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To cite this article:

Zhou, L., Liu, J., & Lo, J. (2022). A comparison of U.S. and Chinese geometry standards through the lens of van Hiele levels. *International Journal of Education in Mathematics, Science, and Technology (IJEMST)*, 10(1), 38-56. <https://doi.org/10.46328/ijemst.1848>

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Article Info

Article History

Received:

5 June 2021

Accepted:

23 November 2021

Keywords

Comparison

Descriptive geometry

Curriculum standards

van Hiele levels

Abstract

The present study aims to compare the geometry standards in U.S. Common Core State Standards of Mathematics (CCSSM) and Chinese Compulsory Education Mathematics Curriculum Standards (CMCS) through the lens of van Hiele levels. The study considered a standard unit as one or multiple learning expectation(s) and placed each learning expectation into van Hiele levels. By examining the van Hiele level distributions of the learning expectations and major topics, this study investigated how CCSSM and CMCS propose the students' development of geometric thoughts. The findings reveal different emphases in geometry learning expectations of CCSSM and CMCS. Specifically, (a) Geometry standards in CCSSM are more general, whereas geometry standards in CMCS are more specific; (b) CMCS shows a more consistent progression through grade bands than CCSSM; (c) CCSSM makes more connections between various aspects within a topic and across multiple topics than CMCS; (d) CCSSM highlights Dynamic Geometry Environments (DGEs), whereas CMCS underscores abstract and rigor at the upper level. Implications of this study and suggestions for future revisions for both standards are discussed.

Introduction

Mathematics standards are a set of learning goals that an authoritative body has set to guide mathematics instruction in K–12 educational settings (Reys et al., 2003). In particular, mathematics standards determine what topics students are exposed to (Stein et al., 2007) and how deep the topics should be learned (Schmidt et al., 2001), which impact students' mathematical learning opportunities and, therefore, their mathematics achievement. Given the critical role standards play, many countries design and implement standards to stimulate curriculum reform. A crucial stage of designing or revising a standard is to refer to or compare with the standards in other countries. For instance, the comparisons with standards in top performance countries in the Trends in International Mathematics and Science Study (TIMSS) catalyzed the development of the Common Core State Standards of Mathematics (CCSSM) (Common Core State Standards Initiative [CCSSI], 2010; Reys, 2014). As a result, CCSSM was released in response to align with the standards in the top-performing countries

in 2011. In this study, we compared geometry standards in CCSSM with the Chinese mathematics standards—Compulsory Education Mathematics Curriculum Standards (CMCS) (Ministry of Education, 2011). The reason we chose the Chinese standards for comparison is that CMCS has remarkable similarities when compared to the performance of the top-performance countries, such as Singapore, South Korea, and Japan, regarding topic coverage and coherence, though mainland China did not participate in TIMSS (e.g., Wang et al., 2012).

Based on the practical significance of standards, analyses of curriculum standards can inform educational practice in improving student learning opportunities (Tran et al., 2016). Tran and colleagues called for more research to engage in curriculum standards analysis that enhances students' learning opportunities. Specifically, they called for sensitive enough methods to monitor what content is emphasized, how the content emphasis differs from one set of standards to another, and the level of rigor of standards. The van Hiele theory of geometric understanding is used as a framework to examine geometry curriculum standards (e.g., Dingman et al., 2013; Newton, 2011). Clements (2003) studied K–12 teaching and learning geometry in the United States and suggested that the curriculum should promote learning and conceptual change to promote students through van Hiele levels. To respond to Tran et al.'s (2016) call and Clements's (2003) suggestions, this study adopts an indirect method, using a framework that integrates Learning Expectations (LE) method and van Hiele levels to compare descriptive geometry standards in CCSSM and CMCS. From extant literature, little is known about the van Hiele levels of the curricula of the TIMSS top-performing countries, which CMCS closely resembled, regarding how geometric topics proceed across grade levels. To fill this gap, in this study, we compare CCSSM and CMCS through the lens of van Hiele levels. Specifically, this study investigated *How are the standards in CCSSM and CMCS framed in van Hiele levels in terms of grade bands and grade level?* (Research Question #1). In order to provide specific curricular recommendations to improve the student learning opportunities, this study also sought to answer the following research question: *What are the similarities and differences between CCSSM and CMCS in their treatments of selected geometric topics?* (Research Question #2). The selection of the topics was based on both their prevalent presence and contrasting treatments in both standards.

Relevant Literature

Analysis Approaches of CCSSM

Tran et al. (2016) reviewed various methods used in analyzing and comparing CCSSM with other standards and identified two main categories: the direct method and the indirect method. The direct comparison approach directly compares sets of standards regarding topics coverage, cognitive depth of topics, and topic distribution across grades. The indirect comparison approach uses a predefined framework to code standards. Different methods are designed for their own particular purposes and might reach different conclusions. For instance, using a direct method, Schmidt et al. (2005) analyzed top-performance countries' mathematics standards, identified the commonalities of learning expectations in these standards, and developed an international benchmark called A+ standards. Because standards might be structured differently, content matching standards may not contain equal information. Therefore, indirect methods purposefully use a framework to analyze standards that capture explicated information. Later, via an indirect method, Schmidt and Houang (2012) used the A+ standards as a model to examine the quality of CCSSM by comparing the coverage of topics in each

grade level in the CCSSM and A+ model and found that CCSSM had a very high degree of similarity to the A+ model regarding coherence and focus.

Learning Expectation (LE) method is another example of an indirect approach that identifies a standard as a learning expectation or multiple learning expectations if the standard includes multiple mathematical ideas (Tran et al., 2014). For instance, Dingman et al. (2013) used the LE method to examine the similarities and differences between K–8 CCSSM standards and pre-CCSSM K–8 standards. Their comparison focused on changes in learning expectations and mathematical topics within and across grade levels. They concluded that the current CCSSM represents significant changes regarding the grade placement, learning trajectory, and emphasis on particular content.

Geometry Standards in CCSSM and CMCS

Compared to pre-CCSSM standards, CCSSM makes considerable changes in geometry. For example, it places great emphasis on geometric concepts (Porter et al., 2011), emphasizes conceptual understanding and extensive thinking (Conley et al., 2011) and proposes more learning expectations in informal deduction (Dingman et al., 2013). Researchers also found that throughout grades 6–8, CCSSM increased attention to transformational geometry, including the verification of congruence and similarity through a series of transformations (e.g., Teuscher et al., 2015; Tran et al., 2014). Dingman et al. (2013) found geometry learning expectations at van Hiele level 3 in CCSSM appear over seven times more than other levels in the state standards. They thus expected that “CCSSM will represent a shift for many students from an expectation of lower levels of geometric reasoning in prior state standards to a higher level (i.e., more informal deduction) in CCSSM” (Dingman et al., 2013, p. 551). Through comparing CCSSM with A+ curriculum standards, Schmidt and Houang (2012) found that CCSSM introduces many geometry topics in grades earlier than the A+ standards. For instance, Perimeter, Area & Volume is introduced in Grade 3, while the A+ standards start to introduce this topic in Grade 4.

CMCS is an updated version of previous standards. In contrast to the non-centralized U.S education system, the Chinese education system is highly centralized. Since its release, CMCS is the only set of national standards. Most content and structure are inherited from the previous standards (Wang et al., 2017). The geometry curriculum in CMCS highlights three core concepts, *spatial thinking*, *deduction*, and *geometric intuition*, which prepare students to develop geometric experiences toward abstract thinking and rigorous proof (Kong & Shi, 2012; Ministry of Education Basic Education Curriculum Committee, 2012). *Spatial thinking* requires students to abstract figures from real models, to imagine real objects according to the descriptions, to image relative positions of figures, to describe motion and transformation, and to draw figures according to the descriptions (Ministry of Education, 2011). CMCS proposes that the development of *deductive reasoning* should be addressed throughout the mathematics learning process. *Geometric intuition* means using geometric visualization to describe and analyze problems. It is an overall understanding and intuition about the geometric attributes of an object. Developing geometric intuition requires effective long-term observation, thinking, and accumulated experiences (Kong & Shi, 2012).

Theoretical Frameworks

The primary theoretical lens for this study is a set of literature on standards curriculum and van Hiele's theory on the development of geometric thoughts (van Hiele, 1984; van Hiele-Geldof, 1984). The literature on standards curriculum helps to navigate the perspective on standards. The van Hiele levels approach formulates the way of examining and analyzing standards.

Theoretical View of Standards

Standards play an essential role in influencing student learning opportunities. Mathematics standards are a set of learning expectations that an authoritative body has set to guide mathematics instruction in K–12 educational settings (Reys et al., 2003). How content is presented reflects the developers' theory of how students learn mathematics. As Stein et al. (2007) wrote,

If one believes that mathematics is best learned through student construction based on active exploration, one set of criteria will be selected to guide one's analyses. On the other hand, if one believes that mathematics is best learned through direct instruction and skills practice, then review criteria will be of a different sort. (p. 327)

Furthermore, mathematics standards determine the mathematics curriculum about certain topics and the depth of topics for teaching and learning (Schmidt et al., 2001; Stein et al., 2007). As most mathematics teachers rely on curriculum materials as their primary tool for teaching mathematics, teachers are unlikely to cover a topic excluded in standards (Stein et al., 2007). As teachers transform the written curriculum into the implemented curriculum (Remillard et al., 2019), students often do not have access to learn content without exposure. Thus, the topics covered in a given set of curriculum materials are of fundamental importance for teaching and learning (Schmidt et al., 2001).

Another criterion to evaluate standards is coherence. Schmidt et al. (2005) defined standards coherence by considering how the standards are sequenced and to what extent such sequence reflects the sequential and hierarchical nature of mathematical topics. Coherent standards are “articulated over time as a sequence of topics and performances that are logical and reflect, where appropriate, the sequential and hierarchical nature of the disciplinary content from which the subject matter derives” (Schmidt et al., 2005, p. 528) toward deeper ideas. They also suggested that a set of standards might have multiple coherent sequences that reflect the inherent structure of the subject. For instance, Winn et al. (2016) compared the terms used in CCSSM and the Next Generation Science and reported that terms of reasoning in CCSSM are inconsistent across grade bands. This study examines the coherence of CCSSM and CMCS through the van Hiele levels lens.

van Hiele Levels

Pierre Marie van Hiele and his wife Dina van Hiele-Geldof (1984) identified five levels of thought in geometry. These original five levels are called the van Hiele levels. Mayberry (1983) described the five levels as follows:

- At level 1, a figure is perceived as a whole, recognizable by its visible form, but the properties of a figure are not perceived.
- At level 2, properties are perceived, but they are isolated and unrelated.
- At level 3, definitions become meaningful, with relationships being perceived between properties and between figures. Logical implications and class inclusions are understood. The role and significance of deduction, however, are not understood.
- At level 4, a deduction is meaningful. The student can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions.
- At level 5, a student should understand the role and necessity of indirect proof and proof by contrapositive. (Adapted from Mayberry, 1983, p. 59)

Since the 1980s, researchers have used the van Hiele levels as a framework to investigate geometry teaching and learning and have suggested that van Hiele theory was a useful tool to examine the geometric concept development of students (Fuys et al., 1988; Mayberry, 1983; Senk, 1989; Usiskin, 1982).

Clements (2003) studied K–12 teaching and learning geometry in the United States and suggested that the curriculum should promote learning and conceptual change to develop students' van Hiele levels. van Hiele theory is also used as a framework to examine mathematics curriculum standards (Dingman et al., 2013; Newton, 2011). For example, Newton (2011) investigated the K–8 learning expectations of geometric thinking in pre-CCSSM state standards through the lens of van Hiele levels. She concluded that the expectations and experiences in K–8 geometry in most state standards at van Hiele level 3 are insufficient to prepare students to achieve van Hiele level 4. Thus, she suggested that standards should address more learning expectations at van Hiele level 3. Later, Dingman et al. (2013) used van Hiele theory to compare geometry standards in pre-CCSSM state standards and CCSSM. They found that CCSSM addresses far more learning expectations at van Hiele level 3 than the state standards. For instance, the K–8 descriptive geometry standards are 30% at van Hiele level 3 in CCSSM compared to 4% in the pre-CCSSM state standards. They concluded that geometry in CCSSM presents a significant change. These previous studies on van Hiele theory suggested the illuminating power of this framework for comparing curriculum standards.

Methods

Data Source

We adopted the LE method used by Newton (2011) and Tran et al. (2014). The data in this study are descriptive learning expectations in CCSSM and CMCS in three grade bands. Below we define descriptive learning expectations and grade bands in detail.

Descriptive Learning Expectation (DLE)

Drawing from Newton (2011) and Tran et al. (2014), we consider a learning expectation to be a standard unit or a component of a standard unit. If a standard includes multiple mathematics foci, the standard is split into

multiple learning expectations. For instance, the standard unit “Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions” (CCSSI, 2010, p. 76) is divided into three learning expectations because it contains three criteria for triangle congruence. In doing so, two content matching learning expectations in CCSSM and CMCS contain roughly equal information. Thus, relatively comparable data were collected. We consider geometry standards focusing on figures, including figures’ properties, the relationships between figures, theorems, and proofs (van Hiele, 1984; van Hiele-Geldof, 1984;). We adopted Newton’s (2011) definition of descriptive geometry, which refers to “geometry that is that concerned with figures and their relationships, but not with coordinates and exact quantities” (p. 80). Unlike Newton’s original definition of descriptive geometry that excludes measurement-related topics like the perimeter, area, volume, and angle entirely, our descriptive geometry includes the fundamental concepts of perimeter, area, volume, and angle. This study uniquely interprets these concepts not from a quantitative perspective but as measurable attributes of figures. In this study, a descriptive learning expectation (DLE) is defined as a learning expectation representing descriptive geometry, while the learning expectations related to measurement and coordinates are not considered.

The two examples below were used to illustrate the inclusive and exclusive of the definition. Example 1 is a DLE included in the study because it represents understanding the concepts of area and area measurement. Example 2 is a non-DLE that is excluded from the study because it focuses on measurement rather than understanding the area.

Example 1. Recognize area as an attribute of plane figures and understand concepts of area measurement. (CCSSI, 2010, p. 25)

Example 2. Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real-world and mathematical problems and represent whole-number products as rectangular areas in mathematical reasoning. (CCSSI, 2010, p. 25)

Grade Bands

CCSSM and CMCS have different structures. CCSSM is organized by grade level through K–8 and by combined content-based standards in high school. CMCS consists of Grades 1–9 standards and is organized into three stages. The first stage is Grades 1–3, the second stage is Grades 4–6, and the third stage is Grades 7–9. The first and second stages are the elementary level, and the third stage is the middle school level. CMCS does not contain high school standards. At the high school level, Chinese geometry standards include four parts: solid geometry, plane analytical geometry, trigonometry, and plane vectors, which do not share much common content with CCSSM. Therefore, we did not include Chinese standards at the high school level in this study. Regarding comparing between grades or grade bands, Porter et al. (2011) suggested that comparing grade aggregation standards makes more sense than comparing each single grade level because the same content may not appear in the same grade but may appear in the surrounding grades. Hence, in this study, we defined three grade bands for comparison: Lower-Grade Band includes K–3 in CCSSM and Grades 1–3 in CMCS; Middle-Grade Band includes Grades 4–6 in both CCSSM and CMCS; Upper-Grade Band includes Grades 7–8 and high school in CCSSM and Grades 7–9 in CMCS (see Table 1).

Table 1. Grade Bands

	Lower-Grade Band	Middle-Grade Band	Upper-Grade Band
CCSSM	K–3	Grades 4–6	Grade 7, 8, & high school
CMCS	Grades 1–3	Grades 4–6	Grade 7-9

Coding

There were three stages of coding. In the first stage, two researchers who are literate in both English and Chinese languages worked together to identify the DLEs from all geometry learning expectations in CCSSM and CMCS. In this stage, the two researchers had a few divergences. In the second stage, drawing on the literature of the descriptors and characteristics of van Hiele levels (e.g., Fuys et al., 1988), the two researchers identified topics of each van Hiele level to help categorize the level of each learning expectation. The topics also reveal geometric topics or geometric ideas explicitly. By comparing a specific topic in a particular grade band, detailed information in the CCSSM and CMCS can be explored further. Initial topics were modified in the coding process, and new topics emerged while coding. Because both documents do not include learning expectations toward van Hiele level 5, this study considers only levels 1-4. Table 2 shows all topics at each level. The order of the topics in the same van Hiele level does not matter. Particularly, measurable attributes of figures (i.e., perimeter, area, volume, and angle) are recognized under the topic of *understanding definitions*.

Table 2. Topics at van Hiele Levels

van Hiele Levels	Topics
Level 1	Recognizing figures Composing and decomposing figures
Level 2	Recognizing figures by their specified attributes Understanding relationships between 3D and 2D figures empirically Recognizing transformation empirically Recognizing definitions empirically
Level 3	Classifying figures Understanding definitions Developing informal proofs Constructing figures Understanding transformation Understanding relationships between 3D and 2D figures
Level 4	Writing formal proofs

In the third stage, the two researchers coded DLEs in CCSSM and CMCS in terms of topics independently and then met to compare codes and negotiate differences. At this stage, the agreement reached 87%. With the discrepancies not reaching a consensus, the third author was involved in the discussion until a consensus was reached.

Findings

The total number of DLEs in CMCS was 132, while in CCSSM it was 114. Table 3 shows both frequencies of DLEs and the percentages of each level in the total number of DLEs. Both documents have the largest proportion of DLEs at level 3—68% in CCSSM and 58% in CMCS.

Table 3. Frequencies and Percentages of DLEs of Each Level

Standards	Level 1	Level 2	Level 3	Level 4	Total
CMCS	7 (5%)	21 (16%)	76 (58%)	28 (21%)	132
CCSSM	6 (5%)	14 (12%)	77 (68%)	17 (15%)	114

DLEs by van Hiele Levels Distribution in Grade Bands

We delved into grade/grade bands and examined the distribution of van Hiele levels in the two documents to get detailed information about differences and potential causes for differences. Table 4 shows van Hiele level distributions among different grade bands.

Table 4. Frequencies and Percentages of DLEs in Grade/Grade Band

van Hiele Level	Lower-Grade Band		Middle-Grade Band		Upper-Grade Band	
	CMCS	CCSSM	CMCS	CCSSM	CMCS	CCSSM
Level 1	6 (60%)	6 (32%)	1 (6%)	0	0	0
Level 2	4 (40%)	6 (32%)	13 (81%)	4 (33%)	4 (4%)	4 (5%)
Level 3	0	7 (37%)	2 (13%)	8 (67%)	74 (70%)	62 (75%)
Level 4	0	0	0	0	28 (26%)	17 (20%)
Total	10 (100%)	19 (100%)	16 (100%)	12 (100%)	106 (100%)	83 (100%)

CMCS has a distinct pattern based on the percentages of DLEs in each grade band: Lower-Grade Band focuses on developing van Hiele level 1 (60%), Middle-Grade Band emphasizes van Hiele level 2 (81%), and Upper-Grade Band concentrates on developing van Hiele level 3 (70%). This pattern was not found in CCSSM. Overall, DLEs in CCSSM at van Hiele level 2 and level 3 are proposed earlier than CMCS. Its DLEs at van Hiele level 3 have the greatest percentage in all grade bands. DLEs at van Hiele level 4 come later for both

CMCS and CCSSM at the Upper-Grade Band. In subsequent sections, we present more details of the DLE distribution in each grade band in these two documents respectively.

Lower-Grade Band DLEs in CMCS and CCSSM

Overall, in the Lower-Grade Band, CCSSM included six (32%) DLEs at level 1, six (32%) DLEs at level 2, and seven (37%) DLEs at level 3. CMCS has no DLEs at level 3; it has six (60%) DLEs at level 1 and four (40%) DLEs at level 2. Table 5 shows the topic details of DLEs of topics at each van Hiele level.

Table 5. Lower-Grade Band DLEs at Level 1-3

van Hiele Level	Topics	CMCS	CCSSM
Level 1	Composing and decomposing figures	1 (10%)	4 (21%)
	Recognizing figures	5 (50%)	2 (11%)
Level 2	Recognizing figures by their attributes	1 (10%)	6 (32%)
	Understanding transformation empirically	3 (30%)	0
Level 3	Understanding definitions	0	4 (21%)
	Classifying figures	0	3 (16%)

Note. The total number of DLEs for CCSSM at Lower-Grade Band is 19 and for CMCS is 10.

The two documents presented DLEs about composing figures quite differently. CCSSM addresses the DLEs explicitly and expects students to produce new shapes from the compositions and “create a composite shape and compose new shapes from the composite shape” (CCSSI, 2010, p. 16). CMCS proposes one DLE to expect students to use specific two-dimensional shapes (rectangles, squares, triangles, parallelograms, and circles) to compose new shapes.

From the percentages of the two topics at level 1, we infer that CCSSM expects students to develop an initial understanding of the 2D and 3D shapes by manipulation. In contrast, CMCS expects students to recognize figures by recognizing the visual clues from real-life objects. At level 2, CMCS proposes the three DLEs address transformation by observing and manipulating to perceive reflection, translation, and rotation. In contrast, CCSSM does not propose DLEs about transformation. At level 3, CMCS proposes no DLEs. Still, CCSSM addresses seven DLEs, including four DLEs about understanding conceptions of area and perimeter and three DLEs about understanding quadrilaterals’ hierarchical relations.

Middle-Grade Band DLEs in CMCS and CCSSM

There are significant disparities in the Middle-Grade Band regarding van Hiele level distributions (see Table 6). Overall, CCSSM addresses four DLEs (33%) at level 2 and eight DLEs (67%) at level 3. CMCS puts one DLE (6%) at level 1 about recognizing line, half-line, and line segment; 13 DLEs (81%) at level 2; and two DLEs (13%) at level 3. CMCS focuses on promoting students’ geometric thinking at level 2, while CCSSM tends to develop geometric thinking at level 3. Table 6 shows the frequencies of DLEs of topics at level 2 and level 3.

Table 6. Middle-Grade Band DLEs at Level 2 and Level 3

van Hiele Level	Topics	CMCS	CCSS M
Level 2	Recognizing definitions empirically	4 (25%)	0
	Understanding transformation empirically	5 (31%)	0
	Recognizing the relation between 2D-3D empirically	2 (13%)	0
	Recognize figures by their attributes	2 (13%)	4 (33%)
Level 3	Understanding definitions	0	4 (33%)
	Classifying figures	0	4 (33%)
	Developing informal proof	2 (13%)	0

Note. The total numbers of DLEs for CCSSM at Middle-Grade Band is 12 and for CMCS is 16.

The disparities of topic distributions are obvious. At level 2, CCSSM only proposes DLEs about recognizing figures by their attributes, whereas CMCS has DLEs across all four topics. At level 3, CCSSM proposed DLEs about understanding definitions and classifying figures, while CMCS includes DLEs about informal proof.

At level 2, CMCS does not propose formal definitions but expects students to understand parallel and perpendicular relations from real life. Both documents present *symmetry* and *symmetric figures*. CCSSM addresses two DLEs that recognize *symmetry* as an attribute of some figures and expects students to identify line-symmetric figures and draw lines of symmetry. CMCS describes *line-symmetry*, which is equivalent to *reflection*, as a form of transformation. Thus, unlike how *symmetry* in CCSSM indicates an attribute of some figures, *symmetry* in CMCS has two meanings: an attribute of some figures and also a form of transformation.

At level 3, similar to DLEs in the Lower-Grade Band, CCSSM addresses the definition of area and perimeter, and four DLEs present the definitions of volume and angle. The other four DLEs at van Hiele level 3 are about 2D figures based on their attributes. CCSSM describes a general criterion of classifying figures. Distinctively, the two DLEs in CMCS at van Hiele level 3 expect students, through observations and manipulations, to understand the Triangle Angle Sum Theorem and Triangle Inequality Theorem.

Upper-Grade Band DLEs in CMCS and CCSSM

In the Upper-Grade Band, CCSSM and CMCS have the most similarities in van Hiele level distributions (see Table 7). Both sets of standards emphasize level 3 and move toward level 4.

At level 3, CMCS proposes 21 DLEs on geometric definitions, whereas CCSSM has 14 DLEs on geometric definitions. The two documents share five common definitions: *congruence*, *angle*, *circle*, *perpendicular line*, and *parallel line*. CCSSM has more transformation-related geometric concepts included as formal definitions, such as *similarity*, *rotation*, *reflection*, and *translation*. On the other hand, CMCS formalizes basic geometric concepts, such as *isosceles triangle*, *right triangle*, *angular bisector*, *line-symmetric figure*,

parallelogram, rectangle, rhomboid, square, and tangent line of a circle; these geometric figures do not have definitions in CCSSM. CCSSM and CMCS similarly define the common definitions except when defining *congruence*. In terms of the definition of *congruence*, CCSSM defines congruence from a geometric transformation perspective as “the second can be obtained from the first by a sequence of rotations, reflections, and translations” (CCSSI, 2010, p. 55). CMCS expects students to understand the congruent triangle from the quantitative perspective that corresponding pairs of sides and angles are congruent. The next section discusses these in greater detail.

Table 7. Upper-Grade Band Frequencies and Percentages of DLEs in Level 3

Topics	CMCS	CCSSM
Understanding definitions	21 (20%)	14 (17%)
Developing informal proofs	28 (26%)	21 (25%)
Understanding transformation	6 (6%)	10 (12%)
Constructing figures	17 (16%)	14 (17%)
Understanding relationships between 3D and 2D figures	1 (<1%)	3 (4%)
Classifying figures	1 (<1%)	0
Total	74 (70%)	62 (75%)

Note. The total number of DLEs for CCSSM at Upper-Grade Band is 83 and for CMCS is 106.

Regarding developing informal proof, CCSSM has 21 DLEs, while CMCS has 28 DLEs at level 3, which expects students to understand and explore theorems and postulates informally. They shared the most common DLEs in informal proof, including the *Pythagorean theorem and its converse; the criteria for triangle congruence; supplementary, complementary, vertical, and adjacent angles; the relationship between central, inscribed, and circumscribed angles; and the criterion for similar triangle*.

At level 4, CCSSM addresses 17 DLEs of formal proof in high school. In CMCS, there are 28 DLEs of formal proof in the Upper-Grade Band. The two documents share common theorems, including segment perpendicular bisector theorem and its converse, parallel lines theorems, and the converse of parallel lines theorems, parallelogram theorems, and converses. CMCS includes angle bisector theorem and its converse, theorems of equilateral triangles, theorems of rectangles, rhombi, squares, and perpendicular diameter theorem, which are not in CCSSM. Simultaneously, CCSSM has a DLE of using triangle similarity to prove the Pythagorean theorem, which does not appear in CMCS.

Comparisons of Selected Geometric Topics within and Across Grade Bands and Levels

In this section, we focus on reporting three geometric standards that are presented in most distinct ways by CCSSM and CMCS, including (a) recognizing and classifying figures, (b) transformation, and (c) constructing

figures. We selected these three topics because they were prevalent in both standards and they provided contrasting treatments to illuminate the significant differences between these two standards.

Recognizing and Classifying Figures

CCSSM proposes six DLEs at van Hiele level 2 and expects students to recognize figures by their attributes, while CMCS addresses one DLE for recognizing figures by attributes: “through observing and manipulating, recognize the attributes of squares and rectangles” (p. 19). Specifically in CCSSM, students are expected to “analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/“corners”) and other attributes (e.g., having sides of equal length)” (CCSSI, 2010, p. 12) as early as in Grade K. In Grade 2, students are expected to “recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces” (CCSSI, 2010, p. 20). Like DLEs at van Hiele level 1, at level 2 CCSSM continues to emphasize manipulation and encourages students to build and draw. CMCS focuses only on 2D shapes and specifies the attributes of squares and rectangles. CMCS proposes that the other three DLEs address transformation by observing and manipulating to perceive reflection, translation, and rotation. In contrast, CCSSM does not propose DLEs that address transformation. We will discuss more details about the differences later.

Classifying figures is another topic that is presented in the two documents with significant disparity. CCSSM addresses seven DLEs across the Lower-Grade Band and Middle-Grade Band, while CMCS addresses only one DLE in the Upper-Grade Band, which expects students to understand hierarchical relationships among parallelogram, rectangle, rhombus, and square. In Grade 3, CCSSM proposes DLEs at level 3 to expect students to classify quadrilaterals according to their attributes (angles and sides) and “understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides) and that the shared attributes can define a larger category (e.g., quadrilaterals)” (p. 26). CCSSM addresses six DLEs of *recognizing figures by their attributes* to promote students’ understanding of figures’ attributes (i.e., angles, sides, and relations between them).

Later in the Middle-Grade Band, in Grade 4, CCSSM proposes two DLEs: “classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size” (p. 31), to continue developing students’ understanding on classifying figures. In Grade 5, CCSSM addresses one DLE to promote students’ understanding of hierarchical relations among figures and expects students to “understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category” (p. 38).

If we consider only the distribution of DLEs of this particular topic, they are consistent and coherent across grade levels in CCSSM. However, according to Mayberry (1983), unless students achieve van Hiele level 3, they should be expected to understand hierarchical relationships among figures and class inclusions. Therefore,

further investigation is needed to determine whether DLEs of *classifying figures* placed in the Lower-Grade Band are too early in CCSSM, as students haven't yet achieved van Hiele level 3.

Transformation

Transformation in both documents consists of reflection, translation, rotation, and dilation. Regarding *transformation*, CCSSM proposes three DLEs in Grade 8 at van Hiele level 2 and 10 DLEs at level 3 in high school and does not set DLEs before Grade 7. In Grade 8, CCSSM poses DLEs about transformation to expect students to recognize transformation empirically. The DLEs expect students to recognize transformations from motions of their attributes. After the first introduction of transformation in Grade 8, CCSSM reinforces the topic in high school at level 3:

Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch). (CCSSI, 2010, p. 76)

Similarly, CMCS provides formal definitions of transformation in the Upper-Grade Band at van Hiele level 3. The difference is that CMCS prioritizes students' understanding and prepares students to experience transformation in the Lower-Grade Band and Middle-Grade Band. For instance, in the Lower-Grade Band, CMCS states, "experience translation, rotation, and reflection from real life" (Ministry of Education, 2011, p. 19). In the Middle-Grade Band, CMCS is more explicate: "through observations and manipulations, recognize figures' translation and rotation on grid paper, translate basic figures vertically or horizontally on grid paper, rotate basic figures to 90° on grid paper" (Ministry of Education, 2011, p. 24). Transformation in CMCS is consistent and coherent throughout the three grade bands.

In brief, the two documents emphasize *transformation* differently. CMCS presents *transformation* crossing grade bands and van Hiele levels, reflecting curricular coherence (Schmidt et al., 2005). CCSSM introduces *transformation* in grade 8. Based on empirical understanding of *transformation*, in high school, CCSSM introduces the concepts of *congruence and similarity*. Differently, *transformation* in CMCS does not have an explicate connection with *congruence and similarity*. *Transformation* in CMCS is most likely an independent topic, while in CCSSM it serves as a foundation for *congruence and similarity*.

Constructing Figures

Constructing figures is the topic in which CCSSM and CMCS have many common DLEs but which are presented differently. CCSSM has 14 DLEs, and CMCS has 17 DLEs on constructing figures. They are all in the Upper-Grade Band at van Hiele level 3. In particular, the two documents share common constructions, including copying a segment, copying an angle, bisecting a segment, bisecting an angle, and constructing a tangent line from a point outside a given circle to the circle. Regarding constructing tools, DLEs in CCSSM are more open and flexible than CMCS. For example, CCSSM expects students to draw "freehand, with ruler and protractor, and with technology" (CCSSI, 2010, p. 50), while CMCS expects students to "use straightedge and

compass” (Ministry of Education, 2011, p. 35), indicating CMCS has a higher expectation on students’ geometric construction. However, while using a compass and straightedge or ruler to construct figures could reinforce students’ geometric understanding, various tools can also provide insightful visualization, particularly in a dynamic geometry environment. Using geometric software features could promote geometric visualization and then improve students’ geometric intuition (Sinclair, 2008).

The DLEs in CCSSM tend to provide more opportunities for students to explore geometric theorems. For instance, students are expected to “focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle” (CCSSI, 2010, p. 50). Here, the standards do not specify the three measures of angles or sides, which offered students more space to discover the principles for constructing a unique triangle and would be able to explain why some conditions can be criteria for triangle congruence and others cannot be. In contrast, CMCS expects students to construct triangles by giving “three sides, two sides and inner angle, and two angles and inner side” (Ministry of Education, 2011, p. 35). This may direct students to the theorems but impedes students’ autonomy in learning.

The DLEs on constructing figures in both CCSSM and CMCS aim to prepare students’ further geometric understanding. For example, the formal geometric construction of the perpendicular bisector of a line segment aligns with the segment perpendicular bisector theorem and its converse, which are at van Hiele level 4 in both documents. However, the DLEs in CMCS are more specific, which often correspond to definitions, postulates, and theorems. In comparison, the DLEs in CCSSM are less focused and inconsistent. To illustrate this inconsistency, we again used the example of constructing triangles from three measures of angles or sides as above. This DLE is in Grade 7. However, no DLEs about congruence appear within grade 7. Further, no DLEs about criteria for a triangle appear in grade 8. By high school, students are expected to “explain how the criteria for triangle congruence follow from the definition of congruence in terms of rigid motions” (CCSSI, 2010, p. 76). It may prove difficult for students to meet this expectation without knowing congruence and criteria for a triangle.

Discussion

The findings revealed the distribution of van Hiele levels and topics in CCSSM and CMCS in grade bands and grade levels. In terms of geometry content, the two documents show great similarity, and both present Euclidean geometries. However, from the lens of van Hiele levels, there are divergences regarding the emphasis in geometry learning. CCSSM emphasizes getting experiences in an informal deduction from multiple ways, including understanding geometry concepts in an earlier grade, using geometry software, and providing an opportunity for students to explain their thoughts. In contrast, CMCS highlights promoting geometric intuition through accumulating experiences from observations and manipulations (Kong & Shi, 2012). Although both documents share major geometric topics, such as congruence and similarity, transformation, and geometric construction, they present differently based on their own particular emphases. We discuss these differences in the following four ways.

First, geometry standards in CCSSM are more general, while geometry standards in CMCS are more specific. CMCS specifies learning expectations, for instance, in a certain grade band, what figures students need to recognize, and what approaches students would use. For instance, in the Middle-Grade Band, CMCS proposed that students "through observation and manipulation recognize parallelogram, trapezoid, and circles, know sectors, and can use a compass to draw circles" (p. 25). Standards in CCSSM are more general, and approaches are broader. For example, in Grade 2, CCSSM expects students to "recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces" (p. 20). The standard does not indicate specific figures and tools to draw figures. Another example is classifying figures; CMCS expects students to understand hierarchical relationships between parallelogram, rectangle, rhombus, and square, while CCSSM addresses a broader expectation that students "classify two-dimensional figures in a hierarchy based on properties" (p. 32).

Second, while both CMCS and CCSSM have the greatest number of DLEs at level 3, the distributions of the four levels across grade bands are quite different. CMCS shows more consistent progression through grade bands with the majority of DLEs at the Lower-Grade Band on level 1, Middle-Grade Band on level 2, and Upper-Grade Band on level 3, while CCSSM is less consistent, with the majority of DLEs in Kindergarten at level 1, from Grades 1–2 the majority of DLEs at level 2, and from Grades 3–8 and high school the majority of DLEs at level 3. Compared to CMCS, CCSSM does not set sufficient DLEs at van Hiele level 2. Previous research revealed that pre-CCSSM geometry standards did not propose sufficient DLEs at van Hiele level 3 to prepare students in developing informal proof (Newton, 2011). Our finding aligns with Dingman et al.'s (2013) claim that CCSSM puts most effort at van Hiele level 3. However, for some topics, there is a lack of preparation at van Hiele level 2. For instance, CCSSM proposes DLEs at level 3 in Grade 3 to expect students to develop a primary hierarchical understanding of different categories of quadrilaterals. However, these DLEs may not be effective unless students are prepared well in van Hiele level 2 to perceive the relations between a figure and its attributes through manipulation or real-life models. Without sufficient preparation in level 2, students may not be grounded in basic geometric concepts and may not understand the language used in level 3 (Fuys et al., 1988). As such, memorization may be their only resource, but it does not help with understanding reasoning and geometric language at a higher level, which may cause frustration and discouragement (Burger & Shaughnessy, 1986).

Third, CCSSM makes more connections between various aspects of the same topics or between different topics. An example of CCSSM's focus on connections can be seen from its treatment of Transformation, Congruency, Similarity, and Symmetry. These concepts are built on the perspective of geometric transformation. Congruence, similarity, and symmetry represent one or more combinations of geometric transformation. Conversely, geometric transformations can explain these concepts or decide if two figures are similar, congruent, or symmetric. Transformation in CMCS is most likely an independent topic and less related to congruence and similarity. Historically, there are many debates around these two approaches of definitions of congruence and similarity. Usiskin and Coxford (1972) argued that the definitions of congruence as the same shape and same size, while mathematically precise, lacked the generality that intuition normally ascribes to the related ideas. Some researchers have suggested transformation should be the basis of high school geometry

because the transformation approach emphasizes intuitive thinking and visual processes (e.g., Sinclair, 2008; Usiskin & Coxford, 1972). To seek the influence of the two approaches, a further discussion around definitions is needed.

Fourth, CCSSM and CMCS use different means to prepare students for formal proofs. CCSSM highlights Dynamic Geometry Environments (DGEs), while CMCS underscores abstract and rigor at the upper level. CCSSM encourages infusing technology in mathematics teaching and learning, in particular, creating DGEs by incorporating geometric software. In high school, CCSSM suggests using dynamic geometric software to make formal geometric construction. Although CMCS also emphasizes the importance of technology in mathematics, using technology is absent in geometry. Researchers have argued that technology can be used as an intervention to provide visualization and constructive techniques that allow students to develop and validate conjectures and stimulate students to provide deductive explanations (e.g., Sinclair, 2008; Sinclair et al., 2017; Wares, 2007).

The discussed differences between CCSSM and CMCS reflect the difference in the geometry curriculum philosophy in the two nations. The descriptions of geometry standards in CCSSM are more general, while geometry standards in CMCS are relatively explicit. Both CMCS and CCSSM underscore learning expectations at van Hiele level 3. Through van Hiele levels, CMCS is more coherent throughout three grade bands, while CCSSM makes more connections between various aspects of the same topics or between different topics. CCSSM holds open attitudes on various multiple geometric learning tools, while CMCS stresses abstract and rigor aspects of geometry. Overall, CCSSM and CMCS expect students to achieve formal deduction from different means of preparation. In terms of proceeding through van Hiele levels, CCSSM proposes higher-level DLEs earlier than CMCS. CMCS is more consistent than CCSSM in moving from a lower to a higher van Hille level when introducing the same topic over multiple grade bands.

Conclusion

The results of this study show that CCSSM and CMCS have significant differences. They both support van Hiele theory in their own ways. CCSSM expects students to make adequate preparation at van Hiele level 3 to reach formal deduction, which aligns with van Hiele's theory that prepares students enough in informal proof to move forward to the formal deduction. CMCS emphasizes abstract thinking and expects students to develop theoretical reasoning through constructing definitions, postulates, and theorems systems, which supports van Hiele's proposal that students should understand the meaning of deduction and construct mathematical proofs using propositions, axioms, and theorems (van Hiele, 1984; van Hiele-Geldof, 1984).

One limitation of this study is that it did not compare the quantitative aspect of the geometric measurements, which is also an essential area in the geometry curriculum. In future studies, general geometry should be considered to make a strong case of the standards' influence. CCSSM is structured by grade level from K–8 and integrated into high school, while CMCS is organized by grade bands. CCSSM consists of high school standards, but CMCS does not. Though these two documents share the most common content, it is hard to

compare through grade level. We acknowledge this is another limitation of this study. Future investigation of textbooks by grade level would shed additional light of the geometry learning in these two curricula.

The present study contributes to a growing body of literature on standard analyses. Specifically, the study established the feasibility of using topics at each van Hiele level to support examining geometry standards. The findings also add to the comparison literature between the U.S and China in mathematics education. Although this study does not focus on explaining the causality of differences between standards, textbooks, teaching, and learning in these two nations, the cross-national comparison of standards could stimulate curriculum development. For the further revision of CCSSM and CMCS, we suggest CCSSM add more DLEs in van Hiele level 2 to better prepare students for formal geometric concepts. CMCS could add multiple geometric learning tools. Particularly, given the importance and innovation of DGE in geometry learning, CMCS should consider integrating DGE in the geometry curriculum to promote students' visualization, conjecture, and reasoning.

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