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A Curriculum-Based Approach to Learning Trajectories in Middle School Algebra

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A Curriculum-Based Approach to Learning Trajectories in Middle School Algebra

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Abstract

Our aim is to contribute to the body of research on learning trajectories (LTs) in mathematics by making visible a process for articulating a hypothetical learning trajectory implicit in a widely adopted, reform-based, middle-grades mathematics curriculum. In doing so, we highlight considerations, decisions, and challenges we faced as part of this work. By describing our LT articulation process, our aim is to highlight ways in which curriculum-specific LTs can be articulated to serve as a more proximal and instrumental tool for teachers' instructional practice. Furthermore, to illustrate we describe how the products of the work were used in practice-based professional learning experiences with middle-grades mathematics teachers.

Keywords: learning trajectories, algebra, curriculum analysis, professional development.

Un Enfoque Basado en el Plan de Estudios para las Trayectorias de Aprendizaje en Álgebra de la Escuela Intermedia

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Resumen

Nuestro objetivo es contribuir al cuerpo de investigación sobre trayectorias de aprendizaje (TA) en matemáticas al hacer visible un proceso para articular una trayectoria de aprendizaje hipotética implícita en un currículo de matemáticas de grado medio ampliamente adoptado y basado en la reforma. Al hacerlo, destacamos las consideraciones, decisiones y desafíos que enfrentamos como parte de este trabajo. Al describir nuestro proceso de articulación de TA, nuestro objetivo es resaltar las formas en que se pueden articular las TA específicas del plan de estudios para que sirvan como una herramienta más próxima e instrumental para la práctica educativa de los maestros. Además, para ilustrar, describimos cómo se usaron los productos del trabajo en experiencias de aprendizaje profesional basadas en la práctica con profesores de matemáticas de grados medios.

Palabras clave: trayectorias de aprendizaje, álgebra, análisis curricular, desarrollo profesional.

Recent discussions in the mathematics education community have increasingly argued that effective teaching requires teachers to continually adjust their instructional decisions, over the course of instruction, and in response to students' mathematical thinking (Jacobs & Empson, 2015; Sztajn, Confrey, Wilson & Edgington, 2012). An important component of effective instruction is knowing what students' current understandings are and where these understandings are located along learning trajectories (LTs) in order that teachers' next instructional steps can be tailored accordingly. As such, LTs provide one avenue for strengthening teachers' instruction in that teachers become familiar with how content develops over time, potential hypothetical LTs individual students may pursue, conceptual landmarks and obstacles students might meet along the way, criteria for identifying where students are on an LT, and for determining appropriate next steps for individual students. Thus, LTs have the potential to enhance teachers' instructional practice as they describe pathways students are likely to follow in developing their understanding of core concepts.

However, research on LT-based professional development has demonstrated that it is often challenging for teachers to integrate LTs into their curricula and everyday instructional practice. In particular, Wilson, Mojica & Confrey (2013) describe how teachers' reported struggling to incorporate the LT into their mathematics curricula. Indeed, most current approaches to LT development are curriculum independent, which may constitute a contextual gap between the articulation of a particular LT and its usefulness in teachers' everyday instructional practice. If an aim of LT articulation is to support teachers' interpretation of students' conceptual development along a trajectory—and to subsequently make appropriate next instructional decisions—teachers are left to do the demanding and highly conceptual work of connecting theoretical LTs to their daily curriculum. Consequently, we posit that a curriculum-specific approach to LT development might be more productive for enhancing teachers' instructional practice.

In this paper, we contribute to the body of research on LTs in mathematics by making visible a process for articulating a hypothetical learning trajectory implicit in a widely adopted, reform-based, middle-grades mathematics curriculum. In doing so, we highlight considerations, decisions, and challenges we faced as part of this work. Specifically, we make this process visible using the case of a hypothetical LT for middle school algebra, with a particular focus on linear functions and equations as part of the iFAST Project

(<https://ifast.uic.edu>). By describing our LT development process and the corresponding components, our aim is to highlight ways in which curriculum-specific LT approaches can be designed in ways that are more proximal and instrumental to teachers' instructional practice. We conclude with an illustration of how our curriculum-based HLT was taken up by teachers in their discussions of concepts within the HLT.

Conceptual Background

Learning Trajectories Definitions in School Mathematics.

A variety of definitions of the LT construct exist in the research literature, with substantial differences in focus and intent (see e.g., [Clements & Sarama, 2004](#); [Corcoran, Mosher, & Rogat, 2009](#); [Simon, 1995](#)). According to the Consortium for Policy Research in Education (CPRE) Report on Learning Progressions for Mathematics ([Daro et al., 2011](#)), LTs are empirically grounded and testable hypotheses about how, with appropriate instruction, students' understanding of, and ability to use, core concepts and explanations and related practices grow and become more sophisticated over time. These hypotheses describe the pathways students are likely to follow to develop mastery of core concepts. Specifically, in our work we ascribe to the definition of LT proposed by [Confrey et al. \(2014\)](#): A researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representations, articulation, and reflection, towards increasingly complex concepts over time. We further ascribe to the idea of a conceptual corridor ([Confrey et al., 2014](#)), which incorporates the possibility of multiple pathways toward learning, as well as attention to the landmarks and obstacles that students typically encounter along those pathways.

Approaches to learning trajectories in mathematics.

In the mathematics education community, LT researchers differ in how they conceptualize LTs, including the grain size of descriptions of levels of student

thinking, how students move among the levels, and ways in which LTs are validated (Confrey, Maloney, Nguyen & Rupp, 2014). Such conceptual differences also include different approaches to LT development.

In an effort to identify a set of taxonomies that distinguish and describe these varied approaches, Lobato and Walters (2017) conducted an extensive literature review in the field of learning trajectories in mathematics and science. Their work is the first of this type in that their object of study is the concept of learning trajectories/progressions (LT/Ps) and corresponding approaches. The resulting taxonomies show how these approaches differ in a variety of dimensions including: objects of learning (e.g., cognitive conceptions, forms of discourse, observable strategies, or textbook tasks), theoretical perspectives (e.g., Piagetian schemes and operations, hierarchic interactionism, emergent perspective, etc.), scale (e.g., they can vary from addressing a single concept to spanning multiple topics and grade levels), and learning focus (e.g., individuals, mathematical practices of a collective classroom, or intertwining of teaching and learning). The authors identified seven distinct LT/P approaches: (1) cognitive levels (e.g., Barrett, Clements, Klanderma, Pennisi, & Polaki, 2006; Bishop, Lamb, Phillip, Whitacre, & Schappelle, 2014; Clark, 2006; Mitchelmore & White, 2000), (2) levels of discourse (e.g., Berland & McNeill, 2010; Gunckel, Mohan, Covitt, & Anderson, 2012; Jin & Anderson, 2012), (3) schemes and operations (e.g., Hackenberg, 2014; Hackenberg & Tillema, 2009; Moore, 2013; Nabors, 2003; Tillema, 2014; Weber & Thompson, 2014), (4) hypothetical learning trajectories (e.g., Clements & Sarama, 2004; Clements, Wilson, & Sarama, 2004; Meletiou-Mavrotheris & Papanastasiou, 2015), (5) collective mathematical practices (e.g., Bowers, Cobb, & McClain, 1999; Cobb, 1999; Cobb & Whitenack, 1996; Cobb & Yackel, 1996; Gravemeijer, Bowers, & Stephan, 2003; Rasmussen & Stephan, 2008; Stephan & Akyuz, 2012), (6) disciplinary logic and curricular sequence (e.g., Bernbaum, Wilmot, Schoenfeld, Wilson, Champney, & Zahner, 2011; Common Core Standards Writing Team, 2013a, 2013b), and (7) observable strategies and learning performances (e.g., Confrey & Maloney, 2010; Confrey, Maloney, & Corley, 2014; Vermont Mathematics Partnership's Ongoing Assessment Project, 2013, 2014a, 2014b; Steffe & Thompson, 2000).

In the first four approaches described by Lobato and Walters (2017), researchers capture the evolution of increasing mathematical sophistication at the level of the individual learner. These approaches draw primarily from

cognitive and constructivist perspectives on mathematics learning, focusing on how mathematical ideas develop in individual learners as a developmental progression through and construction of conceptual schemas. In the *collective-mathematical practices* approach, the focus turns toward shared knowledge production and the socially constructed learning that occurs at the level of the *community*. This perspective draws from the *emergent perspective* of mathematics learning (Cobb & Yackel, 1996), where individual learning is coordinated within collective constructs. A learning trajectory in this framework takes the form of a sequence of classroom mathematical practices together with conjectures about the means of supporting their evolution from prior practices. One of the main analytical tools used in this approach involves *documentation of collective activity* (Rasmussen & Stephan, 2008), a multi-stage process where the unit of analysis is students' collective discourse in the classroom video.

The next two approaches consider trajectories as developing according to rational or hypothesized sequences of disciplinary concepts, rather than directly through empirical observation of student learning. In the *disciplinary logic and curricular coherence* approach, LTs are developed through researchers' reflection upon expert knowledge of the domain. In this approach, LTs are constructed from synthesized findings from studies on students' learning and from articulated learning goals in standards documents. The *hypothetical learning trajectories* (HLT) approach was conceived originally as part of a model of teachers' decision making (Simon, 1995) within the context of implementing instructional supports for learning. The concept of HLT was introduced to capture the result of a process in which a teacher posits a conjecture regarding her students' current understanding of a targeted concept and then develops learning activities that will support them in constructing more sophisticated ways of thinking towards a particular goal. Since its initial development, this approach has been adapted by researchers and curriculum developers as a way to construct an initial, hypothesized learning trajectory as a series of learning activities, based on informed conjectures about student learning related to a particular conceptual pathway. The process has two basic stages: first researchers collect baseline data via interviews, written assessments, etc.; second, an initial HLT is constructed based on the data collected in the first stage and on literature to guide instruction in a teaching experiment setting. Finally, in the *Observable*

Strategies and Learning Performances approach, as its name indicates, each level is described in terms of observable behaviors, strategies and/or learning performances.

While all of these framings offer useful perspectives into LT development that serve their particular sets of purposes very well, the approaches are typically removed from teachers' everyday practices and resources, which hinders the potential of LTs from becoming an instrumental tool for teachers to enhance their daily teaching practice. Even though we posit that LTs crafted as independent from the curriculum are useful for the research community, when using such an approach in working with teachers, it may not prove so beneficial. This is because, in order for teachers to integrate LTs into their instruction and formative assessment practices, they would first need to do the foregrounding conceptual work of making explicit connections between the curriculum and a given LT. Moreover, curriculum-independent approaches may introduce pathways that are not closely aligned with how concepts develop across lessons, units, and grade levels in a given curriculum program. Specifically, the process of mapping curricular tasks to a learning trajectory—or conversely, of articulating the HLT inherent within a sequence of curricular activities—is not addressed as a pivotal element in the LT literature, yet it may be a missing link in leveraging LTs as a useful tool in teachers' everyday instructional practices. In other words, links between researcher-produced LTs and curricular tasks are treated as a black box in curriculum independent LTs approaches. Without a direct connection between LTs and curriculum, teachers' professional learning related to mathematics LTs develops as largely separate from their learning related to curriculum implementation.

As it is the case that curriculum materials are generally highly influential in teachers' decisions about the mathematics they are going to teach (Brown, 2009; Collopy, 2003, Remillard, 2005)—which in turn has considerable implications for pathways of student thinking—we pursued an approach to LT articulation that would directly support teachers' inquiry into student learning as it occurs along the LTs within the curriculum materials. Further, we established as our central focus the connections among these elements, considering their interdependence and collective value in support of teachers' everyday instructional practices. This approach afforded an integrated approach to supporting teacher learning around LTs, framing it as an inquiry into how understanding of LTs within a curriculum support a more robust implementation of that curriculum with students. In this model of professional

learning, teachers and researchers concurrently investigate how, at a relatively fine level of detail, conceptual pathways develop across a series of tasks and lessons in the curriculum. In this part of the process, there is specific attention to the conceptual understandings involved in particular tasks, and then to the conceptual connections and transitions that arise in subsequent tasks. Integrated with this investigation, teachers also consider how these understandings can inform their teaching practices as they implement the curricular activities: the questions they ask, the student reasoning they watch for, the ideas that they might highlight in students' discussions, etc. As they implement the activities in their classrooms, they have opportunities to reflect, revise, and plan for subsequent instruction based on what they and the researchers observed in the classroom.

An Alternative Approach to Learning Trajectory Articulation

In our work as part of the iFAST Project, we use mathematics curriculum materials as the starting point for articulating the hypothetical LTs contained therein. Briefly, the iFAST Project is a multi-year project focused on articulating LTs in middle school algebra to inform the design of LT-based professional development for teachers. Thus, two main components of the project are: 1) to better understand students' learning pathways within middle school algebra, and 2) to enhance teachers' understanding of LTs to inform their use of effective assessment practices in the classroom. A central premise underlying our work is that high-quality formative assessment practices depend on teachers having a clear sense of learning goals, curricular LTs, criteria for locating students along the trajectories, sharing this information with students, and using it to inform instructional decisions.

The development of proficiency in algebra holds a unique role in students' success in mathematics, serving as a gatekeeper to more advanced mathematics and affecting mathematics achievement in high school and beyond. The Common Core State Standards for Mathematics has reconfigured the sequencing of algebra content across grade levels, introducing it in Grades 6 and 7 with a major focus in Grade 8, and calls for students to learn algebra earlier and at more advanced levels than has traditionally been the case. As a result, whether or not middle school mathematics teachers are teaching a course designated as Algebra 1, they are being held accountable for all

students' learning of rigorous content related to strands in algebraic functions and equation-solving. In the iFAST Project, our LT work is centered on linear functions and linear equations topics in middle school algebra.

We focus on the Connected Mathematics Project 3 (CMP3) curriculum as it is widely used and the treatment of linear functions and equations topics is consistent with other functions-based curricula in the U.S. As our main focus is understanding students' learning pathways within CMP3, we needed first to generate a map tracing the hypothetical learning opportunities of algebra concepts embedded in the curriculum. Of course, this is only a hypothetical description as the actual learning opportunities students encounter are mediated by multiple other factors (e.g., school, teacher, curriculum implementation, instruction, etc.). Thus, our approach stands in contrast with other approaches in at least two ways: it is specific to a curriculum program (CMP3) and it included the explicit mapping of an HLT connected to that curriculum. Our use of the term *hypothetical* differs from the widely known approach Hypothetical Learning Trajectories originally developed by Simon (1995), in which a teacher posits a conjecture regarding her students' current understanding of a targeted concept and then develops learning activities that would support students' learning. In fact, this approach has been adapted by many researchers such that HLTs are now often developed as a product of research and not of teachers' analyses. One example is that of Clements and Sarama (2004) on a variety of domains including number and operations, measurement, and geometry in K-5. In the iFAST project we started by conducting an analysis of the curriculum and use the term hypothetical to describe the hypothesized trajectory students might follow according to the tasks the curriculum lays out.

Moreover, in our approach landmarks in the HLTs are anchored on the math tasks included in the curriculum materials which is one of the main elements used to plan and enact instruction in teachers' everyday practices. Our approach brings together the hypothetical and the observable strategies approaches together. Using the curricular tasks, we generate levels and describe them by illustrating with common strategies as depicted by student written work and video.

In the sections that follow, we describe the process we developed to generate such a map and make visible the process by which we developed a curriculum-based hypothetical LT for middle school algebra.

Articulating a Hypothetical Learning Trajectory

Initial considerations.

As we embarked on the process of instrument design to understand what students learn and what obstacles they encounter, the need to understand what learning pathway were intended for students to follow in the curriculum became evident. In order to map the opportunities provided by the curriculum to learn about specific algebra concepts we had to devise a process to understand and represent them. We aimed to produce a map of such opportunities as presented in the curriculum, thus we decided to start working with the unit goals provided by the curriculum materials.

Based on recognition that we would have finite opportunities to engage with teachers in the professional development work and based on our intention to delve deeply into the process of HLT articulation, we narrowed our content focus considerably over the course of the project. Ultimately, we focused on the articulation of proportional linear function concepts and the subsequent transition into non-proportional relationships. Of particular interest within this strand was the writing of algebraic rules from other function representations (i.e., verbal situations, diagram, graphs, and tables). From our initial work with teachers, we identified this topic as a high-leverage area within the linear functions and equations domain. Thus, we selected curricular units (i.e., focus units) that focus on proportionality, functions, proportional functions and linear functions. Within CMP3, we selected the following units: Grade 6 – Comparing Bits and Pieces (from now on CBP), Grade 6 – Variables and Patterns (from now on VP), Grade 7 – Comparing and Scaling (from now on CS), Grade 7, Moving Straight Ahead (from now on MSA), Grade 8 – Thinking with Mathematical Models (from now on TWMM), and Grade 8 – Say It With Symbols (from now on SIWS).

In addition, we set out to generate a product that would explicitly link curricular math tasks used by teachers, curricular learning goals as presented in the curriculum and mathematical themes related to linear functions and equations topics as presented in extant research literature.

Generating a Curricular Map.

The following process is presented in a distilled way but it took several iterations for it to emerge in a way that addressed limitations and implemented desired improvements. We will note some of these improvement cycles in our description of the final version of the process.

Assigning goals to tasks.

As mentioned earlier, since we aimed to articulate the curricular map as an explicit link between the HLTs and teachers' everyday practice, we used the mathematical tasks provided by the curriculum as anchors in the trajectory. By systematically reviewing the unit learning goals included in the curriculum and the mathematics tasks for each unit, we proceeded to assign unit goals to each task included in each unit. Through this process, tasks could be explicitly linked to learning goals, and vice versa. Each of two researchers would independently assign goals to tasks, then come together compare and resolve any discrepancies. We recorded this information using database software.

Clustering process.

Once we had goals mapped to each and every math task in the curriculum, the first challenge we faced in creating this map was to come up with a useful *grain size* to describe the new content that students were offered an opportunity to learn about. We first considered working directly with the curricular goals corresponding to focus units, but a map spanning across Grades 6-8 turned out to be swarming and difficult to be used by others. Therefore, we decided to group unit curricular goals linked by a common mathematical theme. This process is what we coined the *clustering* process.

One focus unit at a time, three researchers first independently grouped goals into clusters; second, researchers identified discrepancies and discussed the clusters by looking both at the math tasks and the goals as stated in the curriculum until an agreement was reached. We iterated this process throughout all focus units. Table 1 provides a sample of the product for the focus unit Moving Straight Ahead in grade 6 including clusters, goals and math tasks and their correspondence.

Table 1

Clusters, Goals and Math Tasks for unit Moving Straight Ahead in 6th grade

Cluster within Unit	CMP3 Goals	CMP3 Problems
<i>Equation Solving</i>		
Gr7-MSA-3: Show and write equivalent expressions with the purpose of equation solving	Show that two expressions are equivalent	3.3 A-B, 3.4 B
	Write and interpret equivalent expressions	
	Solve linear equations in one variable using symbolic methods, tables, and graphs	
<i>Linear Functions</i>		
Gr7-MSA-1: Recognize and translate linear relationship among its multiple representations	Translate information about linear relationships given in a contextual setting, a table, a graph, or an equation to one of the other forms	1.3, 1.4, 2.2 A, 2.4 A
	Construct tables, graphs, and symbolic equations that represent linear relationships	
	Solve problems and make decisions about linear relationships using information given in tables, graphs, and equations	
Gr7-MSA-4: Represent linear relationships by writing equations, and describe what the variables and numbers in the equations represent.	Write equations that represent linear relationships given specific pieces of information, and describe what information the variables and numbers represent	1.3, 1.4, 2.2., A.3, 4.4 A-B
Gr7-MSA-2: Make connections between slope and rate of change in the context of linear relationships	Identify the rate of change between two variables and the x- and y-intercepts from graphs, tables, and equations that represent linear relationships	1.1, 1.2, 1.3 A4, 2.2 A4, 2.4 A4, 4.1
	Make a connection between slope as a ratio of vertical distance to horizontal distance between two points on a line and the rate of change between two variables that have a linear relationship	
	Recognize that $y=mx$ represents a proportional	

Table 1 (continue)

Clusters, Goals and Math Tasks for unit Moving Straight Ahead in 6th grade

Cluster within Unit	CMP3 Goals	CMP3 Problems
<i>Linear Functions</i>		
	relationship	
	Identify and describe the patterns of change between the independent and dependent variables for linear relationships represented by tables, graphs, equations, or contextual settings	
Gr7-MSA-5: Bridge between an ordered pair (input, output) in a functional context and the solution to a single unknown equation	Recognize that the equation $y=mx+b$ represents a linear relationship and means that $mx+b$ is an expression equivalent to y	2.4 A-E, 3.1
	Recognize that linear equations in one unknown, $k=mx+b$ or $y=m(t)+b$, where k , t , m , and b are constant numbers, are special cases of the equation $y=mx+b$	
	Recognize that finding the missing value of one of the variables in a linear relationship, $y=mx+b$, is the same as finding a missing coordinate of a point (x, y) that lies on the graph of the relationship	

External validation of clustering process.

Once the researchers had completed the clustering process for all focus units, we convened a group of external reviewers comprised of mathematics education researchers with expertise in algebra and CMP who conducted an external validation of the clustering process. This work entailed: (1) assessing the relationship between lesson problems and assigned goals, (2) assessing how the goals were group together forming a cluster, and (3) assessing whether the lesson problems selected were considered representative. External reviewers agreed with most of our work but provided some minor suggestions for us to consider. One of the main contributions of this round of feedback pushed our team to think about the math tasks in two different but complementary uses. One as an exemplary *lesson problem* where students have the opportunity to learn and laser in a concept vs. *professional development lesson problems* that would afford the most learning

opportunities for teachers in the limited time an after-school professional development workshop affords, for example.

Connecting clusters across grades.

Thus far in the process, we had goals, clusters and math tasks at the unit level without an explicit connection across units and grades. In connecting the clusters among themselves two unforeseen processes unfolded: (1) in order to be able to express successive refinement in a trajectory we found it necessary to refine the language of the clusters, and (2) we revised some of the clusters and re-grouped the goals and tasks that comprise the clusters. Several elements were used in determining the curricular map across grades. The first organizing element is time given that clusters are organized in columns according to a specific grade moving from 6th grade on the left to 8th grade on the right. Within a specific column (i.e., grade), we followed the order of the units chronologically according to the curriculum. To organize clusters and decide how they relate to each other we made decisions by looking at the math tasks, the goals and clusters; we pay specific attention to successive refinement of a same concept. The current version of the CMP3 Based Hypothetical Learning Trajectories Map has been included in Figure 1.

Future Directions: A Vision for a Learning Trajectories and Teacher Professional Development Framework

HLTs and Conceptual Fields.

As previously discussed, one of the underexplored aspects of LTs is whether and how they can become directly useful for teachers in classroom instruction. In this project, we articulated an HLT associated with an existing mathematics curriculum, CMP3. A rationale for this approach is that daily classroom instruction is influenced in large part by the tasks and activities prescribed in the curriculum being implemented in the classroom (Brown, 2009; Collopy, 2003, Remillard, 2005). With a curriculum-specific approach, the HLT can be interpreted by teachers in direct connection to their day-to-day implementation of CMP3 lessons, without the additional step of aligning a generalized HLT with their daily plan of instruction. This step has already

been built into the mapping process described above. As such, articulating an HLT specific to the classroom curriculum represented the first step in making LTs useful to teachers’ practice.

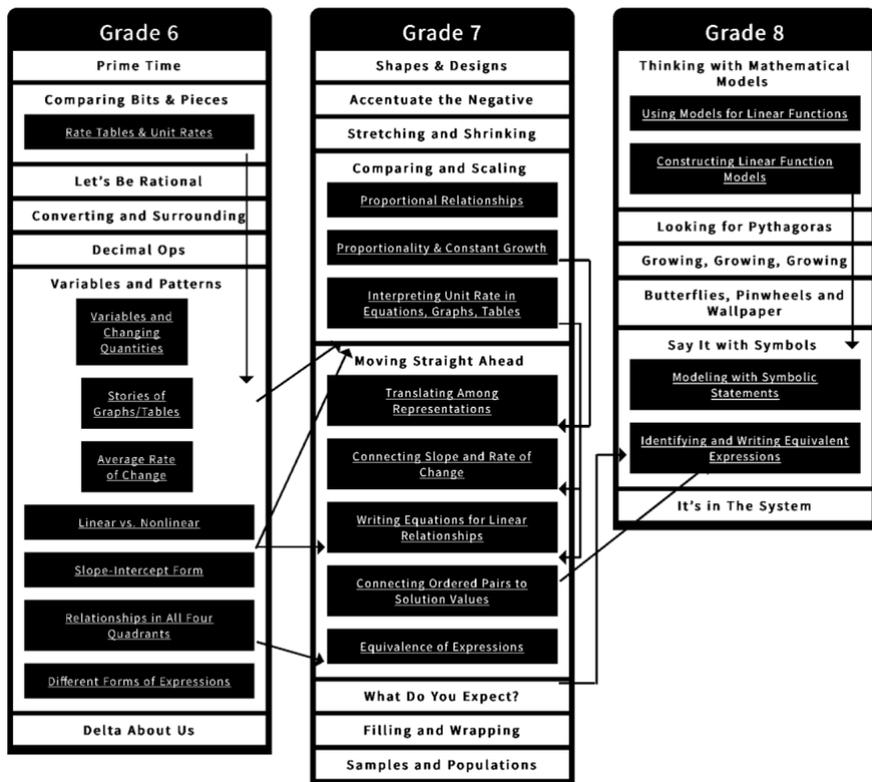


Figure 1. CMP3-Based Hypothetical Learning Trajectory Map

The subsequent steps involved a great deal of conceptual work on the part of teachers, not only to explore the specifics of how the HLT advanced through the lessons and units of the curriculum, but also to develop an orientation toward instruction as a progression along an HLT. We conceptualize teachers’ ability to make use of learning trajectories in their teaching as a shift from what we refer to as a *lesson orientation* to a *learning-trajectory orientation*

toward instruction. A lesson orientation implies that the lesson is the primary unit of instruction, and that student learning is interpreted in terms of performance on tasks, activities, and procedures, largely as it occurs within self-contained daily lesson activities. A learning-trajectory orientation, on the other hand, views instruction as an advancement through a pathway of interrelated concepts over time, where student learning is interpreted in terms of the representations students formulate and reasoning they communicate in relation to those concepts.

In conceptualizing the learning-trajectory orientation, we draw from Vergnaud's theory of *conceptual fields*. A conceptual field represents the complex tying together of related mathematics concepts and the variety of situations in which those concepts arise. A concept does not arise solely in a single situation, nor does any mathematical situation typically involve only a single, isolated concept. Rather, concepts and situations are meshed together in an intricate though interpretable web. If we consider, for example, the concept of rate of change, a collection of possible situations come to mind where rate of change is implicated—cost per unit of items bought, speed as distance per unit time, the constant of proportionality in a linear proportional relationship, slope of a linear function, etc. It becomes clear that each of these situations allows for a particular aspect of rate of change to come to light. The concept cannot be fully articulated within a single situation but rather develops fully over a variety of situations, encountered over time. Conversely, a single situation generally cannot be described mathematically in terms of a single concept. A task involving finding the cost of multiple apples given the unit price per one apple may be “about” unit rate, but it may also be about a number of other concepts as well (e.g., function relationships, multiplicative reasoning, place value, etc.) depending on the specifics of the mathematical situation and task. When viewing an LT from a perspective of conceptual fields, concepts build through a series of situations as they arise in the curricular tasks, intertwined with other interrelated concepts. An articulated HLT, then, serves to “trace out”—perhaps in a usefully oversimplified fashion—how particular conceptual fields build through the curriculum in order to make them more visible and interpretable by teachers in their practice. Movement toward a learning-trajectory orientation involves seeing lesson activities as more than mathematical ends in themselves where students either “met” or “did not meet” a particular goal, but as iterations of larger concepts

that embody some particular aspect of those concepts developing over time and in varied situations.

In our work, we propose that a learning trajectory orientation may be consequential to teachers' classroom formative assessment practices, especially in how teachers interpret and respond to students' reasoning.

The Teacher Professional Development Context in the iFAST Project.

In this professional development project, we worked with a small subset of three grade-six iFAST Project teachers over the course of one summer and the following school year. The program consisted of three full-day work sessions, one in the summer and two during the following school year, with several observation/coaching visits distributed over the school year. These three teachers had already engaged previously in iFAST professional development that focused on basic formative assessment and LT principles with a larger group of teachers.

This additional PD was designed for a small subgroup of teachers who expressed specific interest in extending their LT-based formative assessment practices in their classrooms. Activities were designed with the specific goals of: 1) providing opportunities for teachers to make detailed connections between the HLT goals/clusters and the specific curricular tasks associated with them, and 2) providing opportunities for teachers to interpret their students' reasoning (captured both in classroom video and in artifacts of written work) in terms of the articulated HLTs. The design was also conceived as a co-development project among researchers and teachers, where researchers established the general structure and direction of inquiry in relation to the HLT, but the detailed processes and products of the PD were largely shaped by the mutual participation of researchers and teachers together.

Connecting HLTs to tasks.

Prior to the first PD session, the research team selected a portion of the HLT on which teachers would focus their attention. Concentrating on a relatively small section of the grade-six portion of the HLT map would afford teachers an opportunity to examine a limited number of interrelated concepts in greater depth and detail, and thus they would be able to focus on LT-oriented

processes as well as content-related issues. Because we intended for teachers to be able to integrate their new learning concurrently with their classroom instruction, we began with the earliest portion of the grade-six map, which included unit rate concepts addressed in the CMP unit “Comparing Bits and Pieces”. Later in the school year, we extended along the HLT to focus on rate-of-change concepts in the “Variables and Patterns” unit.

The initial task for teachers was to develop a detailed “mini-trajectory” for how unit rate and rate-of-change concepts were developed within these two CMP units—and subsequently, how they were connected to other proximal concepts—focusing on three landmark lessons. We thought of “mini-trajectory” as an analysis of how a concept grows in complexity and mathematical sophistication over the course of several lessons related to the specific tasks and representations in each lesson, guided by questions such as: (a) What mathematical ideas would students develop in the specific tasks in each lesson?, (b) What understandings would students need to carry from Lesson A in order to be successful in Lesson B? Lesson B to Lesson C?, (c) What ideas in these lessons are relevant throughout all three of these lessons? and, (d) How does the concept of unit rate and rate of change grow and advance in the specific tasks and representations from Lesson A to Lesson B to Lesson C?

The activity led to the production of a detailed, informal set of artifacts showing how teachers mapped out the advancement across lessons of concepts that were central to the unit-rate and rate-of-change mini-trajectories, as well as their accounting of concepts that were more peripheral. These mappings were then further discussed during one-on-one observation/coaching visits between teachers and project researchers. Key concepts, tasks, and questions that were identified during the group sessions served as the focus for the one-on-one coaching conversations when these lessons were taught during the school year.

This analysis periodically panned outward to the larger HLT map as well, with teachers looking forward to other clusters on the map that developed out of the landmark rate-of-change tasks. In the spring PD session, a similar set of activities progressed into variable concepts, linearity, and slope-intercept form as teachers moved into the “Variables and Patterns” unit with their students.

Interpreting student reasoning through HLT lens.

Another aspect of developing a learning-trajectory orientation involved interpreting student reasoning as progressing in relation to a learning trajectory, rather than as a set of discrete instances of correct versus incorrect thinking. This idea guided the second major focus of the professional development activities involving analysis of student work and classroom video.

As an example, teachers were given blinded examples of a subset of their collective students' work on assessment items directly related to the landmark lessons being studied. They were then prompted to analyze the samples in relation to the mini-trajectories they had developed earlier, and then characterize them in response to the question, "What does each of these students understand about rate of change, related to the concepts included in the mini-trajectory map?" In addition, teachers analyzed and discussed video clips of whole-class conversations in which students grappled with key concepts in landmark lessons. In doing so, they identified different stages of understanding demonstrated by their students in the discussion, and traced how particular conceptual discrepancies were resolved through discourse. Finally, teachers were prompted to consider "instructional responses" related to the written samples of student work, and at key moments during the classroom video excerpts. The goal of this activity was to support listening, interpreting, and questioning strategies teachers can develop to advance students' reasoning along a carefully considered trajectory.

Illustrations of the potential of LTs to support teacher learning.

In this section we share some illustrations of teachers' uptake of a HLT to demonstrate the potential of an articulated LT as a support for teachers' professional learning. We are not to the point of claiming empirical evidence, we did observe what appears to be illustrations of how LT concepts might become integrated into these particular teachers' inquiry into the curriculum, and how these teachers described potentially meaningful connections to their classroom instructional practice. The illustrations we share here are used to show three distinct teachers' potential LTs uptakes that might be indicative of a learning-trajectory orientation: 1) interplay of teachers' *within-lesson* and

across-lesson understandings, 2) threading back to the introduction of a given concept, and 3) identifying the contextual functionality of a math task.

First illustration: interplay of teachers’ within-lesson and across-lesson understandings.

One of the focal activities of the PD workshops involved teachers collaboratively analyzing CMP mathematical activities as “mini-trajectories” of a developing mathematical concept. As part of this activity, teachers were prompted to think about how specific investigations (i.e., VP 1.1 and VP 2.1) were connected in terms of similarities, differences, and important transitions within common conceptual threads. As they began their analysis, teachers first took notes on key mathematical aspects *within* each lesson included in the “mini-trajectory.” After analyzing each lesson this way, they then considered relationships *across* lessons, discussing how tasks being considered in each lesson individually were similar to each other with respect to mathematical concept (i.e., proportional functions), and also how they varied from one lesson to the next.

In one notable instance, the teachers discussed differences in the representations that were given in the task and what representations were being asked for in student responses (i.e., colloquial language, tabular form, algebraic expression, graphical form) as being particularly important to the progression from one activity to the next. The teachers noted that even though the focal mathematical concept remained consistent across two lessons, the given and asked-for representations varied in potentially meaningful ways. To consider this variation, the teachers created a drawing of a “representations triangle” (see Figure 2), adapted from a previous PD session, to track which representations (i.e., algebraic, table, graph, and colloquial language) were associated within and between each lesson. Teachers drew the triangle on paper and affixed it in between their posters of “outcomes” for VP 1.1 and VP 2.1, to indicate the representations that arise between the two lessons.

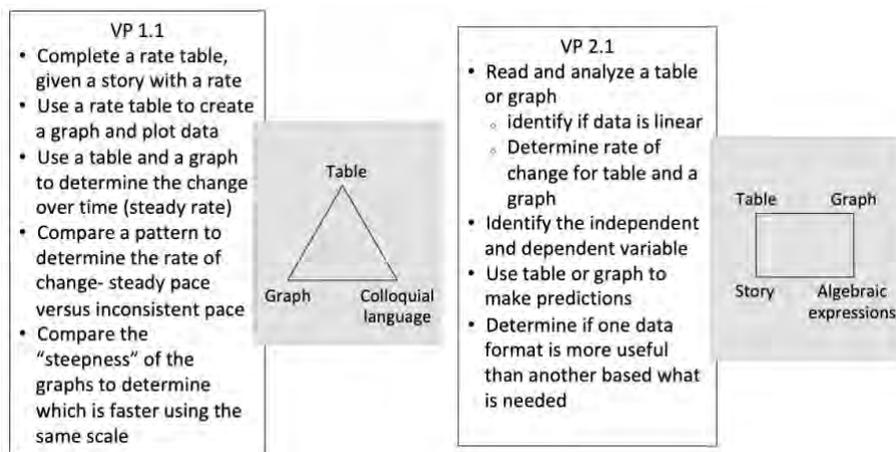


Figure 2. Addition of teacher-developed goals from task analysis in VP 1.1 (a), VP 2.1 (c), and their depiction of the “triangle of mathematics representations” (b) as a key transitional element between the lessons. Teacher 1 posts a second pink note depicting quadrangular relations among multiple representations (i.e., tabular, graph, story, and algebraic expressions)

Later, Teacher 1 drew another figure—a “representations rectangle” that added algebraic expressions as a fourth representation extending out of VP 2.1. Teacher 1 then affixed the figure in the space to the right of VP 2.1, to highlight the idea that algebraic expressions are made “front and center” starting in VP 2.1. In Teacher 1’s words: “... it makes sense because you are throwing an equation and later on making a triangle and then square”.

This instance of teachers’ interplay of thinking *within* and *across* lessons serves as an illustration of a more integrated and contextualized understanding of the mathematical and instructional affordances of a problem, not by itself but as part of a sequence of interconnected problems in the curriculum. At one level, the instance simply shows development of a more detailed understanding of the sequence of tasks and activities—of some of the subtleties of what happens mathematically and where—within a small strand of the curriculum. At another level, the teachers are beginning to consider the “why” of what happens in connection to the larger picture of concept development. In this instance, for example, there is a potentially useful recognition that the way the sequencing of problems affords students with an

expanding set of representations with which to conceptualize key aspects of linear relationships.

Second illustration: threading back to the introduction of a given concept.

As the inquiry work progressed, teachers began to identify the starting point of a conceptual strand as particularly important, and made efforts to track when a concept was introduced in the curriculum specifically by grade level, unit, lesson, and problem. Here for instance, teachers are investigating transitions in the use of representations in the VP unit, assigning particular importance, and dedicating a substantial amount of time during the workshop to locating when a transition to include function equations first arises:

T1: The big jump then is going from the table and the graph to an equation.

T2: Yeah, then when did they introduce an equation for the first time? [Both teachers search through the pages in the teacher guide for VP.]

We interpret this instance as a manifestation of teachers seeing the “starting point” of a conceptual pathway as a particularly important consideration for instruction. This recognition implies that the teachers are tending to which concepts students have (or have not) previously encountered in the curriculum, and that they see this as an important step in contextualizing instruction as explicitly linked to what comes before and what comes after, and perhaps what level of competence to expect of students. Further, this instance suggests that these teachers were actively considering where a specific task “fits” within the strand they are considering. In this example, a particular lesson ultimately takes on a special significance in the teachers’ discussion—a sort of benchmark status—as the place where students first encounter algebraic rule-writing for linear functions. Without a recognition of these conceptual starting points, we could imagine teachers possibly expecting mastery too soon, or else failing to tend to the necessary conceptual leaps students make when first encountering a significant new mathematical idea, representation, or type of problem.

Third illustration: identifying the contextual functionality of a math task

Consideration of how a math task functions—and relatedly, of where it occurs within the broader context of a curriculum—is a key aspect of developing a learning-trajectory perspective. As an example, when a problem involving a non-linear function model arises in a unit that is primarily focused on linear function, it is likely that the problem is intended to advance students' understanding about linear function by setting up a contrast to a non-linear function (rather than as introducing non-linear concepts “out of the blue”.) Teachers who consider the curricular context in this way (i.e., past, current and upcoming units learning goals) will be better positioned to make sense of the inclusion of such a situation and, in turn, orchestrate the discussion on the contrast between a linear situation and a nonlinear situation. This recognition on the part of teachers, for example, would help them more readily identify when students over-extend linearity concepts to non-linear situations, and help them prepare a repertoire of strategies who are overextending the use of a linear function to non-linear situations, and help them focus on non-linear representations in terms of their relationship to linear. In this way, a given math task might play the unique role of a “non-example” of the focal concept in one curricular unit (Mason et al., 2006; Petty, Osmond, & Jansson, 1987), while in another curricular unit, the same task might play the role of “example” of a different concept, as one among many other similar tasks. The analysis of the function of a math task in this way is possible because the curricular context is considered.

During the professional development sessions, teachers engaged in discussion about the role of non-linear math tasks included in a unit that focuses on linear functions. In doing this, teachers thought about what it would look like if students were to overextend the application of linearity to a non-linear situation, what feedback they would consider providing and what representations they would use in the process, and how they might facilitate such a discussion. As an example, one of the teachers here anticipates how her students might approach a math task involving a “non-example” of linearity:

T1: I think that students assume it is linear. Especially, at first, they just see the table and just assume it is linear. I think that some of them don't even think of making connections and go figuring out “Oh well, um, I am looking at the unit rate,” and they look at the 5

by 4 hundred dollars. Other things that some of them do is that they figure out unit rate and then add two of them to the value of thirty, and multiply that by 32.

In thinking on how to orchestrate the discussion with their students, teachers suggest the following ideas:

T1: Well, I think that the part of what they start answering these questions- you know that the cost of 20 bikes and then the cost of they have to answer the cost of 40 bikes that you know. That could be the part of the discussion.

T2: Having them graph it.

T1: I think that going back to what T1 just said is “what would the table look like if it was linear?”

T1: And I think I would have my students create a table and what would table look like if it was constant [referring to constant rate of change].

We posit that the fact that teachers are placing a task within a context allows them to think carefully what the role of that task is, namely to shed light on a concept by experimenting in contrast with a non-example. In doing this teachers, teachers can develop a set of a-priori potential interventions to facilitate a conceptually fruitful discussion about a concept not just by identifying that it does not fit but also why.

Concluding Remarks

In the first part of the article, we shared a process for articulating a hypothetical LT implicit in a widely adopted, middle-grades mathematics curriculum. In doing so, we highlighted considerations, decisions, and challenges we faced as part of this work. Specifically, we made this process visible using the case of a hypothetical LT for middle school algebra, with a particular focus on linear functions and equations. By describing our LT articulation process, our aim is to highlight ways in which curriculum-specific LT approaches can be designed in ways that are more proximal and instrumental to teachers’ instructional practice. To do this, we linked statements about learning goals which, draw from both the curriculum and extant research together with lesson problems within a unit, throughout a grade and across grades. As teachers everyday work revolves around lesson problems, bringing LTs to the problems they already use seems to be the more

ecological solution that enables us to capitalize on teachers' knowledge of the curriculum in the form of math tasks. LTs' landmarks and obstacles are anchored to already familiar curricular math tasks. This is of utmost importance since research on LT-based professional development has demonstrated that it is often challenging for teachers to integrate LTs into their curricula and everyday instructional practice. In particular, Wilson, Mojica & Confrey (2013) describe how teachers' reported struggling to incorporate LTs into their mathematics curricula. Most current approaches to LT development are curriculum independent, which may constitute a contextual gap between the articulation of a particular LT and its usefulness in teachers' everyday instructional practice. If an aim of LT articulation is to support teachers' interpretation of students' conceptual development along a trajectory—and to subsequently make appropriate next instructional decisions—teachers are left to do the demanding and highly conceptual work of connecting theoretical LTs to their daily curriculum. Consequently, we posit that a curriculum-specific approach to LT development might be more productive for enhancing teachers' instructional practice.

Even though the main objective of this paper is to put forth a process for articulating HLTs in an already existing curriculum, we wanted to provide the context and motivation of our work along with illustrations of potential teachers' uptakes. We hope that in sharing our vision on how HLTs can be used in the context of PD we, together with the broader research community would take interest in further examining the illustrations put forth here and further develop the construct of how teachers incorporate knowledge about LTs in their practice which is beyond the scope of this report. These illustrations are anecdotal data points and not results of research. We shared them as working hypothesis in the way that mathematical conjectures are shared. In conclusion, we believe these illustrations of potential teachers' uptakes cast a possible path to link Learning Trajectories with Teachers' everyday practice. We hope these PD illustrations become a food for thought in trying to think how to examine teachers' LTs uptake in a systematic way in real classroom practice.

References

- Barrett, J.E., Clements, D.H., Klandermañ, D., Pennisi, S.J., & Polaki, M.V. (2006). Students' coordination of geometric reasoning and measuring strategies on a fixed perimeter task: Developing mathematical understanding of linear measurement. *Journal for Research in Mathematics Education*, 37(3), 187-221.
<https://doi.org/10.2307/30035058>
- Berland, L.K., & McNeill, K.L. (2010). A learning progression for scientific argumentation: Understanding student work and designing supportive instructional contexts. *Science Education*, 94(5), 765-793.
<https://doi.org/10.1002/sce.20402>
- Bernbaum Wilmot, D., Schoenfeld, A. H., Wilson, M., Champney, D., & Zahner, W. (2011). Validating a learning progression in mathematical functions for college readiness. *Mathematical Thinking & Learning*, 13(4), 259-291. <https://doi.org/10.1080/10986065.2011.608344>
- Bishop, Alan J., Lamb, L. L., Phillip, R. A. , Whitacre, I., & Schappelle, B. (2014). Using order to reason about negative numbers: The case of Violet. *Educational Studies in Mathematics*, 86(1), 39-59.
<https://doi.org/10.1007/s10649-013-9519-x>
- Bowers, Janet, Cobb, P., & McClain, K. (1999). The evolution of mathematical practices: A case study. *Cognition and Instruction*, 17(1), 25-66. https://doi.org/10.1207/s1532690xci1701_2
- Brown, Matthew W. (2009). The Teacher–Tool Relationship: Theorizing the Design and Use of Curriculum Materials. In J. Remillard, B. Herbel-Eisenmann & G. Lloyd (Eds.), *Mathematics Teachers at Work*. Routledge.
- Clark, D. B. (2006). Longitudinal conceptual change in students' understanding of thermal equilibrium: An examination of the process of conceptual restructuring. *Cognition and Instruction*, 24(4), 467-563.
https://doi.org/10.1207/s1532690xci2404_3
- Clements, D.H., & Sarama, J. (2004). Learning Trajectories in Mathematics Education. *Mathematical Thinking & Learning*, 6(2), 81-89.
- Clements, D.H., Wilson, D. C., & Sarama, J. (2004). Young children's composition of geometric figures: A learning trajectory. *Mathematical Thinking & Learning*, 6(2), 163-184.

- Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking & Learning*, 1(1), 5-43.
- Cobb, P., & Whitenack, J. W. (1996). A method for conducting longitudinal analyses of classroom videorecordings and transcripts. *Educational Studies in Mathematics*, 30(3), 213-228.
<https://doi.org/10.1007/BF00304566>
- Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, 31(3/4), 175-190.
<https://doi.org/10.1080/00461520.1996.9653265>
- Collopy, Rachel. (2003). Curriculum Materials as a Professional Development Tool: How a Mathematics Textbook Affected Two Teachers' Learning. *The Elementary School Journal*, 103(3), 287-311.
<https://doi.org/10.1086/499727>
- Common Core Standards Writing Team. (2013a). *Grades 3-5, numbers and operations, fractions: A progression of the Common Core State Standards in Mathematics*. Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013b). *Preface for the draft progressions*. Institute for Mathematics and Education, University of Arizona.
- Confrey, J., & Maloney, A. (2010). *The construction, refinement, and early validation of the equipartitioning learning trajectory*. Paper presented at the International Conference of the Learning Sciences, Chicago, US.
- Confrey, J., Maloney, A. P., & Corley, A. K. (2014). Learning Trajectories: a Framework for connecting standards with curriculum. *ZDM-The International Journal on Mathematics Education*, 46(5), 719-733.
<https://doi.org/10.1007/s11858-014-0598-7>
- Corcoran, Tom, Mosher, Frederic A., Rogat, Aaron, & Consortium for Policy Research in, Education. (2009). *Learning progressions in science: an evidence-based approach to reform*. Consortium for Policy Research in Education.
- Daro, Phil, Mosher, Frederic A., Corcoran, Tom, Barrett, Jeffrey, & Consortium for Policy Research in, Education. (2011). *Learning trajectories in mathematics: a foundation for standards, curriculum, assessment, and instruction*. Consortium for Policy Research in Education.

- Gravemeijer, K., Bowers, Janet, & Stephan, Michelle. (2003). A hypothetical learning trajectory on measurement and flexible arithmetic. In M. Stephan, J. Bowers, P. Cobb & K. Gravemeijer (Eds.), *Supporting students' development of measurement conceptions: Analyzing students' learning in social context* (Vol. 12, pp. 51-66). National Council of Teachers of Mathematics.
- Gunckel, K.L., Mohan, L., Covitt, B. A., & Anderson, C. W. (2012). Addressing challenges in developing learning progressions for environmental science literacy. In A. C. Alonzo & A. W. Gotwals (Eds.), *Learning progressions in science: Current challenges and future directions* (pp. 39-75). Sense.
- Hackenberg, A. J. (2014). Musings on three epistemic algebraic students. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing* (Vol. 4, pp. 81-124). University of Wyoming.
- Hackenberg, A. J., & Tillema, E.S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior*, 28(1), 1-18. <https://doi.org/10.1016/j.jmathb.2009.04.004>
- Jacobs, V., & Empson, S. (2015). Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. *ZDM: The International Journal on Mathematics Education*, 48(1-2), 185-197. <https://doi.org/10.1007/s11858-015-0717-0>
- Jin, H., & Anderson, C. W. (2012). A learning progression for energy in socio-ecological systems. *Journal of Research in Science Teaching*, 49(9), 1149-1180. <https://doi.org/10.1002/tea.21051>
- Lobato, J., & Walters, C. D. (2017). A Taxonomy of Approaches to Learning Trajectories and Progressions. In J. Cai (Ed.), *The Compendium for Research in Mathematics Education*: National Council of Teachers of Mathematics.
- Mason, J., Dreyfus, Tommy, Bills, L, Tsamir, P, Watson, A, & Zaslavsky, Orit. (2006). Exemplification in mathematics education. In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds.), *Proceedings of the 30th conference of the international group for the Psychology of Mathematics Education* (Vol. 1, pp. 125-154). Prague, Czech Republic: PME.
- Meletiou-Mavrotheris, M, & Paparistodemou, E. (2015). Developing students' reasoning about samples and sampling in the context of

- informal inferences. *Educational Studies in Mathematics*, 88(3), 385-404. <https://doi.org/10.1007/s10649-014-9551-5>
- Mitchelmore, Michael, & White, Paul. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics*, 41(3), 209-238. <https://doi.org/10.1023/A:1003927811079>
- Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics*, 83(2), 225-245. <https://doi.org/10.1007/s10649-012-9450-6>
- Nabors, W.K. (2003). From fractions to proportional reasoning: A cognitive schemes of operation approach. *The Journal of Mathematical Behavior*, 22(2), 133-179. [https://doi.org/10.1016/S0732-3123\(03\)00018-X](https://doi.org/10.1016/S0732-3123(03)00018-X)
- Petty, O., & Jansson, L. (1987). Sequencing Examples and Nonexamples to Facilitate Concept Attainment. *Journal for Research in Mathematics Education*, 18(2), 112-125. <https://doi.org/10.5951/jresmetheduc.18.2.0112>
- Rasmussen, Chris, & Stephan, Michelle. (2008). A methodology for documenting collective activity. In A. E. Kelly, R. Lesh & J. Y. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (pp. 195-215). Erlbaum.
- Remillard, Janine T. (2005). Examining Key Concepts in Research on Teachers' Use of Mathematics Curricula. *Review of Educational Research*, 75(2), 211-246. <https://doi.org/10.3102/00346543075002211>
- Simon, M. (1995). Reconstructing Mathematics Pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26, 114-145. <https://doi.org/10.5951/jresmetheduc.26.2.0114>
- Steffe, L., & Thompson, P. (2000). Teaching Experiment Methodology: Underlying Principles and Essential Elements. In R. Lesh & A. E. Kelly (Eds.), *Research Design in Mathematics and Science education*. Erlbaum.
- Stephan, Michelle, & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education*, 43(4), 428-464. <https://doi.org/10.5951/jresmetheduc.43.4.0428>
- Sztajn, P., Confrey, J., Wilson, H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational*

Researcher, 41(5), 147-156.

<https://doi.org/10.3102/0013189X12442801>

Tillema, E.S. (2014). Students' coordination of lower and higher dimensional units in the context of constructing and evaluating sums of consecutive whole numbers. *The Journal of Mathematical Behavior*, 36, 51-72.

<https://doi.org/10.1016/j.jmathb.2014.07.005>

Vermont Mathematics Partnership's Ongoing Assessment Project. (2013). *OGAP proportional reasoning framework*. Montpelier, VT.

Vermont Mathematics Partnership's Ongoing Assessment Project. (2014a). *OGAP fraction framework*. Montpelier, VT.

Vermont Mathematics Partnership's Ongoing Assessment Project. (2014b). *OGAP multiplicative framework*. Montpelier, VT.

Weber, E., & Thompson, P.W. (2014). Students' images of two-variable functions and their graphs. *Educational Studies in Mathematics*, 87, 67-85.

Wilson, P. H., Sztajn, P., Edgington, C., & Confrey, J. (2013). Teachers' use of their mathematical knowledge for teaching in learning a mathematics learning trajectory. *Journal of Mathematics Teacher Education*, 17(2), 149–175. <https://doi.org/10.1007/s10857-013-9256-1>

Wilson, P., et al. (2013). Learning trajectories in teacher education: Supporting teachers' understandings of students' mathematical thinking. *The Journal of Mathematical Behavior* 32, 103–121.

<https://doi.org/10.1016/j.jmathb.2012.12.003>

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