

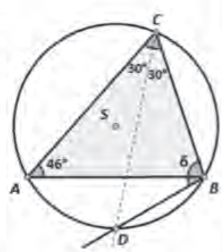
Figure 3

Context-based task with drawing – Task 3

Young astronomer and mountain climber Sanjin (marked as point S) is observing 4 mountain peaks (marked as A, B, C and D), located at the edge of the horizon, as shown in the drawing.

After climbing the mountain peak A, with the help of astronomical tools, he measured that from the point A, the imaginary line segment $\overline{BC}$ is seen at an angle of 46°.

He also climbed the mountain peak C. He measured that from the point C the imaginary line segment $\overline{AB}$ is seen at an angle of 60° and the imaginary line segment $\overline{BD}$ at an angle of 30°.



Sanjin did not make any more measurements but instead he drew a drawing (as shown) and he applied his knowledge of mathematics and determined at which angle the line segment $\overline{CD}$ is seen from point B.

Explain how Sanjin determined at what angle the line segment $\overline{CD}$ is seen from point B; i.e. what is the measure of the angle δ in the drawing?

All versions of the task included an instruction for the participants to illustrate their work (drawings, computations) and to explain how they reached the result, enabling the qualitative analysis of participants' work. Participants were also allowed to use various geometric tools for their drawings, except the protractor.

Data Collection

To fulfil all the preconditions of the SA problem, the research was conducted at the end of the school year, following the completion of the curriculum. At the beginning of the class session, the participants were given the task and a 15-minute time-frame in which to solve it. The participants were not additionally prepared to solve the task. With a random selection, the participants were divided into three groups. Each group received one version of the task.

Data Analysis

Following data collection and the review of participants' work, the assessment criteria were defined, and three areas were agreed upon:

(1) The creation of the drawing in Task 1 and supplement of the drawing in Task 2,

(2) The correctness of the final answer, as well as the presented solution,

(3) The alignment of the task text, drawing and symbolic notation.

The drawing in Task 1 was assessed according to the emphasis of all the given and required elements, whereas the drawing in Task 2 was assessed according to the emphasis of the required elements. The codes of the given elements are: Type of triangle: Scalene (ST), Equilateral (ET), and Isosceles (IT); Given measure of Angles (GA), Circumscribed Circle (CC), Bisector of angle (BA), Point of Intersect (PI). The codes of the required elements are: Ray or segment BD (R) and Corresponding arc of angle CBD (A).

Regarding the correctness of the final answer, each work was sorted into the following categories: T – if the answer is correct, F – if the answer is incorrect, and O – if the answer is not given. Additionally, the task-solving process was assessed (textual description and symbolic notation) using another five categories: without any explanation (1), procedure completely incorrect (2), procedure partially correct (3), procedure fully correct, but unfinished (4) and procedure fully correct (5). Thus, each work was appointed one marking T1–T5, F1–F4 or O1–O4. For example, T4 indicated that the participant's answer was correct and with a fully correct procedure, but unfinished. There were no F5 and O5 markings as each fully correct procedure resulted in the right solution.

Students' assignments were also assessed for the alignment of the drawing with the task text (highlighting of given and required elements), the highlighting of additional elements and markings in the drawing and the quality of the connection between the drawing and the symbolic notation (Table 1). Thus, each Task 1 assignment was appointed a marking D11–D15, each Task 2 assignment was categorised as D21–D24, and each Task 3 assignment was categorised as D31–D34. Although Task 2 and Task 3 contained pre-set drawings, several students sketched their drawings, resulting in two parallel categories (D2 and D2* for Task 2; D3 and D3* for Task 3).

Table 1

Alignment code for the assessment of the task text, drawing and symbolic notation

Code	Description	Explanation
		Coding for Task 1
D11	without anything	There was no drawing and the computation has not been done.
D12	misaligned	The drawing was misaligned with the task text and the computation, i.e. certain highlighted elements were mismatched with the task conditions, and the calculation was incorrect accordingly.
D13	partially aligned	The drawing was correctly, but partially aligned with the text (not all the given elements were highlighted and there were no required elements), the drawing was partially supplemented with other elements and markings, and it was partially aligned to the conducted computation.
D14	aligned but unfinished	The drawing was correctly, but partially aligned with the text (all the given elements were highlighted, but not all the required ones), the drawing was partially supplemented with other elements and markings, and the computation was conducted accordingly.
D15	completely aligned	The drawing was fully aligned with the task text (all the given and required elements were highlighted), it was supplemented with additional elements and markings, and the computation was conducted accordingly.
		Coding for Task 2
D21	without anything	Nothing was highlighted or marked in the drawing and the computation was not conducted.
D22	misaligned	Nothing was highlighted or marked in the drawing, but the computation was conducted.
D23	partially aligned	The drawing was partially supplemented as per conditions of the task text, including additional elements and markings, and the computation was done accordingly.
D24	completely aligned	The drawing was fully supplemented as per conditions of the task text, including additional elements and markings, and the computation was done accordingly.
		Coding for Task 3
D31	without anything	Nothing was highlighted or marked in the drawing and there is no computation.
D32	misaligned	Nothing was highlighted or marked in the drawing, but the computation was conducted.
D33	partially aligned	The drawing was partially supplemented with additional elements and markings, and the computation was done accordingly.
D34	completely aligned	The drawing was supplemented with additional elements and markings, and the computation was done accordingly.

Results and Discussion

Out of 182 randomly selected participants, 66 (36.26%), 63 (34.62%), and 53 (29.12%) solved Task1, Task 2 and Task 3, respectively. The analysis and discussion of the observed factors contributing to the (lack of) problem-solving success were performed with respect to the features of Polya's four phases. This approach emphasised participants' strategies towards finding the solution. Three variations of the task (Task1-3) also allowed for the analysis of the problem-posing as a possible factor and contributor to the (lack of) problem-solving success.

According to Duval, a true understanding and a success of problem-solving in geometry are achievable when a student is capable of establishing connections among the spoken language (task text), visual representation (drawing, whether sketching a drawing or recognising elements in the given drawing), and the symbolic notation (computation) (Duval, 1999, p. 25), where the visualisation skills are essential. Therefore, to answer the research questions, we also examined the quality of the established connections among the task text, the visual representation, and the symbolic notation and computation. The analytical focus was placed on the visualisation skills as one's sketching or reading of the given drawing, not as a matter of choice but a requirement of the SA problem-solving process that provides insight and understanding of the described situation.

The Phase of Understanding the Problem

According to Polya, students understand a problem if they are able to identify all the given and required elements and at least anticipate the connection that will ensure their path to the solution (Polya, 1966, p. 6). The chosen SA problem emphasised (the lack of) this ability in sketching, identifying the given and required elements, as well as establishing the connections among them.

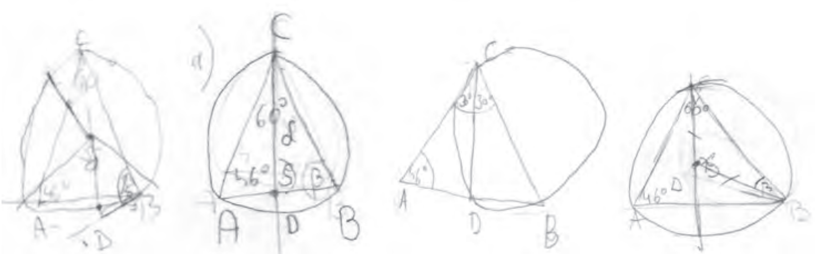
To solve Task 1, the participants had to draw a scalene triangle ΔABC, highlight or mark the given measure of angles at vertices A and C and draw a circumscribed circle. Also, they had to draw the angle bisector at vertex C and highlight point D in the intersection with the circle. As such, all given elements were visually represented (Figure 2). Finally, the participants had to highlight the leg (ray or segment) of BD and then mark the angle at vertex B, namely the angle $\angle CBD$, to visualise the required element (Figure 3).

Besides sketching, it is equally important to mark the drawing adequately. These markings contribute to the understanding of the problem as 'the student is forced to observe the objects to mark' (Polya, p. 6). Additionally, these markings are used as the symbolic notation for the observed concepts and their relations, enabling the computation.

In solving Task 1, many participants experienced immediate difficulties in drawing (Figure 4), making numerous mistakes in the process. Several participants drew the triangle ΔABC as isosceles or equilateral (for instance, Figure 4 (a)), and, as a result, equalised the angle bisector with the bisector of line segment AB and highlighted the centre of the circumscribed circle S on the angle bisector (for instance, Figure 4 (a)), where point D was on the AB side (for instance, Figure 4 (b)). This approach distinctively changed the situation of the given problem, where the required angle became the angle of the triangle ΔABC at vertex B. In this context, some participants drew the circumscribed circle to triangle ΔBCD instead of triangle ΔABC (for instance, Figure 4 (c)). Additionally, 29% (17 of 66) of participants mismatched the point of the circle and angle bisector intersection with the centre of the circumscribed circle (for instance, Figure 4 (d)).

Figure 4

Incorrect participants' drawings of an SA problem



(a) (b) (c) (d)

While solving Task 1 and Task 2, participants experienced significant difficulties when asked to highlight the required angle, which was crucial for the phase of understanding the problem (Table 2). Only 36.43% of participants (47 of 129) highlighted the required angle on the drawing. Among them, 48.94% (23 of 47) of participants answered correctly, 29.79% (14 of 47) answered incorrectly, and the remaining 21.28% (10 of 47) provided no answer.

In comparison, among the remaining 63.57% (82 of 129) of participants who did not highlight the required angle in the drawing, 24.39% of participants (20 of 82) answered correctly, 30.49% (25 of 82) answered incorrectly, and 45.12% (37 of 82) provided no answer. However, the majority of participants who failed to highlight the angle answered correctly by using the estimation based upon the drawing or by relying on the offered answer.

The poor results in Task 1 could be explained by untidy or incorrect drawings that led to an incorrect identification of the required angle. However, the results of Task 2, a task in which the drawing was given, indicate other underlying causes, such as a lack of understanding of the concept of angle and the symbolic notation '∠CBD' for the required angle (Linchevski & Gal, 2010). Additionally, these results also indicate the lack of ability to establish the relevant connections in the drawing, which is a pre-requisite of the visualisation process (Duval, 2000).

Table 2
The correlation between the required angle and the correct answer

Angle∠CBD	Task 1		Task 2		ALL	
Designed	N	%	N	%	N	%
T	15	22.73	8	12.70	23	17.83
F	7	10.61	7	11.11	14	10.85
O	1	1.52	9	14.29	10	7.75
S1	23	34.85	24	38.10	47	36.43
Not designed	N	%	N	%	N	%
T	18	27.27	2	3.17	20	15.50
F	13	19.70	12	19.05	25	19.38
O	12	18.18	25	39.68	37	28.68
S2	43	65.15	39	61.90	82	63.57
S1 + S2	66	100	63	100	129	100.00

To solve Task 2 and Task 3, 22.22% (14 of 63) and 13.21% (7 of 53) of participants sketched their drawings, respectively, in addition to the given drawing. In comparison to the drawings for Task 1, these drawings were neater, containing more correct elements, yet still reflected the aforementioned drawing-related difficulties. A possible explanation of the need for additional drawing could be the predominance of the triangle ΔABC on the given drawing and its effect on those with weaker visual skills, who usually rely on 'the thing they see first'. Another possible explanation could be that the participants found it easier to understand the connections between the given and required elements by sketching their drawing.

All these results suggest that many participants have underdeveloped visual skills (drawing/reading of the drawing as per the task text and highlighting of the required element), resulting in difficulties during the phase of understanding of the problem. These visual skills are crucial for the process of SA problem solving; lacking the necessary skills, the participants were unable to find the path to the solution. These results are comparable with other researchers' findings, which also emphasise that visualisation skills are not innate but have to be learned and developed (Duval, 1999; Rellensmann et al., 2017; Sinclair et al., 2018). Although many use the saying 'Geometry is the art of reasoning well from badly drawn figures', this ability to reason well cannot be expected from students lacking visualisation skills and the confidence in their knowledge. Not only are the process of drawing and the further use of the drawing not simple, but the reading and understanding of the representations made by others are quite complex processes (Duval, 2000, p. 59).

Although a necessary part of the understanding phase, the recognition of given and required elements alone is insufficient to find a solution, which indicates other factors affecting the (lack of) success in the problem-solving process.

The Devising a Plan Phase and the Carrying-out Phase

After reading the text, making the drawing, or identifying the elements on the drawing, one needs to develop an idea that would be the basis for the creation of the realisation plan, and 'the path from understanding the task to implementing the plan can be quite long and curvy' (Polya, 1966, p. 7). After the implementation of the plan, 'the execution of the plan is much easier' (Polya, 1966, p. 11).

The process that consists of creating/reading the drawing, observing the necessary configuration in the drawing, creating an adequate symbolic notation in line with the drawing and computing, also occurs as a nonlinear, interchangeable process. Such a process constantly evolves via looking back to the drawing, (re)creating notations and (re)computing, which requires a significant amount of knowledge, skills and experience (Duval, 1999; Polya, 1966).

Although encompassing several basic geometry concepts, the conceptual design of the chosen SA task revolved around three features: the equal inscribed angles at the same arc, the angle bisector (dividing into two equal parts), and the sum of interior angles of triangles. If utilising only these features, participants were able to determine the measure of the required angle. One possible approach (a good idea) was to notice that the required angle was one of the angles in ΔBCD triangle, meaning that the measure of the required angle could be determined using known measures of the other two angles of the triangle. Another good idea was to notice that the required angle consisted of two angles with the B vertex, meaning that the measure of unknown angle could be determined from the measures of these angles with the B vertex.

In general, our research sample consisted of participants who had a good idea that resulted in an elegant way to the solution (Figure 5), but also of those who got lost along the path to the solution, despite having a good idea (Figure 6). Additionally, some participants started with a good idea but failed to find their path to the solution (Figure 7), whereas several participants had no idea and came to the (correct) solution by guessing and estimation or incorrect computation (Figure 8).

Figure 5

Solution to Task 3 with the code D34 and T5

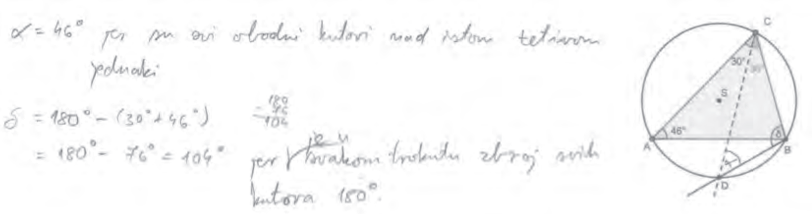


Figure 6

Solution to Task 2 with the code D23 and F3

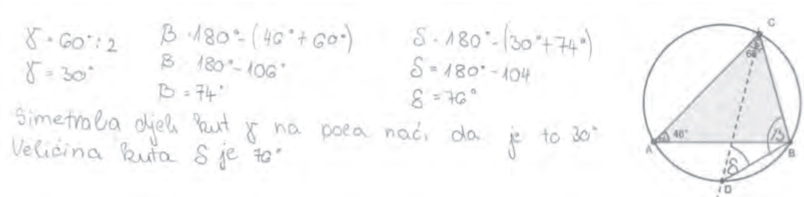


Figure 7

Solution to Task 2 with the code D24 and O4

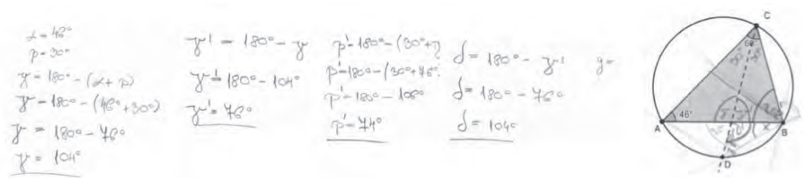
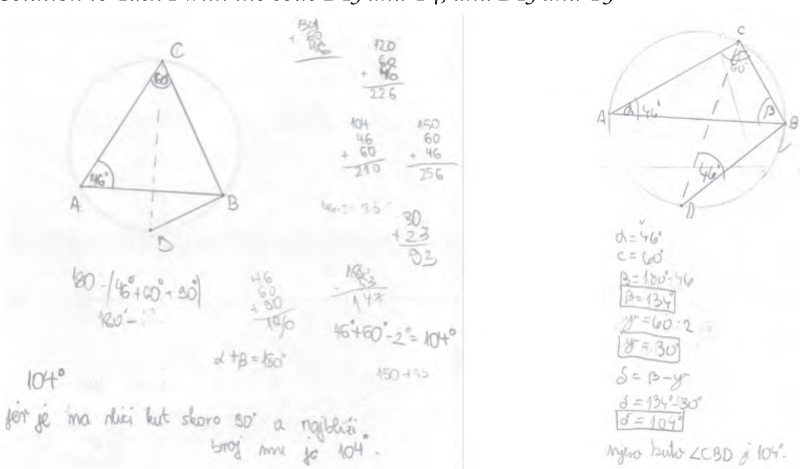


Figure 8

Solution to Task 1 with the code D13 and T4, and D13 and T3



In the problem-solving process of the chosen SA problem, participants utilised various configurations but experienced tremendous difficulties when observing the configuration with the required element and utilising this particular configuration to determine the measure of the required angle. Hence, participants experienced difficulties while discovering the path plan (e.g., Figure 8). The inability to determine the necessary configuration is a reflection of underdeveloped visual skills (Duval, 1999), whereas the inability to find one's way inside the chosen configuration reflects a poor knowledge of adequate concepts and insufficient experience in working with these concepts (Polya, 1966, p. 7).

Many participants also had difficulties in making symbolic notations of the observed elements in the drawing, their connections, and the applicable rules (e.g., Figure 6), all of which were necessary preconditions for the successful realisation of the plan. As they lack ideas and the path plan towards the solution, participants tend to note and compute everything they observe and to perform computations irrespective of the drawing and the highlighted markings (Figure 8). Consequently, many participants used the same marking for different angles or a marking that was not highlighted for the computation. Also, some participants utilised one marking in the drawing, and another in the computation, although both markings represented the same object.

The observed difficulties indicate the existence of important factors that affect the (lack of) success in determining the problem solution: the visualisation skill while reading complex drawings, the knowledge and experience when working with certain concepts, the skill to apply symbolic notations to the observed elements in the drawing and the conduction of computations based upon applicable rules.

The analysis results of the problem-solving process and its respective connections between the task text, the drawing and the symbolic notations, as a basis of the computation, are given in Table 3.

Table 3

Establishing connections between the text, drawing, notation and computation

	Task 1						Task 2						Task 3						All together	
matching text, image, notation and computation		**T**	**F**	**O**	**S1**	**%**		**T**	**F**	**O**	**S2**	**%**		**T**	**F**	**O**	**S3**	**%**	**S**	**%**
completely aligned	D15	14	5	1	20	30.30	D24	8	7	9	24	38.10	D34	14	5	1	20	31.75	64	35.16
aligned but unfinished	D14	6	3	2	11	16.67													11	6.04
partially aligned	D13	1	2	1	4	6.06	D23	0	5	8	13	20.63	D33	0	10	6	16	25.40	33	18.13
not aligned	D12	11	10	8	29	43.94	D22	2	7	15	24	38.10	D32	4	5	2	11	17.46	64	35.16
without anything	D11	1	0	1	2	3.03	D21	0	0	2	2	3.17	D31	0	0	6	6	9.52	10	5.49
SUM		33	20	13	**66**	100.00		10	19	34	**63**	100.00		18	20	15	**53**	100.00	**182**	100.00

In solving Task 1, 30.30% of participants (20 of 66) completely and correctly highlighted all the given elements and the required element in their drawings, connecting drawings to the computations. Only two participants (3.03%) did not have a drawing, and the remaining 66.67% of the participants (44 of 66) had difficulties highlighting the required angle, or they misrepresented the required elements, misaligning the required angle with the task's conditions. As a result, the misrepresentation of the drawing's conditions had a significant impact on the relevant notations and computations.

Although the drawing was given as a part of Tasks 2 and 3, participants failed to make full use of it. In solving Task 2, 38.10% of the participants (24 of 63) filled the given drawing completely and correctly, resulting in the aligned computation. In solving Task 3, this was accomplished by 37.74% of participants (20 of 53).

Since participants did not take care to use systematic, precise and correct symbolic notation, nor for matching a symbolic notation with the drawing, and were often impatient in the problem-solving process, we conclude that the finality of the answer was of greater importance to them than the process itself. Without a plan, participants turned to the guessing method utilising what they 'see' on the drawing, especially when the answers were offered, or they simply 'tuned' their computation accordingly. As members of the true *click generation:* students gravitated towards the 'first instance' solution, meaning the solution 'at the first click'.

A general strategy applied by our sample participants was to calculate whatever was possible, whatever came first to mind and then attempt to find a connection with what was required while hoping to be lucky enough to be successful.

Insufficient patience and perseverance in the problem-solving process are important factors affecting the (lack of) success. The offered solutions have a disastrous impact on the problem-solving process, especially when students lack an idea of how to do so; instead, they align their answers with the offered solution.

All the aforementioned prevents participants from mastering the problem-solving process and finding the path to the final solution. It also hinders those who assess participants' work to clearly and fully recognise the flow of thoughts and the degree of participants' understanding in this process.

The Looking Back Phase

The problem-solving process does not end with the solution; the looking back phase requires checking whether the obtained solution makes sense, whether the conducted procedure is correct, and whether there is another, more economical path to the solution.

Because the participants mostly failed to work and develop a plan, computing despite their partial understanding of the context, it is possible that their problem-solving process ended with the solution. Their notations showed writing, then erasing, then writing again, but more as an on-the-fly process, which stopped after reaching the solution.

To summarise, the third phase of the problem-solving process was predominantly seen in participants' assignments, along with participants' intention to reach the final solution as quickly as possible. Failing to check the meaningfulness of the obtained solution and to monitor their process, participants passed over the possibility of finding possible mistakes, as well as to utilise the important and instructive working phase for strengthening one's knowledge and task-solving skills fully (Polya, 1966, p. 12).

Problem-posing and Solving Success

Finally, it is important to single out participants who fully connected the task text (all the given and required elements), the visual notation (whether they drew or used the given drawing) and symbolic notation (code D11, D21, D21*, D31 or D31*), for which the computation was correct (code T5). There were 19.70% (13 of 66), 11.11% (7 of 63), and 22.64% (12 of 53) of such participants who solved Task 1, Task 2 and Task 3, respectively.

These results suggest that the problem-posing impacted solving success. Participants who solved Task 3 were the most successful (22.64%), providing the largest number of the most elegant solutions and more than one way of solving the task. Therefore, we conclude that the realistic context of Task 3 greatly motivated participants to complete the solving process. Also, the completeness of the given drawing possibly contributed to the process, guiding participants' attention in the right direction. Surprisingly, participants who solved Task 2 were the least successful (only 11.11%). Task 2 offered no answers to lead participants or an interesting context to motivate them. The drawing of Task 2 was less helpful, as the required angle was not highlighted.

Moreover, Task 2 had the highest percentage of assignments without an answer (53.97%, 34 of 63). To solve Task 1, participants were required to draw, which contributed to the poor results, as those participants who were insecure and possessed underdeveloped visual skills could easily draw using the offered solutions 'as a last resort'. Hence, Task 1 had the least number of assignments without an answer (19.70%, 13 of 66), which clearly indicates that the offered answers directed participants toward guessing or aligning the solution.

To conclude, numerous factors improve or hinder a successful path to the required solution. The most prominent factors are the visualisation process (making or reading of the drawing, observing the relevant elements and connections), the understanding of the concepts and the connections necessary to solve the problem, the correct symbolic notation aligned with the drawing and text, the gradualism and patience in conducting of the procedure (through the four phases), control over one's process, testing the meaningfulness of the obtained solution, and the posing of the problem.

Conclusion

The obtained results allow us to conclude that the selected sample of students lacked fully developed problem-solving skills, the understanding of certain geometrical concepts, and the skill to identify and connect conceptual properties, resulting in students' inability to find a systematic way to the required solution (which is in agreement with the previous State Matura results, as well as the PISA 2015 results). The underdeveloped visualisation skills were observed as a particular issue, as fully-developed visualisation skills are required for the problem-solving process of geometrical tasks. The poor connections among the task text, the visual representation, symbolical notations and computation were noted as another issue; without these connections, it is impossible to find the path to the solution (Duval, 1999). In the task-solving process, a student often conducted only the third phase (the notation that is usually not connected to the drawing and computation), and lacking a plan, resolved to use of 'calculate whatever you can' strategy and, if possible, 'at first click'.

All aforementioned difficulties that students experienced throughout the problem-solving process indicate that the learning and teaching of geometry should emphasise the visualisation skills (drawing, interpretation, formation of connections among different notations, etc.) (Duval, 2014; Sinclair et al., 2018) and systematic notetaking. This skillset can be learned and developed by solving geometry problems of different cognitive requirements. Additional emphasis should be on the contribution of the visualisation towards the development of geometric thinking, imagination and creativity, not only in mathematics but also in other areas (Duval, 2014). Hence, it is of utmost importance to increase the awareness among teachers and pre-service teachers about the role of the visualisation in the teaching process.

In addition, the assessment of the problem-solving process, especially of the SA problem-solving process, is a highly complex and demanding endeavour. However, this assessment provides remarkably useful information about the flow of thoughts and the background processes relevant to students' understanding. Teachers can utilise these cognitions to effectively change their teaching and the implementation of problem tasks. By involving students in the assessment process, teachers can create an environment where each student will have the chance to explore, discover and relate various mathematical concepts and thus improve their mathematical literacy, mathematical communication skills, and problem-solving culture (Odluka, 2019, p. 94). A discovery of horizontal and vertical connections among the mathematical knowledge can consequently strengthen students' interest in mathematics (Yanhui, 2018).

For further insights, it is desirable to conduct more research, for instance, with secondary school students or with pre-service teachers, and to examine the extent to which the aforementioned factors affect the success in problem-solving. It would be important to study how teachers approach geometric problems, how they help students solve the aforementioned problems, how they utilise the visualisation in teaching, and similar issues. Moreover, it would be useful to enrich professional teacher training with workshops for the SA problem design and further examine whether such problem tasks incite students' interest or improve their success in solving problem tasks.

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References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, *52*(3), 215–241. https://doi.org/10.1023/A:1024312321077

Baranović, N. (2019). Pre-service primary education teachers' knowledge of relationships among quadrilaterals, In J. Milinković & Z. Kadelburg (Eds.), *Research in Mathematics Education* (pp. 112–127). Mathematical Society of Serbia.

Bingolbali, E. (2011). Multiple solutions to problems in mathematics teaching: Do teachers really value them? *Australian Journal of Teacher Education*, *36*(1), 18–31. https://doi.org/10.14221/ajte.2011v36n1.2

Boonen, A. J. H., Van Wesel, F., Jolles, J., & Van der Schoot, M. (2014). The role of visual representation type, spatial ability, and reading comprehension in word problem solving: An item-level analysis in elementary school children. *International Journal of Educational Research*, *68*, 15–26. https://doi.org/10.1016/j.ijer.2014.08.001

Cavanagh, M. (2008). One secondary teacher's use of problem-solving teaching approaches. In M. Goos, R. Brown & K. Makar (Eds.), *Proceedings of the 31st Annual Conference of the Mathematics*

Education Research Group of Australasia (Vol. 1, pp. 117–124). MERGA http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.507.6272

Cohen, L., Manion, L., Morrison, K. (2007). *Metode istraživanja u obrazovanju* [Research methods in education]. NAKLADA SLAP.

de Villiers, M. (1998). To teach definitions in geometry or teach to define? https://www.researchgate.net/publication/255605686

Dreyfus, T. (1991). On the status of visual reasoning in mathematics and mathematics education. In *Proceedings of the fifteenth annual meeting of the international group for the psychology of mathematics education* (pp. 33–48). PME, Assisi.

Državna matura 2011/2012 – ljetni rok [State Matura 2011/2012 - Summer Assessment Term] (DM ljeto 2012) (2012). Nacionalni centar za vanjsko vrednovanje. https://www.ncvvo.hr/dm-2011-2012-ljetni-rok/

Duval, R. (1999). Representation, vision and visualisation: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st annual Meeting of the North American, Chapter of the international group for the psychology of mathematics education* (pp. 3–26). PME.

Duval, R. (2000). Basic issues for research in mathematics education, Plenary. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th international conference for the psychology of mathematics education*, 1 (pp. 55–69). PME, Hiroshima University.

Duval, R. (2014). Commentary: Linking epistemology and semi-cognitive modelling in visualization. *ZDM - International Journal on Mathematics Education*, *46*(1), 159–170. https://doi.org/10.1007/s11858-013-0565-8

Foong, P. Y. (2002). Using short open-ended mathematics questions to promote thinking and understanding. *National Institute of Education, Singapore*. http://math.unipa.it/~grim/SiFoong.PDF

Fujita, T., & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals: Towards a theoretical framing. *Research in Mathematics Education*, *9*(1), 3–20. https://doi.org/10.1080/14794800008520167

Gal, H., & Linchevski, L. (2010). To see or not to see: Analysing difficulties in geometry from the perspective of visual perception. *Educational Studies in Mathematics, 74*(2), 163–183. https://doi.org/10.1007/s10649-010-9232-y

Hodnik Čadež, T., & Manfreda Kolar, V. (2018). Monitoring and guiding pupils' problem solving, *Magistra Iadertina*, *12*(2), 115–139. https://doi.org/10.15291/magistra.1493

Hsu, W.-M. (2013). Examining the types of mathematical tasks used to explore the mathematics instruction by elementary school teachers. *Creative Education, 4(6),* 396–404. http://dx.doi.org/10.4236/ce.2013.46056

Klavir, R., & Gorodetsky, K. (2011). Features of creativity as expressed in the construction of new analogical problems by intellectually gifted students. *Creative Education*, *2*(3), 164–173. https://doi.org/10.4236/ce.2011.23023

Klavir, R., & Hershkovitz, S. (2008). Teaching and evaluating 'open - ended' problems. *International*

Journal for Mathematics Teaching and Learning, *20*(5), 23.

Kurnik, Z. (2000). Matematički zadatak [Mathematical task]. *Matematika i škola*, *II*(7), 51–58. https://mis.element.hr/fajli/545/07-02.pdf

Leikin, R., Lev, M. (2007). Multiple solution tasks as a magnifying glass for observation of mathematical creativity. *In the Proceedings of the 3th conference of the international group for the psychology of mathematics education*. 3 (pp. 161–168). PME, Seoul.

MZOS [Ministry of Science, Education and Sports]. (2006). Nastavni plan i program za osnovnu školu [The educational plan and programme for elementary school]. Ministry of Science, Education and Sports.

Odluka o donošenju kurikuluma za nastavni predmet Matematike za osnovne škole i gimnazije u Republici Hrvatskoj [Decision on the adoption of the curriculum for the subject of Mathematics for elementary schools and gymnasiums in the Republic of Croatia]. (Odluka) (2019). Narodne novine 7/2019.

PISA 2012 Matematičke kompetencije za život [Mathematical Competencies for Life] (2013). Nacionalni centar za vanjsko vrednovanje.

Polya, G. (1966). *Kako ću riješiti matematički zadatak* [How to solve mathematical task]. Školska knjiga.

Presmeg, N. C. (2006). Research on visualisation in learning and teaching mathematics. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 205-235). Sense Publishers.

Presmeg, N. (2014). Contemplating visualisation as an epistemological learning tool in mathematics. *ZDM Mathematics Education, 46*, 151–157. https://doi.org/10.1007/s11858-013-0561-z

Priručnik za stručne radne skupine koje izrađuju ispite državne mature s primjerima zadataka iz ispita na državnoj maturi [Handbook for professional working groups preparing state Matura examinations with examples of tasks from the state Matura exam] (Priručnik DM). (2017). Nacionalni centar za vanjsko vrednovanje.

Priručnik za unapređivanje nastave matematike s primjerima zadataka iz međunarodnog istraživanja TIMSS 2015 [Handbook for the advancement of teaching mathematics with examples of tasks from TIMSS 2015] (Priručnik TIMSS). (2017). Nacionalni centar za vanjsko vrednovanje.

Rellensmann, J., Schukajlow, S., & Leopold, C. (2017). Make a drawing. Effects of strategic knowledge, drawing accuracy, and type of drawing on students' mathematical modelling performance. *Educational Studies in Mathematics*, *95*(1), 53–78. https://doi.org/10.1007/s10649-016-9736-1

Schukajlow, S., Leiss, D., Pekrun, R., Blum, W., Müller, M., & Messner, R. (2012). Teaching methods for modelling problems and students' task-specific enjoyment, value, interest and self-efficacy expectations. *Educational Studies in Mathematics*, *79*(2), 215–237. https://doi-org.nukweb.nuk.uni-lj.si/10.1007/s10649-011-9341-2

Sinclair N., Moss J., Hawes Z., Stephenson C. (2018). Learning through and from drawing in early years geometry. In K. Mix, & M. Battista (Eds), *Visualizing mathematics. Research in mathematics education*. (pp. 229–252). Springer. https://doi.org/10.1007/978-3-319-98767-5

Sullivan, P. (2009). Constraints and opportunities when using content-specific open-ended tasks. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides: Proceedings of the 32nd annual conference of the mathematics education research group of Australasia,* 1 (pp. 726–729). MERGA. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.544.9781&rep=rep1&type=pdf

Van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education*. Academic Press

Yang, K. L., & Li, J. L. (2018). A framework for assessing reading comprehension of geometric construction texts. *International Journal of Science and Mathematics Education, 16*(1), 109–124. https://doi.org/10.1007/s10763-016-9770-6

Yanhui, X. (2018). On "one problem multiple change" in Chinese "Bianshi" mathematics teaching. *Teaching of Mathematics*, *21*(2), 80. http://www.teaching.math.rs/vol/tm2123.pdf

Biographical note

Branka Antunović-Piton is a PhD student at the Faculty of Education at the University of Ljubljana, Slovenia and an assistant in the field of mathematics and didactics of mathematics at the Faculty of the Educational Sciences at Juraj Dobrila University of Pula, Croatia. Her research interests include developing mathematical thinking, particularly geometrical thinking. She is also interested in problem solving in mathematics, action research, and teacher education and professional development.

Nives Baranović is a senior lecturer in the field of mathematics and teaching mathematics at the Faculty of Humanities and Social Sciences (Department for Teacher Education) at the University of Split, Croatia, and a PhD student at the Faculty of Mathematics at the University of Belgrade, Serbia. Her research interests include teaching and learning mathematics, particularly developing geometrical and functional thinking, visualization and problem solving in mathematics, as well as professional development of teachers.