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# Multiple Approaches to Problem Posing: Theoretical Considerations Regarding its Definition, Conceptualisation, and Implementation

Ioannis Papadopoulos*[1], Nafsika Patsiala[2], Lukas Baumanns[3] and Benjamin Rott[3]

The importance of mathematical problem posing has been acknowledged by many researchers. In this theoretical paper, we want to capture different meanings and aspects of problem posing by approaching it from three different levels: (1) by comparing definitions, (2) by relating it to other constructs, and (3) by referring to research and teaching settings. The first level is an attempt to organise existing definitions of problem posing. The result of this analysis are five categories, which shows that there is no consensus regarding the conceptualisations of problem posing. In the second level, we examine how problem posing is conceived by the research community compared to other mathematical constructs, such as problem solving, mathematical creativity, or modelling. Finally, in the third level, we summarise possible ways of implementing problem posing in research and teaching settings as they are depicted in the relevant literature. Given this broad variance regarding the conceptualisations of problem posing, we attempt to provide some arguments as to whether there is a need for consensus on a commonly accepted concept of problem posing.

**Keywords:** problem posing, definition, conceptualisation, implementation

1 *Corresponding Author. Faculty of Elementary Education, Aristotle University of Thessaloniki, Greece; ypapadop@eled.auth.gr.
2 Faculty of Elementary Education, Aristotle University of Tessaloniki, Greece.
3 University of Cologne, Germany.

# Večdimenzionalni pristop k zastavljanju problemov: teoretični premisleki glede njegove opredelitve, konceptualizacije in izvedbe

Ioannis Papadopoulos, Nafsika Patsiala, Lukas Baumanns in Benjamin Rott

Pomen zastavljanja matematičnih problemov je obravnavalo že več raziskovalcev. V tem teoretičnem prispevku želimo zajeti različne opredelitve in vidike zastavljanja problemov, h katerim smo pristopili na treh različnih ravneh: 1) primerjava opredelitev; 2) povezave z drugimi konstrukti; 3) izvedene raziskave in poučevalne prakse. Na prvi ravni smo želeli organizirati različne obstoječe opredelitve zastavljanja problemov. Rezultate te analize smo uvrstili v pet kategorij, s čimer pokažemo, da ni konsenza glede opredelitve zastavljanja problemov. Na drugi ravni povzamemo, kako je zastavljanje problemov sprejeto v skupnosti raziskovalcev in kako se ta povezuje z drugimi konstrukti, kot na primer z reševanjem problemov, s kreativnostjo, z modeliranjem. Nazadnje, na tretji ravni, iz relevantnih raziskav povzamemo mogoče načine raziskovanja zastavljanja problemov in implementiranja v poučevanje. Upoštevajoč precejšnja odstopanja glede konceptualizacije zastavljanja problemov, poskušamo navesti nekaj argumentov za potrebo po soglasju o splošno sprejetem konceptu zastavljanja problemov.

**Ključne besede:** zastavljanje problemov, opredelitev, konceptualizacija, implementacija

## Introduction

Problem posing is considered an important topic that, from time to time, attracts the attention of the research community (Cai & Hwang, 2020; Stanic & Kilpatrick, 1988). For example, Einstein and Infeld (1938) emphasise the importance of problem posing by claiming that 'the formulation of a problem is often more essential than its solution' (p. 95).

The potential of problem posing to enhance students' learning in mathematics has been acknowledged by many researchers, thus confirming its importance (English, 1998; Silver, 1994). They attribute this potential to the fact that problem-posing activities are cognitively demanding tasks (Cai & Hwang, 2002), which require students to expand their thinking beyond already known procedures to improve their understanding by reflecting on the structure of the given problem. In this sense, problem-posing activities are considered an ingredient in doing high-quality mathematics (Hadamard, 1954). In the 'Principles and Standards for School Mathematics' by the National Council of Teachers of Mathematics (NCTM, 2000), it is considered important for students to 'formulate interesting problems based on a wide variety of situations, both within and outside mathematics' (p. 258), and it was recommended that students make and investigate mathematical conjectures in order to learn how to generalise and extend problems by posing follow-up questions. These standards explicitly emphasise that problem posing (in combination with problem solving) leads to a more in-depth understanding of the mathematical contents, as well as the process of problem solving and, thus, to a better grasp of doing mathematics itself.

Moreover, problem posing is important for teachers who regularly formulate and pose worthwhile problems for their students, no matter whether they are selecting and modifying standard textbook problems or developing self-generated problems (NCTM, 1991).

As a reasonable consequence of the research interest on this topic, there are many publications on problem posing. However, these are not homogeneous; instead, a wide range of different approaches on the meaning of problem posing and of its relations with other mathematical constructs can be identified in the relevant literature. On the one hand, there are papers in which the need for 'a clear distinction between problem posing and the general practice in raising questions in mathematics' (Mamona-Downs & Downs, 2005, p. 392) is emphasised. On the other hand, there are papers in which the importance of training students and teachers in problem posing, the use of problem posing as a measure of creativity, and the role of technology in problem-posing activities are examined (Cai et al., 2015). There are also papers in which the

respective authors discuss methodological issues (among others) about problem posing and speculate about the future directions of the relevant research (Cai & Hwang, 2020).

Against this background, this paper is an attempt to organise the different aspects of problem posing. We do not provide a full systematic literature review but rather want to present and discuss a broad spectrum of literature on problem posing. We aim to stimulate reflection and initiate discussion rather than to propose irrefutable answers. This narrative synthesis is a summary of the current state of knowledge in relation to the following three research questions:

1. In which way can existing definitions of problem posing be categorised?
2. How is problem posing conceived by the research community in relation to other mathematical constructs?
3. What are the possible ways of implementing problem posing in research and teaching settings?

## Theoretical Considerations on Problem Posing: Definitions, Conceptualisation, and Implementation

This section constitutes an attempt to examine the wide diversity of problem-posing aspects. We aim to sort the broad spectrum of literature thematically. Through an extensive examination of representative literature compiled by two research groups working on problem posing, three focal points were identified. First, the paper navigates the most common definitions found in the research literature and their differences (subsection 2.1). The second aspect concerns the connection of problem posing to further constructs at the research level (subsection 2.2). Finally, the third aspect deals with the implementation of problem posing in research and teaching settings (subsection 2.3). The theoretical considerations within each subsection are the product of an inductive category development in a qualitative content analysis (Mayring, 2014). Various definitions of several papers were analysed with regard to their content-related key aspects.

### Problem posing: Examining the existing definitions

The analysis of the definitions found in the relevant literature resulted in five categories: Problem posing as (1) only the generation of new problems, (2) only the re-formulation of already existing or given problems, (3) both the generation and/or re-formulation of problems, (4) raising questions, and (5)

an act of modelling. Please note that these five categories are not necessarily disjunctive; there might be definitions that fit into more than one. It is not our goal to provide distinct categories but to initiate a discussion.

This effort was initiated by the acknowledgement that there are many different perspectives on and definitions of problem posing (Silver & Cai, 1996). Of course, there have been other efforts to organise problem-posing definitions. For example, Olson and Knott (2013) organised the existing definitions into two groups according to whether they focus on students or teachers. The main difference between these two categories is the aiming goal. Students (mainly college students) pose problems to exhibit their conceptual understanding. Teachers pose problems to cultivate the mathematical thinking of their students. Similarly, Cai and Hwang (2020) specify problem posing separately for students and teachers.

In the same paper, looking across different perspectives, Cai and Hwang (2020) propose the following:

> By problem posing in mathematics education, we refer to several related types of activity that entail or support teachers and students formulating (or reformulating) and expressing a problem or task based on a particular context (which we refer to as the problem context or problem situation). (p. 2)

We attempt to deepen this effort and elaborate the categorisation of Cai and Hwang (2020) by using the categories (1) to (5) mentioned above.

Please note that in this paper, the focus is only on definitions of *mathematical* problem posing. There are definitions from other domains, such as Freire's (1970), who sees problem posing as a way to make students 'critical thinkers' (p. 83), extending the concept of problem posing to various domains of knowledge. However, such definitions are beyond the scope of this paper.

***Problem posing as generating new problems***

As Kilpatrick (1987) mentions, in real life outside of school, many problems, if not most, must be created or discovered by the solver, who gives them initial formulations. He also adds that in some cases, problems emerge from the exploration of ill-defined problems with a given mathematical input. This is in accordance with Lakatos (1976), who said that a problem never comes out of the blue; it is always related to our background knowledge. Kilpatrick (1987) provides the following example: Let's say that one is looking at the divisors of various numbers. It is easy to notice that the number of divisors varies; therefore, considering numbers with very few divisors might be interesting. It

is reasonable to look at extreme cases and think that numbers with 0 or 1 divisors are likely non-existent and therefore uninteresting. A further observation might result in the hypothesis that numbers with 3 divisors always seem to be squares of numbers with 2 divisors and, therefore numbers with 2 divisors are of special interest. Then it is possible to examine additional examples of primes and factorisations of numbers into primes and to ask whether the relationship between a number and its divisors is a function. Then, the new problem is generated: Any integer greater than 1 can be expressed as a product of primes in essentially only one way (Fundamental Theorem of Arithmetic).

In the classroom context, the generation of new problems might result from proposing a problem-posing situation. In Kwek's (2015) study, students are initially presented with the following situation: 'A gardener is planting a new orchard. The young trees are arranged in the rectangular plot, which has its longer side measuring 100 m' (p. 279). Then, they are asked to use the information above to pose a mathematical problem.

In these examples and, therefore, in this category, problem posing could exclusively be seen as the creation of new problems. Stoyanova and Ellerton (1996) describe it as a 'process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems' (p. 518). The same idea of problem posing as the generation of problems based on given situations or mathematical expressions or diagrams can be found in the work of Cai et al. (2020). This definition holds for both teachers and students. This line of thinking is also present in the context of teacher education, where problem posing is seen as a task designed by teachers asking students to generate word problems (Kwek, 2015).

***Problem posing as reformulating already existing or given problems***

Definitions in this category consider problem posing to be re-formulations of problems that already exist – out of interest after finding interesting problems or because problems were given by someone (e.g., a teacher), with the request to find new problems. In this sense, this could immediately be related to Pólya's (1957, p. xvi-xvii) suggestions of *Devising a Plan*: 'Could you restate the problem?', 'Could you restate it still differently?', and 'Try to solve first some related problem … more accessible, … more general, … more special, … analogous.'

Kilpatrick (1987) provides an example of what might be considered a re-formulation of an existing problem. Students are given a practical problem: a cloth-drying rack for the backyard must be made, for which there are two options. The clotheslines are string between two parallel supports (Figure 1, left)

or between crossbars (Figure 1, right). The students are asked to find how many feet of clothesline are needed for each case, given the length of the outer side and the separation between adjacent lines.

**Figure 1**
*The cloth-drying rack task*

These two models presented above seem realistic, but they can be questioned by the solvers. They do not include the clothesline necessary for tying the ropes to the supports. Indeed, the original mathematical model is quite simplified and, therefore, this is an opportunity for students to attempt a second model that takes account of the extra clothesline. This new model constitutes a reformulation of the initial problem (cf. Verschaffel et al., 1994).

The re-formulation of an existing problem is very often connected to the sense of ownership of the new problem (Kilpatrick, 1987). Moreover, this re-formulation can be a series of transformations of the original problem. In this case, each re-formulation indicates progress towards a solution and provides possibilities for further expanding the scope of the original problem (Cifarelli & Sevim, 2015).

***Problem posing as both generating new and/or reformulating given problems***

This category refers to definitions that include both the generation of new problems and/or the re-formulation of existing ones, thus combining the previous two categories. Very early, Duncker (1945) used such a definition. However, perhaps the most frequently used among the definitions is that of Silver (1994), who defines mathematical problem posing as 'both the generation of new problems and the re-formulation of given problems' (p. 19) and, as a consequence, posing

can occur before, during, or after the solution of a problem. A similar approach is that of Singer and Voica (2015), who suggest that 'problem posing refers to generating something new or to revealing something new from a set of data' (p. 142).

Cai and Hwang (2020) proposed a new one-for-all definition:

> By problem posing in mathematics education, we refer to several related types of activity that entail or support teachers and students formulating (or reformulating) and expressing a problem or task based on a particular context (which we refer to as the problem context or problem situation). (p. 2)

In this definition, they explicitly suggest both formulation and re-formulation of a problem made by either students or teachers, thus bringing forward the issue of teachers' education. A slightly modified version of this definition can also be found in Osana and Pelczer's (2015) working definition that considers problem posing as 'the act of formulating a new task or situation, or modifying an existing one, with a specific mathematical learning objective and a targeted pedagogical purpose in mind' (p. 485).

### *Problem posing as raising questions and viewing old questions from a new angle*

Ellerton and Clarkson (1996) adopt an approach for problem posing that is inspired by Einstein's and Infeld's (1938) perspective of raising new questions and as possibilities to regard old questions from a new angle. One can object that seeing already existing questions from a different perspective is similar to reformulating a problem. However, the focus now is on the questions asked in a problem rather than on its set of data. So, in the cloth-drying rack problem, the focus is on the data, which is on the accuracy of the numbers given for the clothesline length. However, when raising questions, the focus is on the questions themselves, as explained in the example of Gonzales (1996) below.

In the same spirit, Marquardt and Waddil (2004) say: 'Problem posing involves making a taken-for-granted situation problematic and raising questions about its validity' (p. 190). The same is written by O'Neil and Marsick (1994): 'Problem posing involves raising questions that open up new dimensions of thinking about the situation' (p. 22). 'The Principles and Standards for School Mathematics' (NCTM, 2000) are aligned with this spirit, emphasising that 'problem posing, that is generating new questions in a problem context, is a mathematical disposition that teachers should nurture and develop' (p. 117).

Estrada and Santos (1999) studied the concept of variation within a course in a group of Grade 11 students. The students received information

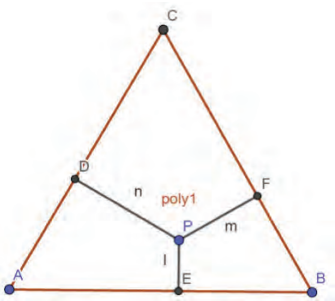
showing the prices of a product, the behaviour of a school of fish, and the fluctuation of a currency (the peso). They were asked to examine the given data and formulate corresponding questions. The information provided to the students was given in formats that included tables, paragraph forms, or actual newspaper texts.

Some other researchers connect this to a specific situation where, for example, a picture is given without any explanation to students who are then asked to generate questions relevant to the situation. Gonzales (1996), for example, presented to the students a mathematical situation found in a newspaper in the form of a statistical graph that contains data but no built-in question. The students were expected to investigate the given situations and to pose several questions that could be answered by referring to the information provided in the graphs.

The issue of raising questions can also be connected to the application of the 'what-if-not' technique of Brown and Walter (1983). In this approach, the main elements of the task are identified, and then the solver starts negating them, asking what would happen if these elements were different. Mamona-Downs and Papadopoulos (2017) exemplify this using the task in Figure 2.

**Figure 2**

*Task used for applying the 'what-if-not' technique.*




Consider a point *P* internal to an equilateral triangle. Its distances from the sides of the triangle are 3, 4, and 5 cm.

Find the length of the altitude of the triangle.

The main elements of the task are that (1) the shape is plane, (2) it is a triangle, (3) the triangle is equilateral, (4) the point *P* is internal to the triangle, and (5) the distance from each side is considered. Negating each element, the solver can raise interesting questions: 'What if point *P* is not internal to the triangle?' (not 4), "What if the triangle is not an equilateral one?" (not 3), "What if we consider the distance from its edges instead of its sides?" (not 5), are some examples of interesting questions that can be generated using the 'what-if-not' technique.

However, as mentioned earlier, Mamona-Downs and Downs (2005) contend there is a need for a distinction between 'problem posing and the general practice in raising questions in mathematics' (p. 392) to avoid distorting effects in the relevant research literature.

***Problem posing as an act of modelling***

Finally, there is a limited number of papers that consider problem posing as an act of modelling. Referring to the definition of problem posing by Stoyanova and Ellerton (1996), Bonotto (2010a) says in one of her papers:

> I consider mathematical problem posing as the process by which students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems. This process is similar to situations to be mathematised, which students have encountered or will encounter outside school. (p. 402)

She also adds that problem posing becomes an opportunity for interpretation and analysis of reality, and this takes place through activities that are quite absent from today's school context and are typical of the modelling process (Bonotto, 2010b). This definition of problem posing lies at the heart of modelling. Bonotto (2010a, 2010b) presents a relevant example. Children were given various menus (products on offer, prices, ingredients, cover charges, etc.). They were asked to compile an order according to their experience outside school, following the structural features of a blank receipt (description of goods, quantity, cost, etc.). In the end, they had to calculate the total amount they would have to pay.

Greer (1992), in an implicit way, sees mathematical problem posing as a task of 'translating from the natural language representation of a problem to the mathematical-language representation of the model' (p. 285). More explicitly, Stillman (2015) says that problem posing in a real-world situation occurs when a problem is formulated in such a way that is amenable to mathematical analysis. There are many situations in the world around us that can be transformed into a problem that can be solved.

## How problem posing is conceived by the research community

This subsection examines how problem posing is conceived by the research community compared to other mathematical constructs. The analysis of several research papers reveals that problem posing is seen as (1) an autonomous mathematical construct viewing it as the (implicit or explicit) aim of

tasks, and/or (2) interwoven with other mathematical constructs such as problem solving, creativity (which might also include giftedness), and modelling.

***Problem posing as an autonomous concept***

Despite problem posing being a counterpart of problem solving, for many years, the latter has attracted much more attention from mathematics education researchers than the former. This realisation initiated the emergence of studies examining a variety of aspects of problem posing. In these studies, students and teachers are engaged in problem-posing activities, and the potential effects of this engagement are examined (Koichu, 2020). Therefore, problem posing is viewed as a goal of an activity.

Being the goal of an activity might entail a variety of approaches. Some studies attempt to determine what kinds of problems are posed by the solvers (Lavy & Bershadsky, 2003; Silver et al., 1996), what the influence of different task formats on problem posing is (Leung & Silver, 1997), who poses problems, for whom and how problems are posed around particular situations (Singer et al., 2011), and what the role of computerised environments in problem posing is (Abramovich & Cho, 2006).

Others attempt to develop a better understanding of the ways pre- and in-service teachers use problem posing: What do they focus on when they pose mathematical problems (Stickles, 2011)? How do the pre-service teachers pose problems to the students? How do their practices change, and what factors contribute to the change (Crespo, 2003)? How do they develop their students' actual problem-posing abilities by explicitly teaching them about what are considered to be key elements of mathematical problem posing (English, 1998)? Moreover, how do they examine the extent to which the problems students pose are mathematical and solvable (Silver & Cai, 2005)?

***Problem posing considered as interwoven with other mathematical concepts***

Even though many research studies focus on problem posing per se, some studies focus on mathematical constructs connected to problem posing. Three common links that are examined are (1) between problem posing and problem solving, which is the most common, (2) between problem posing and creative mathematical thinking, and (3) between problem posing and modelling.

(1) Several researchers have conducted empirical studies examining potential connections between problem posing and problem solving. On the one hand, there are studies considering both as merely different sides of the

same coin. As Kilpatrick (1987) says, 'problem formulation is an important companion to problem solving'. Newell and Simon (1972) characterised problem posing as a process that is embedded within, and which is difficult to separate from, problem solving. Silver (1994) stated that problem posing and problem solving are interwoven activities in the means that problem posing can occur *prior*, *during*, and *after* a problem-solving process. Furthermore, as an extension, problem posing can be considered as a problem-solving process in which the solution is ill-defined since there are many problems that could be posed (Silver, 1995). Gonzales (1998), as well as Wilson et al. (1993), consider problem posing as the fifth step in Pólya's steps of problem solving. For Singer and Moscovici (2008), problem posing is an extension and application of problem solving, both included in a learning cycle in constructivist instruction.

In contrast, there are studies examining the various effects of problem posing on problem-solving skills and competencies. Silver and Cai (1996) identified a high correlation between problem-solving and problem-posing performances. More precisely, good problem solvers generated more and more complex mathematical problems than their less successful classmates did. Silver (1994), reviewing several studies relevant to problem posing, found that they give evidence about the positive influence of problem posing on students' ability to solve word problems. Moreover, there is a relationship between the students' use of abstract problem-solving strategies and their ability to pose extension problems, meaning problems that go beyond the given information (Cai & Hwang, 2003).

(2) Another construct closely connected to problem posing is creative mathematical thinking (CMT) (cf. Joklitschke et al., 2019). Again, there are two approaches here. On the one hand, problem posing is considered as a distinct and creative act (Dillon, 1982) equal to or more valuable than finding a solution or as a form of creative activity that can operate within rich-situated tasks (Bonotto & Dal Santo, 2015; Freudenthal, 1991). Leung (1997), examining the relationship between CMT and problem posing, claims that creativity is in the nature of problem posing and that, in essence, creating a problem is a creative activity.

On the other hand, many researchers use problem posing to promote, facilitate, and evaluate CMT. Jay and Perkins (1997, p. 257) identify problem posing as a key aspect of creative thinking and creative performance, and not only in mathematics. Silver (1997) claims that CMT lies in the interplay between problem solving and problem posing and that is 'in this interplay of formulating, attempting to solve, reformulating,

and eventually solving a problem that one sees creative activity' (p. 76). Kontorovich et al. (2011) use Guilford's (1967) categories of fluency, flexibility, and originality as indicators of creativity in students' problem posing. It seems that students who pose coherent and original problems through changes made in their formulation have creative skills (Singer et al., 2011). Moreover, when students pose mathematical problems, they gradually develop fluency skills, and they generate problems of high-quality elaboration and originality (Van Harpen & Sriraman, 2013).

(3) Despite some research papers in which problem posing is defined as an act of modelling, other researchers view both as two closely interlinked concepts. For example, English, Fox, and Watters (2005) state that in mathematical modelling as a rich problem situation, the generation of problems and questions, as well as solving those, occur naturally. When students attempt to make sense of incomplete, ambiguous, or undefined information as in a modelling situation, numerous questions naturally arise for the children as they try to make sense of this information, elicit and work with the embedded mathematical ideas, and modify and refine their model. This perspective on modelling mirrors problem posing. Barbosa (2003) claims that modelling is strongly linked to problem posing and modelling activities to the act of creating questions/problems. As Barbosa exemplifies:

> Imagine that the teacher proposes that students study the impact of the social contribution tax. This is a tax deducted from people's salaries by the Brazilian Government for the maintenance of social welfare. The students certainly will have to formulate questions, search for information, organise it, draw up strategies, apply mathematics, evaluate the results, etc. (p. 230)

Christou et al. (2005) suggest that problem posing constitutes an integral part of modelling cycles, which require the mathematical idealisation of real-world phenomenon; and Stickles (2011) confirms that mathematical modelling starts with the posing of a problem.

### Problem-posing activities in research and teaching settings

Finally, this third subsection summarises possible ways of implementing problem posing in research and teaching settings.

For the former, Kilpatrick (1987, p. 123) mentions the shift from viewing problem posing not only as a goal of instruction but also to use it as a means of instruction. Koichu (2020) adds to that, highlighting that problem-posing

activities serve genuine mathematical or pedagogical needs. For the latter, the main way of implementation is as a diagnostic tool aiming to deepen our understanding of the difficulties students face in the learning of mathematics.

Below, we present three main ways of its implementation: (1) problem posing as a tool for teacher training leading to the enhancement of their subject didactic competence, (2) problem posing as a pedagogical/educational tool, and (3) problem posing as a diagnostic/assessment tool. Presenting only these three ways does not mean that they are the only ones. These were chosen as the most frequent ways represented in the research literature (Hošpesová & Tichá, 2015).

***Problem posing as a tool for teacher training leading to the enhancement of their subject didactic competence.***

Teachers' training seems to be a promising area for problem posing. Ellerton (2013) highlights the importance of integrating problem posing in teacher training, providing examples on how problem posing can be an integral part of mathematics teacher education programs through the active learning framework, thus contributing to two areas.

The first area is the improvement of the teachers' strategies. An example of this is shown by Crespo (2015), who reported significant improvements in a group of elementary pre-service teachers' problem-posing strategies during a semester-long engagement in problem-posing tasks. The participants were required to pose problems collaboratively for pupils or when open-ended exploration of a mathematical situation precedes problem posing (Crespo & Sinclair, 2008). In a similar spirit, Grundmeier (2015) explored how problem-posing activities can benefit prospective elementary and middle school teachers. After the course provided to them, the participants developed their problem-posing abilities regarding their techniques.

The second area is the enhancement of teachers' subject didactic competence. Tichá and Hošpesová (2013) used problem posing to produce that. The findings from their study indicate that pre-service teachers gained a deeper understanding of concepts realising the need to use various representations. Similar results can be found in the work of Malaspina et al. (2015).

***Problem posing as a pedagogical/educational tool***

Pólya (1981), in his book 'Mathematical Discovery', emphasises that letting students formulate problems not only motivates them to work harder but it teaches them a desirable attitude of mind, thus highlighting the pedagogical value of problem posing. Examples of studies that focus on this aspect include

the work of Silver and Cai (2005) about teachers posing problems related to a specific situation in the class, or the work of English (1998) about teachers using problem posing to enhance students' learning of mathematics.

Downton and Sullivan (2017) used problem posing with Grade 3 students, aiming to prompt them to use more sophisticated strategies. The results of this study reveal that the tasks prompted the use of more sophisticated thinking. Thus, problem posing could be a useful educational tool.

Finally, in the context of teacher training, the case of educators offering examples of posed problems and pointing out flaws, mistakes, and misconceptions (Tichá & Hošpesová, 2013) is also an option for the educational use of problem posing.

***Problem posing used as a diagnostic/assessment tool, which helps teachers/researchers to uncover deficits and obstacles in students' knowledge.***

Problem posing has the potential to be used as a diagnostic/assessment tool. Following Tichá and Hošpesová (2009), according to the posed problems, it is possible to investigate both the level of understanding and the difficulties of a specific mathematical concept. They used problem posing to reveal pre-service primary school teachers' shortcomings in their conceptual understanding of fractions (Tichá & Hošpesová, 2013). Silver and Cai (1996) used problem posing as a tool for evaluating students' performance and revealing the connection between their problem-posing and problem-solving abilities based on a written 'story-problem'. They also suggest problem posing as a useful assessment tool for students' learning. Measuring the mathematical and linguistic complexity of the generated problems, teachers could evaluate students' conceptual understanding (Silver & Cai, 2005). Teachers can design and use problem-posing tasks to understand students' mathematical learning and assess students' understanding, as well as the obstacles to understanding and misconceptions (Caniglia, 2016). As an example, Verschaffel et al. (2009) use problem posing as a diagnostic tool for students' understanding of division-with-remainder problems. Problem posing around certain mathematical concepts can be a useful tool for assessing students' understanding. This provides important information to teachers who then optimise their quality of instruction (Lin, 2004). Cai et al. (2013) also used problem posing to measure the curricular effect of learning on middle-school students in order to compare the implementation of two different curricula.

## Concluding Remarks

The contents of this paper aim to initiate a discussion on questions such as: Does this broad range of definitions constitute an advantage or an obstacle for researchers who work on problem posing and especially for those who start working in this area? Is there a need for consensus on a commonly accepted definition of problem posing? Does the diversity of the existing definitions enhance or reduce the robustness of the research findings?

The broad range of definitions, conceptualisations, and implementations of problem posing in research on mathematics education demonstrates that 'the field of problem posing is still very diverse and lacks definition and structure' (Singer et al., 2013, p. 4). While a differentiated range of definitions offers the opportunity to select suitable ones for respective research interests, underlying definitions and conceptualisations within studies should be made transparent, which sometimes fails to happen. Consequently, studies pursuing a similar research aim are hardly comparable due to different usages of similar terms. Furthermore, comparatively young research fields, such as the field of problem posing, suffer from the abundance of different understandings, which may have a detrimental effect on the actual teaching of problem posing. As Ruthven (2020) says:

> [A]n increasingly diverse range of concerns are finding a place under the banner of problem posing. […] There is a danger, then, of usage of the term becoming so diffuse as to undermine its analytic power and reduce it to a nebulous slogan. (p. 1)

Therefore, researchers must be aware of the spectrum of understandings so that they are able to make a differentiated and reasoned choice from them in order to make their understanding of problem posing comprehensible to recipients of their research.

This paper also illustrates how problem posing is addressed in research: sometimes in isolation and sometimes in connection with other constructs, such as problem solving, Creative Mathematical Thinking (CMT), and modelling. In empirical research on the connection of problem posing to these constructs, it is noticeable that the focus is mainly on the *products*, meaning the problems posed. There is a lack of research to evaluate the *process* of problem posing when investigating connections to problem solving or CMT. We argue that the products can only reflect one component of the activity of problem posing. Looking at CMT, for example, a comparatively insignificant problem may have been posed as a result of a highly creative process for the particular student. A mere evaluation of the problem would not do justice to this.

In practice, the descriptions in this article can be helpful in understanding the enormous spectrum of conceptualisations of *problem posing*. This may enable a targeted selection and assessment of appropriate problem-posing activities for educational purposes to be achieved.

As with this paper, we only intend to stimulate discussion on this far-reaching and complex topic; a future systematic literature review may provide more valid insights into definitions, conceptualisations, and implementations of problem posing in research and practice.

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## Biographical note

**Ioannis Papadopoulos**, PhD, is an associate professor of Mathematics Education at the Faculty of Elementary Education at Aristotle University of Thessaloniki, Greece. His research interests include mathematical problem solving and problem posing and the use of digital technology in mathematics teaching and learning.

**Nafsika Patsiala** is a PhD student in the field of didactics of mathematics at the Faculty of Elementary Education at Aristotle University of Thessaloniki, Greece. Her main research areas include problem posing and problem solving.

**Lukas Baumanns** is a graduate student for mathematics education at the University of Cologne, Germany. His research interests include mathematical problem posing and problem solving as well as mathematical creativity and giftedness.

**Benjamin Rott**, Dr. rer. nat., is a professor of mathematics education at the University of Cologne, Germany. His research interests include mathematical problem solving (with a focus on heuristics and process regulation), mathematical creativity, critical thinking, and epistemological beliefs.