IOJPE

ISSN: 1300 – 915X
www.iojpe.org

International Online Journal of Primary Education

2021, volume 10, issue 2

FACTORS AND MULTIPLES: IMPORTANT AND MISUNDERSTOOD

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Received: May 23, 2021 **Accepted:** December 15, 2021 **Published:** December 31, 2021

Suggested Citation:
Hurst, C., Hurrell, D., & Huntley, R. (2021). Factors and multiples: Important and misunderstood. *International Online Journal of Primary Education (IOJPE), 10*(2), 273-286.

Abstract

Factors and multiples are important aspects of mathematical structure that support the understanding of a range of other ideas including multiplication and division, and later on, factorization. At primary school level, it is important that factors and multiples are taught as a connected enterprise and as vital parts of the multiplicative situation; that is multiplication and division. The primary objective of the study on which this paper is based was to determine the extent of children's understanding of factors and multiples. A written quiz containing questions about factors and multiples and asking for children to explain their responses, was administered. Results suggest that the language involved with factors and multiples may play a role in the extent to which children develop a conceptual understanding of them. Also, most children know some things about factors and multiples but struggled to connect and articulate ideas when factors and multiples were presented in a different context. In conclusion, the inconsistency of participant responses suggests that teaching about factors and multiples needs to emanate from a more conceptual and connected standpoint.

Keywords: Factors, multiples, language, divisibility, connections.

INTRODUCTION

Factors and multiples are important elements of mathematics and underpin an understanding of number, the consolidation of multiplication and division facts, expansion and factorization, and powers, roots, and exponents (Bana, Marshall, & Swan, 2014; Turton, 2007). A factor can be defined as being a whole number that can be multiplied a certain number of times to create a given number, and a multiple is the result of multiplying a number by a natural number (Turton, 2007). Although expressed as being important (Bana, Marshall, & Swan, 2014; Feldman, 2014; Gunes, 2021; Turton, 2007) there is a paucity of extant literature which deals directly with factors and multiples. A search of literature from 2015 to 2022 though a University library database was undertaken. Search parameters for peer reviewed journal articles were entered with the filters of "multiplication & teaching & factors & multiples not higher education and not multiplication facts". A total of 3 426 articles were presented by the database. Through reviewing the abstracts of the first 300 articles, only 17 indicated that the topics of factors and multiples were addressed and of these only 3 were considered proved to be germane. This paucity of research on these two fundamental ideas is concerning.

The importance of factors has been documented in research (Feldman, 2014; Gunes, 2021). Feldman (2014, p. 231), noted that factors are part of number theory, "which includes topics such as prime numbers, composite numbers, divisibility, prime factorization, greatest common factors, and least common multiples." Feldman points to the Common Core State Standards for Mathematics (CCSSM) as indicating that children from kindergarten begin learning aspects of number theory. Specifically, he notes that, "in fourth grade, students find factor pairs of whole numbers less than 100 and recognize that a whole number is a multiple *of* its factors" (Feldman, 2014, p. 231). Two important points arise there – first that factors are in pairs (except for square numbers) and second, that a whole number is a multiple of its factors. These points will be discussed later. Feldman (2014) notes that an understanding of factors underpins many other important ideas, including links with multiples, properties of multiplication, adding and subtracting fractions using the lowest common denominator, and later, simplification of algebraic expressions.

Explicitly making connections between factors, multiples, and divisibility, is central to understanding them. In linking multiples to the division process, Thompson (2012) described the process of 'chunking' to divide larger numbers. He gave an example of generating multiples of 36, initially using 'ten times' (360), then halving to find 'five times' (180), before doubling and doubling again to generate 'two times' (72), 'four times' (144), and 'eight times' (288). The degree of success enjoyed by children with the process "depends on how much work children have done on multiples" (Thompson, 2012, p. 46). In his discussion of the vital development of multiplicative thinking, Drake (2012) noted that 'truly' multiplicative thinkers "understand about factors, multiples, primes, and divisibility and use these ideas in their thinking (p. 49)". McEachran (2008, p. 24), in describing an investigation of prime numbers, noted that it is important for children "to understand what these numbers *are*, not what they are *called.*" We suggest that the same applies to factors and multiples. It is important that children know what they are, and the language used in helping children learn about them is of critical importance. The regular use of appropriate mathematical language and terminology is likely to enhance the development of conceptual understanding of ideas like factors and multiples, as opposed to children knowing them in a procedural way. Conceptual understanding should better place children for learning concepts related to algebra and proportional reasoning (Hurrell & Day, 2015; Siemon et al., 2021)

McKenna (2019, p. 38) described a teaching activity based on exploring factors, noting that the 'traditional' approach would have likely been referred to as "doing factors" and "which would have relied heavily on direct instruction followed by . . . repetitive exercise with not much thinking involved". Consequently, McKenna challenged students to explore their understanding of factors by getting them to prove or disprove the conjecture that bigger numbers have more factors. The end result was that the students disproved the conjecture and importantly, arrived at the realisation that "they all had an even number of factors apart from 36" (McKenna, 2019, p. 39). Further investigation led to children realising that 36 was a square number, and as a square number, would have an odd number of factors. This thoughtful development of understanding about factors echoes an earlier article by Richards (2007) describing how, in response to a challenging question, 'How can we be sure that we have all of the factors [of 36]?' one student responded with, "The factors come in pairs: 1 and 36, 2 and 18, 3 and 12, 4 and 9, and 6" (Richards, 2007, p. 39). The notion of 'factor pairs' is important because, if there is one factor, there must be a 'partner factor' (except for square numbers where a factor is multiplied by itself). The development of the idea of factor pairs is within the scope of children. Rickard and Earle (2019) worked with children between the ages of eight and ten years and found that students were not only able to describe the connections between factor pairs, but could extrapolate these connections to reason how 6 × 7 will give the same result as 3 × 14.

Further to these ideas, it is necessary to explicitly teach children about the inverse relationship between multiplication and division and the notion of the factor-factor-product relationship. Strategies such as 'thinking of multiplication' to derive division facts are important (Siemon et al., 2021). Similarly, Lemonidis (2016) discussed the complementary nature of multiplication and division and how knowledge of products can inform the derivation of factors.

Karp, Bush, and Dougherty (2015) discuss the ‘rainbow’ representation for factors (see Figure 1) which they describe as having shortcomings.

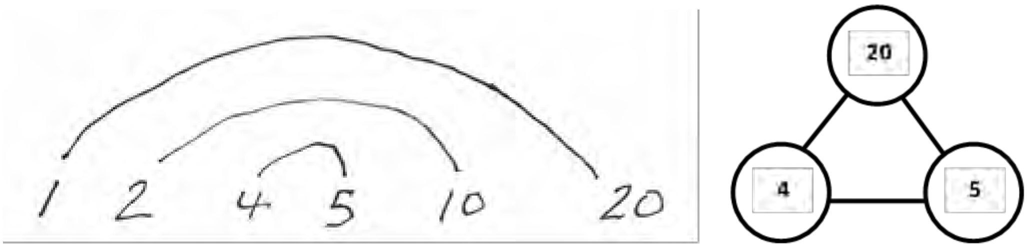

Figure 1. ‘Rainbow’ and ‘triad’ representation for factors of 20

They illustrate this, using 20 as the multiple, “The rule is taught so that once you identify factors that are consecutive numbers (e.g., 4 and 5), you have identified all factors” (Karp et al., 2015, p. 211). They note that the rule does not hold for numbers like 40 where the closest factors are five and eight, nor for square numbers which have an odd number of factors. Also, Parker (2019) described an activity using ‘a triad’, a triangle with three circles at its vertices, the multiple at the apex, and the pair of factors in each of the base circles. Parker (2019) asserts that this is not only a powerful representation of factor pairs but also of the link between factors and multiples.

Rickard and Earle (2019) also described a task using a Venn diagram to show multiples of three and six. A good example of connected knowledge was demonstrated by one child’s comment that, “there won’t be any numbers in there [pointing to the circle for multiples of 6] . . . [because] . . . there are two threes in every six” (Rickard & Earle, 2019, p. 17). However, while Rickard and Earle (2019, p. 15) describe another useful task to explore how the four times table is ‘hidden’ within the eight times table, using number grids, there is no mention made of the use of the term ‘multiples’ to describe that eight is a multiple of four and therefore shares the same factors as four. In a similar way, they describe how children showed an awareness of the commutative law but talked about it in terms of ‘swapping the numbers around’ rather than in terms of the factors being the same and that the order is irrelevant. The paucity of the use of the terms factors, multiples and divisibility embedded and highlighted in the context of teaching and learning of multiples is of concern. Day and Hurrell (2015, p. 20) note that the naming of factors and multiples is “an important and often undervalued piece of mathematical understanding”, and that the idea of ‘number families’ needs to be developed through the use of arrays They continue to say that “students are told that if you know 3×4 you know the associated facts of 4×3, $12 \div 3$ and $12 \div 4$, and many accept this as being the case without ever seeing why it is so” (Day & Hurrell, 2015, p. 20).

The explicit use of the terms factors, multiples and divisibility can be found in the task ‘Thinking of Two Numbers’ (Mathematics Assessment Resource Service – MARS, 2015). MARS (2015, p. 4) provided focus questions and prompts for teachers including, ‘What can you tell me about a factor?’, ‘What do we mean by a multiple?’, and ‘Show what you have written to your neighbour. Do you agree on what a factor is? Have you described it in a similar way? If not, what are the differences?’ In contrast, ‘Demystifying multiplication’ (National Council for Teachers of Mathematics – NCTM, 2013) discussed some important ideas such as using the multiplicative array to demonstrate the distributive property, and then linking that to factorization. However, there was no specific mention of the terms ‘factor’ or ‘multiple’ in this document. The point here is that some sources explicitly mention the terms while other sources do not.

There appears to be a lack of consistency in a number of articles that discuss issues related to factors and multiples. Indeed, it is interesting to note how many times the word ‘factor’ was mentioned as opposed to the term ’multiple’. In discussing perfect numbers, Griffiths (2017), mentioned ‘factor/factors’ 17 times and ‘factorization/s’ twice. ‘Multiple’ was not mentioned at all. In the article by Ollerton and Cooper (2017) about sequences and patterns of number facts, ‘multiple’ was mentioned five times, but ‘factor’ was not mentioned at all. Again, in Richards’ (2014) article on

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ISSN: 1300 – 915X
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division, ‘multiple/s’ had eight references but ‘factor/s’ were not mentioned, and McEachran (2008) discussed prime numbers but mentioned neither ‘factor’ nor ’multiple’. It is difficult to understand how a discussion of prime numbers can be achieved without talking about factors and multiples. Thompson (2012) did something similar in an article on ‘chunking’ in division – ‘multiples’ was mentioned eight times, but there was no mention made of ‘factor/s’. It is suggested here that factors and multiples need to be taught simultaneously and that the terms should be used in tandem wherever possible in order for children to develop a connected understanding of the mathematical structure and be able to articulate that understanding.

The proposition on which this article was based was that children need to hold a strong understanding of factors and multiples and concepts associated with them if they are to grasp the mathematical structures that are underpinned by factors and multiples. Given that, the researchers were keen to determine the extent to which children hold a connected understanding of factors and multiples, and how they relate to multiplication, division, and divisibility. However, as is evident above, while there is material written about factors and multiples, there appears to be inconsistent use of the key terms ‘factor’ and ‘multiple’ and this may impinge on students’ capacity to develop a conceptual understanding of factors and multiples.

METHOD

Research Model

According to the research theorists (Cresswell & Cresswell, 2018), pragmatism “…arises out of actions, situations and consequences rather than antecedent conditions” (p. 10) and is about what is practical and works (Cresswell & Plano-Clark, 2011). Further, because pragmatism as an epistemology welcomes the plurality of methods (Kaushik & Walsh, 2019) and is often related with mixed-methods (Cresswell & Plano-Clark, 2011; Teddlie & Tashakkori 2009), it was deemed suitable for this qualitative study which was supported by descriptive statistics. The data collected for this study came from only one source, namely a questionnaire, but was subjected to some descriptive statistical analysis to look at trends, and then a more qualitative approach was adopted to further refine the decisions made about the meaning of the answers. This dual interrogation of the data satisfied the parameters of a mixed-method study.

In consideration of the established form of symbolic interactionism as articulated by theorists (e.g. Denzin, 2004), people act towards things (and events) on the basis of the self-reflections and meaning these things have for them. From the collected data we formulated and offered narratives of how the respondents’ communicated their level of understanding regarding the role of factors and multiples as indicators of their level of multiplicative thinking. For the methodology, survey research was adopted for this study. Survey method is the act of questioning individuals on one or multiple topics, and then reflecting upon their responses (Check & Schutt, 2012). A questionnaire was considered the most efficient and least obtrusive way in which to collect the data from the students.

Study Group and Data Collection Tools

The research on which this article is based is part of a larger study on Multiplicative Thinking. Data for the study were generated through the use of a Multiplicative Thinking Quiz (MTQ-A). The instrument was developed and refined over three years with multiple age groups and was deemed to be valid as results were consistent across different samples. This quiz (questionnaire) was constituted with eight questions, most of which had multiple parts. The three questions which are examined in this article had a particular focus on factors and multiples:

- In the number sentence $7 \times 5 = 35$, which number/s are factors and which number/s are multiples? Please explain how you know.
- Write as many factors of 30 as you can. Please explain how you know they are factors of 30.
- Write at least four multiples of six. Please explain how you know they are multiples of six

The MTQ was administered by the researchers to whole class groups in primary schools in Perth (Western Australia) and Plymouth (United Kingdom). Each student had her/his own paper and was encouraged to seek guidance if s/he could not read and understand the question. No help was given in helping the students formulate or write their responses. No time limit was set, and students were asked to hand in their quiz at their discretion.

Post the quiz, student responses to questions were recorded on spreadsheets and analysed by the researchers to determine trends and themes. The participant sample on which this article is based consisted of 108 children from four classes, two Year Five classes, with children aged between 10 and 11, and two Year Six classes, with children aged between 11 and 12. One Year Five and one Year Six class came from one primary school in Western Australia, and one Year Five and one Year Six class from one primary school in the United Kingdom.

Data Analysis

A recording tool was constructed by the researchers and the data were transcribed to that tool. Transcription revolved around reading the children's responses and determining if the question was suitably answered in relation to the stated intent for the question. For example, question 3a asked "In the number sentence 7 × 5 = 35, which number/s are factors and which number/s are multiples?" For this to be classified as suitably answered the student had to identify both 7 and 5 as factors and 35 as the multiple. The idea that 35 is a factor of itself was not considered germane in this situation. Each of the researchers was then given the questionnaires and asked to determine if a zero or a one should be ascribed each answer in accordance with the key. One would be an indicator that a reasonable response had been given and zero, that the response was inadequate. No partial credit was given. The researchers coded their data individually, and this coding was then scrutinised to ensure consistency of the application of this coding system, to measure interrater reliability.

The extent to which two or more coders agree is interrelated reliability, and is usually measured as a percentage agreement between the coders. To measure the percentage agreement, the researchers created a template in which the rows represented variables and the columns the coders' response to the collected data. The cells in the matrix contained the zero or one entered by each coder for each of the variables. The interrater reliability proved to be 92%.

RESULTS

Data from three interview questions listed above are now discussed in turn.

- Question 1. In the number sentence 7 × 5 = 35, which number/s are factors and which number/s are multiples? Please explain how you know.

Students used a range of approaches to provide answers to this question. These are summarized in Table 1. Samples of student responses showing their various methods and reasoning are included after Table 1.

Table 1. Student responses to identifying factors and multiples in 7 x 5 = 35

Student response	% n = 108
Recognized the number sentence as containing both factors and a multiple	9
Identified 7 and 5 as factors of 35	50
Identified 35 as a multiple (of 5 and/or 7)	43

It is significant that whilst half of the cohort could identify the factors as being 7 and 5, and just less than half the cohort identified 35 as the multiple, a small percentage recognised that the number sentence contained both factors and a multiple (these students were all in the Year 5 class from the UK). Where students gave correct responses, their explanations suggested they were secure in

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understanding the language of factors and multiples and could link this to their knowledge of multiplication and division. For example, student CRE explained that ‘A factor is a pair or one number that are multiplied to make a multiple’ and student CNE explained that ‘7 x 5 are factors and 35 is a multiple because the word multiple means the one that is answer to the numbers being multiplied’. Student WBL succinctly said ‘Because 2 factors make a multiple. You x 2 factors together to make a multiple’. However, where students gave incorrect responses, they appeared to be confusing the notion of multiplying and the term multiple, suggesting that as the 7 and 5 are being multiplied together, they must be multiples. For example, student MAN said ‘The 7 x 5 is the multiple and the 35 is the factor’. The explanation was ‘The 7 x 5 is multiplying and 35 is the answer’. This confusion in the link between multiply and multiple was also demonstrated by student ESC who said ‘35 are factors and 7 and 5 are multiples, because it says times, so I picked them’. This student seemed to understand that multiply and times are equivalent, so thought that the numbers used in the ‘times’ must be multiples. The confusion with the language is exemplified by the sample from Student TLI (Figure 2).

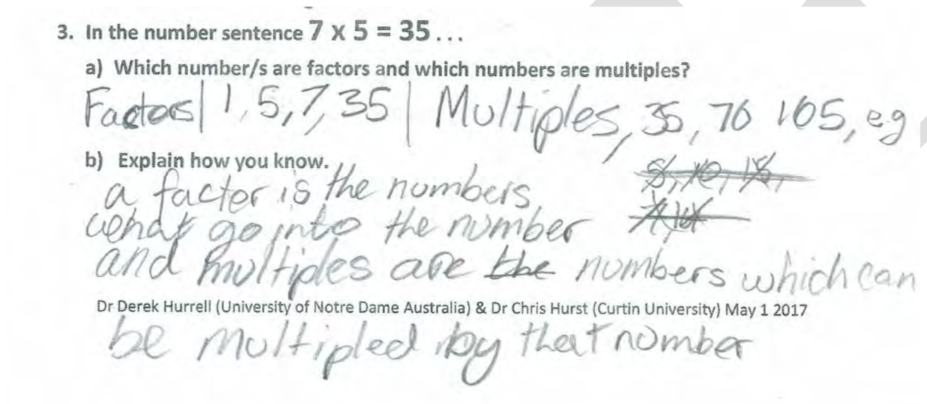

Figure 2. Sample from Student TLI

- Question 2. Write as many factors of 30 as you can. Please explain how you know they are factors of 30.

Students’ responses to this question were varied. These are summarized in Table 2. Samples of student responses showing their various methods and reasoning are included after Table 2.

Table 2. Student responses to writing factors of 30

Student response	% n = 108
Identified some factors of 30	84
Identified factors of 30 in pairs	44
Explained factors in terms of divisibility or multiplication	54

The majority of students across the 4 classes were able to identify some factors of 30, although not necessarily all of the factors. This suggested that students understood what factors are, however their explanations showed misunderstanding with the language of factors and multiples, as in the previous question. For example, student CBR when describing why the numbers given are factors of 30 said, ‘Because they all multiply into 30’. Student RWO also said, ‘They can all times into 30’. These students appeared to understand that 30 can be divided exactly by its factors, and were perhaps thinking that each factor has a multiple which is 30 in their use of the words ‘multiply into’ and ‘times into’. Student SSI gave a comprehensive response to the questions (see Figure 3).

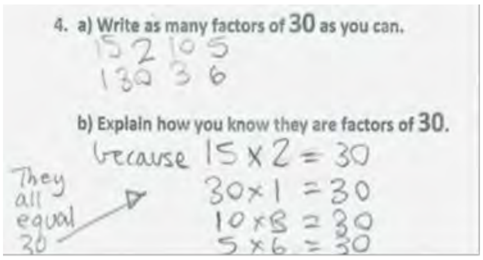

Figure 3. Samples by Student SSI

Other students used the language of division incorrectly but the factors they gave were correct, suggesting they understood there is a link between the factors of a number and divisibility. For example, student JMA's response was '1, 3, 5, 6 10, 15, 30, because if you divide them, they would equal 30'. Student OST explained that 'Also, 5, 6, 10, 2, 3, 15 are all factors of 30 because they are divisible by 30', which suggested some awareness of the idea of divisibility. Student CBO offered an explanation based on the 'rainbow' representation (see Figure 4), explaining that the factors are multiplied.

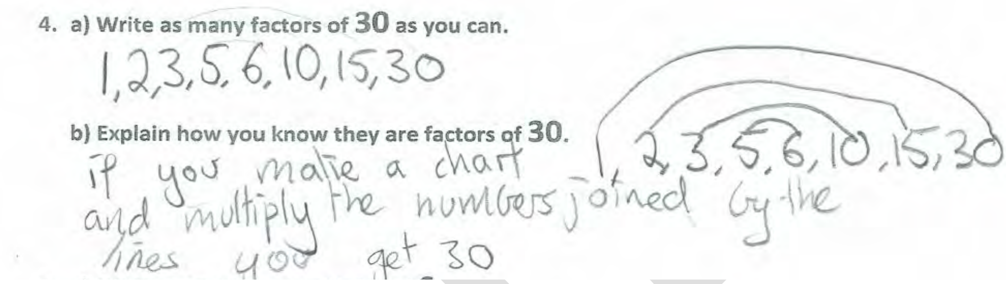

Figure 4. Sample from Student CBO

There was great variation from class to class and school to school in terms of students being able to write factors in pairs. The vast majority of the children wrote some factors of 30 but less than half wrote them in pairs. There was also great variation between the two school sites as to the number of students who wrote all of the factors of 30, even though the question didn't specifically ask for that. A question for further analysis is 'To what extent do these children understand that to have one factor means that there must be another factor (or partner factor)?'

- Question 3. Write at least four multiples of six. Please explain how you know they are multiples of six.

Students offered a range of responses to this question. These are summarized in Table 3. Samples of student responses showing their various methods and reasoning are included after Table 3.

Table 3. Student responses to writing factors of 30

Student response	% n = 108
Identified four multiples of 6	34
Explained multiples in terms of divisibility or multiplication	28

Whilst the percentage of students who could write four multiples of six was low across the sample, where students answered correctly, they had good explanations for their response. For example, student PHO gave examples of 12, 18, 24, 30, 36 and said, 'Because 6 x ? equals those numbers'. Student LNG gave 12, 18, 24, 30 and said, 'They can be divided by 6 to form a whole number'. Student LME wrote them as 6 x 4, 6 x 6, 6 x 9, 6 x 12 and said, 'Because it is six times another number'. These students seemed to understand why the numbers are multiples of six and their explanations suggest that they might have understood that it had to be a multiple of *something*.

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ISSN: 1300 – 915X
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One student, CRE, knew to multiply numbers by six to get the multiples, saying ‘I multiplied random numbers by 6 to make multiples’, whilst student ELA had a more organised approach, saying ‘Because they are different numbers in order multiplied by six, making the six times table’, and showed an understanding that the answers in a multiplication table are multiples. A particularly interesting response came from student JDA who had nothing correct in the quiz to this point and then wrote all the multiples of 6 to 114. He said, ‘They are multiples of six because they can all be divided by six’, which suggested he knew there was a link between multiplying numbers by six to get the multiples, and being able to divide the multiples of six by six. Student IHA made explicit the link between multiples and division (see Figure 5).

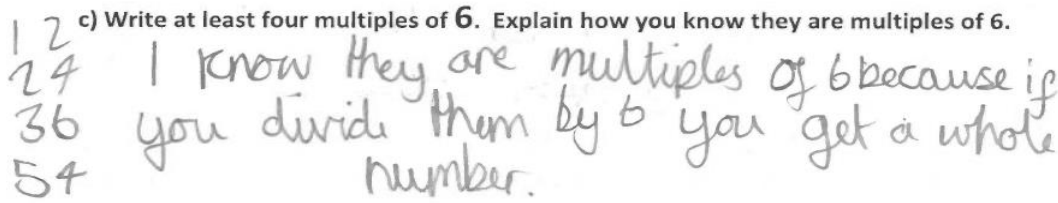

Figure 5. Samples from Student IHA

More widely across the sample for this question was evidence of misconceptions and poor explanations. One of the more common incorrect responses was to confuse multiples with factors, and instead of giving four multiples of six, the students gave four factors of six. Here are a selection of responses that highlighted this error.

Student JDU said, ‘1, 2, 3, 6, and you have to see if it can go into 6’. Student SSI said, ‘They all equal six’ and showed some examples of 2 x 3 = 6 and 6 x 1 = 6. Student IIS said, ‘You can’t do four’ . . . because she had identified 1, 2, 3 as multiples of six. These students seemed to be looking for factors of six instead of multiples, but there were other interesting responses, such as student RCO who said ‘1 and 6, and 2 and 3, because as pairs in columns and rows make 6’, which perhaps invoked the notion of arrays using six counters, and student SBL who said, ‘3, 2, 1, because there [sic] factors of 6 and they add up to six’, identifying factors of six rather than multiples, but then noticed that these factors total six – it would be good to think that an inspiring teacher might have pointed out that this makes 6 a ‘perfect’ number.

The quiz items on factors and multiples showed up some interesting features individually, but a comparison between the items showed that some students had insecure understanding on some aspects, which would seem unlikely given their response to other questions. For example, consider questions 1 and 3 from above. Table 4 shows the percentage of students who responded correctly or incorrectly to both the identification of factors and multiples in 7 x 5 = 35 and the identification of four multiples of six.

Table 4. Student responses to identifying factors and multiples in 7 x 5 = 35 and writing four multiples of six

	Student response	% n = 108
(a)	Correctly identified 35 as a multiple Correctly identified 35 as a multiple	20
(b)	Correctly identified 35 as a multiple Did not identify four multiples of 6	30
(c)	Did not identify 35 as a multiple Correctly identified four multiples of 6	16
(d)	Did not identify 35 as a multiple Did not identify four multiples of 6	27
(e)	No response to identifying 35 as a multiple No response to identifying four multiples of 6	7

Criterion (a) (both correct) and criterion (d) (both incorrect) were reasonably expected. That is, it was expected that children could recognize or identify and write a multiple in both examples, or in neither example. However, only 47% of students did one or the other. Also, it was reasonably expected that some students would not respond to either and 7% did that (criterion (e)). This leaves some 46% of the students who gave different combinations of responses. It begs the question, ‘Why would 30% of the cohort correctly identify 35 as a multiple, yet not be able to write some multiples of six (criterion (b))?’ Similarly, the other question is, ‘Why would 16% of the cohort correctly identify multiples of six, yet not be able to identify 35 as a multiple (criterion (c))?’

Student JLO seemed to have no idea about the factors and multiples in 7 x 5 = 35, but provided all the factors of 30 and showed them in the form of 15 x 2 =30, as justification. Student ZWH did exactly the same thing but also provided correct multiples of six and said that they can be divided by six. These instances were puzzling but far from uncommon across the sample. It suggested that students’ understanding was not sufficiently robust to recognise that the two questions were asking the same things. Some answered correctly when the multiple was in the context of a number sentence but could not do so when simply asked to write some multiples, and vice-versa.

Students TCL and JOA were interesting cases. For the 7 x 5 = 35 question, when asked which were factors and multiples, TCL wrote, 5, 10, 15, 20, 25, 30, 35, with the explanation of, ‘5 x tables’. While he didn’t actually answer the question, his answer suggested that he might know what multiples are. However, why would he then write 6, 1, 3, and 2 for multiples of six? Student JOA showed she had little or no idea of the rest of the questions but then where others have not done so, she showed the multiples of 6 as 6 x 10 = 60, 6 x 2 = 12, 6 x 8 = 48, 6 x 3 = 18, 6 x 4 = 24, with the comment, ‘I know my 6’s’. The apparent confusion is the students’ understanding is typified by Student EMA who was partially correct regarding 35 being a multiple, but could not provide four multiples of 6 (see Figure 6).

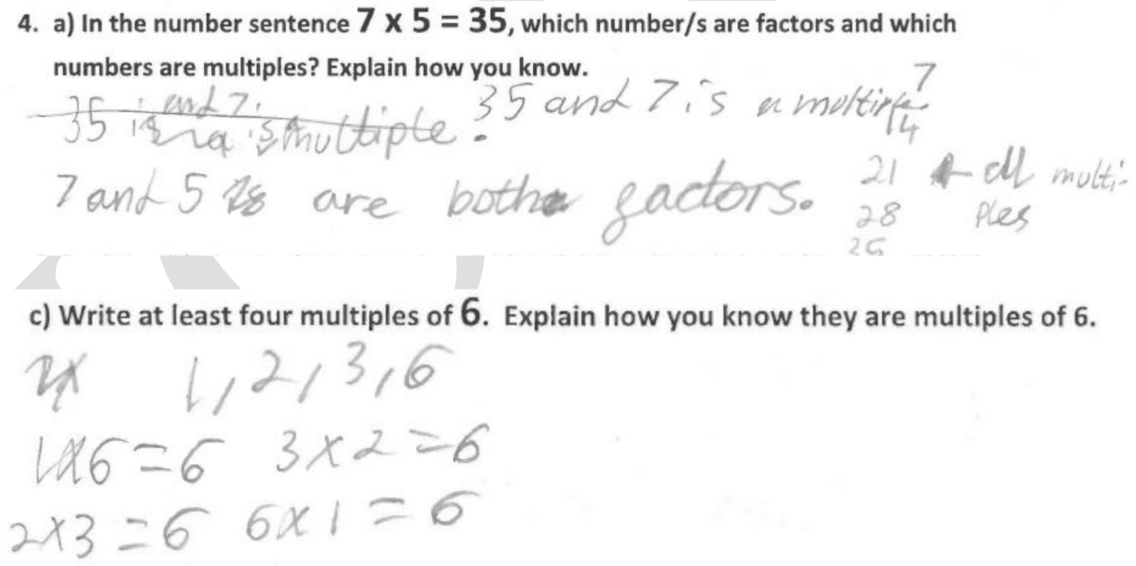

Figure 6. Samples from Student EMA

These comparative results from those two questions raise interesting questions - What is the issue here? Was it to do with the fact that they could or could not recognize the multiple unless it was in a number sentence? However, some (n = 4) did not identify the 35 but did identify multiples of six, and vice-versa (n = 5). Was that something to do with the words ‘multiple’ and ‘multiplied’? Was it linked to the phrasing ‘5 multiplied by 7’ and did that prompt them to say that the 7 is the multiple?

In a similar way to above, consider the link between questions 1 and 2. Table 5 shows the percentage of students who responded correctly or incorrectly to both the identification of factors and multiples in

7 x 5 = 35 and the identification of factors of 30.

Table 5. Student responses to identifying factors in 7 x 5 = 35 and writing factors of 30

	Student response	% n = 108
(a)	Correctly identified 7 and 5 as factors Correctly identified factors of 30	50
(b)	Correctly identified 7 and 5 as factors Did not identify factors of 30	4
(c)	Did not identify 7 and 5 as factors Correctly identified factors of 30	37
(d)	Did not identify 7 and 5 as factors Did not identify factors of 30	5
(e)	No response to identifying 7 and 5 as factors No response to identifying factors of 30	4

Criterion (a) (both correct) and criterion (d) (both incorrect) were reasonably expected. That is, one would have expected that children could recognise or identify and write a factor in both examples, or in neither example. Some 55% of students did one or the other. Also, it might be reasonably expected that some students would not respond to either and 4% did that (criterion (e)). This left some 41% of the students who gave different combinations of responses. This raised the question, 'Why would 37% of the cohort be unable to correctly identify 7 and 5 as factors, yet could correctly identify some factors of 30 (criterion (c))?'

Looking at individual responses, student BPE for 7 x 5 = 35 said, '5 is a factor and 7 is a multiple because the first number you multiply it then you look at the second number and that's a factor'. As with the results from Table 4, there appeared to be confusion between 'multiply' and 'multiple'. Also, when explaining factors of 30, she said that, 'they all add up to 30 equally', and for multiples of 6, she said, 'Because they all add up to 6'. Student LED identified the 7 and 5 as factors and identified some factors of 35, though her explanation that 'factors make the product' is not entirely convincing (see Figure 7).

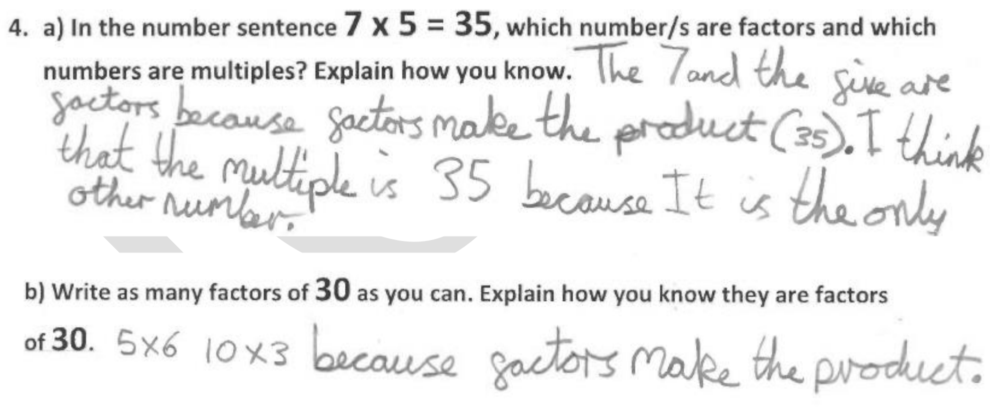

Figure 7. Samples from Student LED

The language around factors, multiples, multiply and divide seemed to be causing many issues for students, highlighted in this example from student OST who said '7 and 5 are factors because they are both divisible by 35'. Also, '5, 6, 10, 2, 3, 15 are all factors of 30 because they are divisible by 30'. The student appeared to have been taught the term 'divisible' but confused the meaning in the quiz.

Finally in this section, a consideration of the extent to which students could explain factors and multiples in terms of multiplication and division. Table 6 shows the percentage of students who could explain why factors are factors and multiples are multiples, in terms of multiplication and/or division.

Table 6. Student responses to explaining factors and multiples in terms of multiplication or division

	Student response	% n = 108
(a)	Explained factors and/or multiples in terms of multiplication and division on all three occasions.	15
(b)	Explained factors and/or multiples in terms of multiplication and division on two of three occasions.	18
(c)	Explained factors and/or multiples in terms of multiplication and division on one of three occasions.	30
(d)	Did not explain factor and/or multiples in terms of multiplication and division on any occasion	37

The questions in the three quiz items discussed here essentially ask the same thing.

- Which number/s are factors and which numbers are multiples? Explain how you know.
- Write as many factors of 30 as you can. Explain how you know they are factors of 30.
- Write at least four multiples of 6. Explain how you know they are multiples of 6.

It is not unreasonable to think that, if a student answered one of the questions correctly, s/he would be able to answer all three correctly. However, only 15% of the whole cohort did so. The same applies to students who did not explain it appropriately in any question – 37% of students did that. This more reflects the reasonable assumption that, if a student cannot explain the situation for one question, s/he would not do it for the other two questions. It is puzzling as to why 48% of the children answered either one or two questions correctly but not the others.

There is also a clear difference between the two Australian classes and the two UK classes, with a much higher proportion of the latter being unable to explain it for any question, which could be due to the curriculum content in the two countries, or the way in which the students' knowledge and understanding are tested at a national level, leading to teachers in one country adopting a different teaching approach.

Some students gave very succinct explanations about factors and multiples, for example, student BRO explained that '7 and 5 are the factors and 35 is a multiple because factors are what 35 can be divided by and the multiple is the answer', and the same student said '1, 2, 3, 5, 6, 10, 15, 30 because 30 can be divided by all those numbers'. Student WBL described the relationship between factors and multiples as '2 factors make a multiple. You x 2 factors together to make a multiple'.

However, explanations from the majority of students showed a lack of understanding about factors and multiples, and the language they used suggested that language itself may be a very significant issue in acting as a barrier to their learning. Student CBO offered '7 is a factor because it being timed by 5 so 5 is a number', while student JCA said 'A factor is something that can be into (sic), and a multiple is 2 numbers x together, and then makes an answer'. Student JMA observed that numbers can be both factors and multiples in the same number sentence when saying '7 and 5 are factors and they are multiples because they equal 35'. Finally, student MGI admitted to not understanding but was happy to suggest '7 and 5 are multiples and 35 is a factor because 7 and 5, you have to multiply and 35 is just a factor. Don't know why but that's what I think'. The confusion between 'multiple' and 'multiply' repeatedly appeared as a possible factor (sic) in students' lack of ability to explain factors and multiples. This has raised questions that require further analysis.

- To what extent does the context of the question affect their reasoning?
- Do they have a sufficiently robust understanding of the concept?

DISCUSSION and CONCLUSION

It is suggested that the sample size (n=108) is sufficiently large to be able to make some generalizations, and a number of observations can be made from the data presented in Tables 1, 2, and 3, and the ensuing discussion. First, with regard to identifying factors (McKenna, 2019; Richards, 2007; Rickard & Earle, 2019) in the context of a multiplication sentence, approximately half of the participants were able to do so. However, when asked to identify factors without such a context, more than three quarters were able to do so. Furthermore, less than half of the participants wrote the factors in pairs (Feldman, 2014; Rickard & Earle, 2019; Parker, 2019), suggesting an incomplete understanding of what factors are. Second, when asked to identify a multiple in the context of a multiplication sentence, less than half of the participants did so, and even less could identify multiples when no context was provided. Third, with regard to explaining why they identified particular numbers as factors or multiples, slightly more than half of the participants offered an explanation for factors in terms of multiplication or division, yet slightly more than a quarter were able to do so for multiples. The variation in these results is surprising as it would be reasonable to expect that, because factors and multiples are inextricably linked, responses for each would be similar.

Tables 4 and 5 contain comparisons between pairs of questions related to the same idea – factors or multiples. Each of the first two questions from the MTQ essentially asked participants the same things that is to identify factors or multiples. It could reasonably be expected that participants would successfully do so for both questions or neither question. In the case of multiples, less than half responded correctly or incorrectly to both questions yet the same proportion responded correctly to one question but not for the other. In the case of factors, the pattern was similar. Slightly more than half of the participants responded either correctly or incorrectly to both questions yet more than 40% responded correctly to one question but not to the other.

Table 6 considered the participants' explanation of what constituted factors and multiples in terms of multiplication or division. There were three opportunities for this to occur. Again, it would be reasonable to expect that participants would offer a correct explanation in all three instances or in none, yet slightly more than half did so. Indeed, slightly less than half provided an appropriate explanation on one or two occasions but not the others.

In seeking an explanation for the inconsistency of participant responses, the following observations and comments are made. First, many participants appear to confuse the terms 'multiple', 'multiply', and 'multiplied', along with the '×' sign. Evidence has been presented to support that. Second, it appears that a large number of participants fail to recognize that factors and multiples do not exist without one another, and that they must be 'a factor of something' and 'a multiple of something'. This is manifest in the fact that many participants did not show factors in pairs and/or did not show all possible factors, further suggesting that they did not realize that to identify one factor means that there must be a 'partner factor'. Third, the inconsistent results might be explained by students failing to see that the questions posed essentially asked the same things, which suggests that their knowledge of factors and multiples is not sufficiently robust to enable them to make connections between ideas, a situation which is concerning if we are trying to develop students who are multiplicative thinkers (Day & Hurrell, 2015; Drake, 2012; Siemon, et al., 2021).

Perhaps these results are hardly surprising given that the review of the pertinent literature (for example Griffiths, 2017; McEachran, 2008; Ollerton & Cooper, 2017; Richards, 2014; Thompson, 2012) pointed out inconsistencies in the use of terminology on the part of contributors to teacher journals. In any case, there are some clear implications for teaching about factors and multiples. First, factors and multiples need to be taught together as part of the multiplicative situation about equal groups. The base for doing this is provided by the multiplicative array, in which the factors are represented by the number of rows and the number in each row. The multiple, product, or total is represented by the whole array. It is of critical importance that the language of factors and multiples is used as part of the daily discourse of the mathematics classroom and that children are given ample opportunities to discuss and explain their thinking. Second, it is suggested that the language used is carefully considered. It appears that there is confusion based on the terms 'multiply' and 'multiplied' so it might be prudent to refer to multiplication and division sentences in terms of 'six groups of five' as opposed to 'six multiplied by five' or 'six times five'. Third, in keeping with the use of the array, it is likely to help if multiplication and division are taught as being different ways of considering the same situation – that is, the multiplicative situation (Downton & Sullivan, 2017; Hurst, 2015; Siemon et al., 2021). This links with the inverse relationship between multiplication and division and the language of factors and multiples. If we know both factors, they are multiplied to find the product or multiple. If we know one of the factors – number of groups or the number in each group, and the total, product or multiple – we divide to find the other factor.

As noted at the end of the literature review, the proposition on which this article was based was that children need to hold a strong understanding of factors and multiples and concepts associated with them if they are to grasp the mathematical structures that are underpinned by factors and multiples. It would appear from the results of this study such an understanding may be held by some children but not by all. Indeed, the proportion of children who did not demonstrate a deep or broad conceptual understanding is of great concern.

Limitations of this study

Although we suggest that the sample size (n = 108) is sufficiently large to be able to make some generalizations, we recognise that this is not an exhaustive sample. It is large enough to be indicative without being conclusive. Secondly, although a questionnaire is an efficient manner in which to collect data, for some students the act of writing may limit their capacity to show the depth of their understanding. A one-to-one interview might be employed to alleviate this issue, but is an intrusive and time consuming exercise.

Ethical Collection of Data

All data was collected following standard protocols as dictated by the University of the lead-researcher. Requirements for ethical research was met for the necessary sector/systems and permission to work in particular schools, was granted by the schools themselves.

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