

# Pre-Service Teachers' Procedural and Conceptual Understanding of Pupils' Mean Value Knowledge in Grade 6

Christina Svensson<sup>1</sup> , Mona Holmqvist<sup>1\*</sup> 

<sup>1</sup>Malmö University, SWEDEN

\*Corresponding Author: [mona.holmqvist@mau.se](mailto:mona.holmqvist@mau.se)

**Citation:** Svensson, C., & Holmqvist, M. (2021). Pre-Service Teachers' Procedural and Conceptual Understanding of Pupils' Mean Value Knowledge in Grade 6. *International Electronic Journal of Mathematics Education*, 16(3), em0649. <https://doi.org/10.29333/iejme/11067>

## ARTICLE INFO

Received: 7 Jan. 2021

Accepted: 2 May 2021

## ABSTRACT

The aim of this study is to explore pre-service teachers' procedural and conceptual understanding of pupils' understanding of mean value, before and after an intervention. Participants are 66 pre-service teachers, and two teacher educators. The data consists of five video-recorded lessons and pre-service teachers' pre- and post-test discussions. Based on variation theory, the analysis aimed to identify pre-service teachers' expressed knowledge of procedural and conceptual understanding, focusing four categories of expressions: procedural (P), procedural didactics (PD), conceptual (C), and conceptual didactics (CD). The pre-service teachers' expressed understanding of pupils' knowledge developed differently in the five teacher student groups. A shift from procedural to conceptual focus occurred in two groups. The number of procedural related words expressed in the first group decreased from 104 (pre-test: 65 P and 39 PD) to 16 (post-test: 9 P and 7 PD), at the same time as the conceptual related words increased (pre 4 C and 0 CD and post 11 C and 16 CD). In the second group, 87 procedure-related words (36 P and 51 PD) identified during pre-test decreased after the intervention (3 P and 9 PD). Instead, concept-related words increased from 16 (12 C and 4 CD) to 29 (10 C and 19 CD). The changes occurred among participants who had got, except of theoretical instruction, challenging hands-on tasks about the Mathematical content in focus. The findings from this study contribute to understand how design of instruction for pre-service teacher contributes to enhance teacher students' didactic perspective on pupils' learning.

**Keywords:** conceptual knowledge, mean value, pre-service teacher education, procedural knowledge, variation theory

## INTRODUCTION

Studies in the field of mathematic didactics have reported a correlation between teachers' didactic subject-related knowledge and the quality of their teaching (Ball et al., 2008; Shulman, 1986). Although instructional methods in pre-service education are thought to influence pupils' learning outcomes, few studies have explored their development (Skott et al., 2018). The development of effective instructional methods requires teachers' capabilities for evaluating pupils' knowledge and using this information to design lessons. However, in some cases, the focus has been more on methodological or procedural issues than the focus on what is needed to learn a specific content. Some previous studies have examined pre-service and in-service teachers' skills for teaching specific content. Rowland (2014), and Skott et al. (2018) examined how teachers' knowledge develops and influences practice and suggested action steps for professional teachers. They argued that the focus has been on general didactic skills, with relatively little interest shown in how content knowledge (CK) can be taught in specific subject fields (Skott et al., 2018).

Studies have indicated that CK, which influences teaching skills, is weak. Goulding et al. (2002), who investigated primary teachers' mathematical knowledge, found that teachers demonstrated inadequate knowledge and limited planning and teaching skills. Moreover, Rowland et al. (2000) found that pre-service teachers who lacked mathematical CK also demonstrated weak teaching performance. They noted that "a high proportion of students scoring poorly on the audit do indeed fall within the weak teaching practice category" (Rowland et al., 2000, p. 10).

Teachers' CK for teaching mathematics has been found to impact on pupils' test results even at a very elementary stage in the first grade (Hill, Rowland, & Ball 2005). Moreover, the findings of Tutak and Adams' (2017) study indicate that pre-service teachers who lack geometric content knowledge can enhance their skills by teaching with pupils. A growing number of studies have highlighted the importance of subject mathematical knowledge (SMK) (Scots et al., 2018), thus supporting the view that subject CK should be augmented within teacher education programs.

In this study, the focus is on conceptual and procedural mathematical knowledge. Conceptual knowledge is a form of knowledge that is based on rich information- relationships that are connected in networks where both informal and formal pieces

of information are essential to identify (Hiebert & Lefevre, 1986). Procedural knowledge is based on a person's ability to become familiar with mathematical symbols and conventions. Simultaneously, rules and procedures are available for solving mathematical rules regulations and procedures such as algorithms for solving averages (a.a.). Conceptual knowledge, however, does not necessarily have to be based on rich information relations about advanced mathematical understanding. For example, a child may possess conceptual knowledge, less developed than an adult but still considered conceptual knowledge (Star, 2005). The construction of tasks is vital for illuminating, capturing interest, and retaining an interest in optimizing learning to promote conceptual knowledge (Chapman, 2013).

## BACKGROUND

Mathematical CK, in this study limited to arithmetic mean value, could imagine being a simple concept because of its form, but it is quite complicated to understand (MacCullough, 2007). Sullivan et al. (2013) argue for representational tasks to enhance conceptual development and understanding of mathematics. However, there are many opportunities for developing conceptual understanding by representative tasks in this topic area in daily society. Tables, diagrams, and arithmetic solutions of different average are common in media and everyday life. The latter, for example, in business stores, where you need CK of calculations and reasoning knowledge about how many shoes in a specific size to be ordered. Further, the need for CK to be able to interpret diagrams for average temperature or interpret diagrams and calculation of averages in falling ill in a pandemic.

This kind of daily mathematical situations give many opportunities contribute to an increased experience and explored statistics. However, it is also a crucial understanding for considering when the mean is a useful measure for different data to be an informed citizen (MacCullough, 2007). The opportunity of real tasks in daily life can be compared to contextualized tasks that engaging and enhance motivation for conceptual knowledge in mathematics (Sullivan et al., 2013). Cobb and Moore (1997) claim that critical for statistical understanding is that "data" sets presented in the tasks are in its context. This because "statistics requires a different kind of thinking, because data are not just numbers, they are numbers with a context" (Cobb and Moore, 1997, p. 801).

A common phrase in statistics education attributed that data are numbers with a context. But in mathematical textbooks, it is common with only numbers without context in this topic area (Liljekvist, 2014). However, understanding of mean value includes both the algorithm for the mathematic arithmetic mean value and the arithmetic mean value as a mathematical point of balance (MacCullough, 2007). The balance between only calculating mean value algorithm and statistical reasoning for developing understanding is claimed for. Therefore, statistic tasks must be about a context to promote statistical reasoning (Cobb & Moore, 1997), and to develop conceptual knowledge requires that the algorithm is related to a context (MacCullough, 2007). Pupils also need to explain their reasoning for creating a more precise mathematically average understanding in the tasks (Maker, 2014), to be able to move from their data to get beyond their classroom context. Therefore, it is also essential to highlight the mathematical content in the context as well as the opposite (Cobb & Yackel, 2011).

If information is limited, e.g., in a diagram, the pupils can use verbal and gestural expressions to compensate for this limitation when they reason (Chen & Herbst, 2012). Burland et al.'s (2000) study of two groups of Grade 6 pupils examined how the gestures related to their use of reasoning strategies according to diagrams, where illustrations were included, played an essential role in developing collaborative mathematical reasoning. Therefore, it is useful to follow the pupils learning process, and in this process observe gestures and informal verbal expressions of the content knowledge (Radford, 2013). This puts demands on having CK for developing pedagogical content knowledge (Ball et al., 2008; Shulman, 1986) as a teacher.

Within higher education, students are expected not only to possess the required content knowledge but also to be prepared for an uncertain future whose knowledge needs are not yet known (Pang & Marton, 2003). Teacher educators try to prepare pre-service teachers to teach the next generation of pupils even though the knowledge requirements cannot be precisely predicted. The provision of tools that can support pre-service teachers in handling new situations within their future professions can facilitate both teachers' and pupils' learning processes (Pang & Marton, 2003).

In some Asian countries, such as Japan, teachers collaboratively develop their knowledge by observing each other's lessons and subsequently share their reflections to enhance the quality of instruction and to facilitate pupils' knowledge development (Holmqvist, 2017). This professional development model, known as lesson study, is widely used and has a long tradition in Japan, beginning in the 19th century. A similar model, known as learning study, has been developed more recently in Sweden and in Hong Kong (Marton, 2015). In this model, teachers follow an outline and engage in collaborative planning of theoretically informed instruction relating to general and subject didactics and in collegial conversations, leading to the revision and re-teaching of lessons. In a learning study, the cycles of activity usually entail cooperation between teachers and researchers (Holmqvist, 2011; Marton, 2014). While this model has been found to be highly effective in developing professional learning (Kullberg et al., 2016), it has mainly been applied in the context of developing pupils' learning in primary and secondary schools. The theory-driven cyclical and iterative process illuminates and encourages the articulation of teachers' own experiences of teaching (Holmqvist, 2011). Teachers can embark on their professional development by systematizing and designing their teaching on the basis of a theoretical understanding rather than implicit decisions (Marton et al., 2004). The structure of the lesson plans reflects collective and cyclical processes for developing an experiential problem (Lewis & Tsuchida, 1997) that is comparable to the structure developed using the lesson study method (Stigler & Hiebert, 1999).

The aim of this study is to explore pre-service teachers' procedural and conceptual understanding of pupils' understanding of mean value, before and after an intervention. The intervention was designed as a learning study, with pre- and post-tests, in which pre-service teachers analyzed pupils' reasoning relating to a defined area of content (mean value) before and after an intervention.

Specifically, the development of pre-service teachers' skills for exploring pupils' mathematical CK relating to the mean value were investigated from a didactic perspective. Based on variation theory, the analysis aimed to identify pre-service teachers' expressed knowledge of procedural and conceptual understanding, focusing four categories of expressions: procedural (P), procedural didactics (PD), conceptual (C), and conceptual didactics (CD).

The following research questions were formulated:

RQ1: What aspects of content knowledge expressed by the pupils, during a video-recorded group discussion about mean value, are critical for their knowledge development?

RQ2: What aspects of the pupils' expressed knowledge were discerned by the pre-service teachers before and after the intervention?

RQ3: What differences, if any, were there in pre-service teachers' understanding of pupils' knowledge before and after the intervention?

## THEORETICAL FRAMEWORK

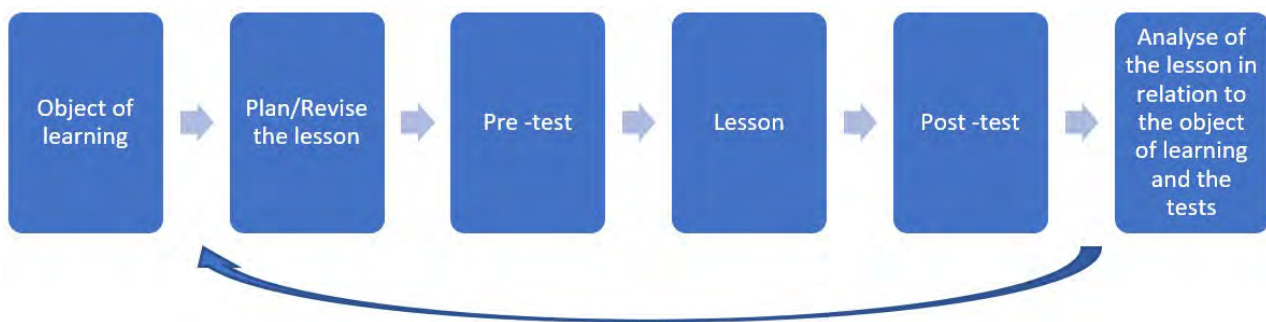
Variation theory, which offers analytical tools for exploring the necessary conditions for learning, has been used as theoretical framework to investigate knowledge development (Marton, 2014). It entails assumptions about learning that can be teaching applied to enhance learners' capabilities to discern aspects of the learning object that have not been previously experienced (Holmqvist & Selin, 2019; Marton, 2014; Marton & Booth, 1997). According to variation theory (Holmqvist, 2011; Lo, 2012; Marton & Booth, 1997), variation enables learning aspects to be discerned on the basis of discerning different aspects of a phenomenon. Through the function of simultaneity, whereby different aspects of the learning object can be discerned at the same time, critical aspects may be rendered visible through their variation. Critical aspects are aspects not yet discerned by the learning, but needed to develop their understanding. Thus, an intervention based on variation theory strives to make these critical aspects to be distinguished. The highlighting of critical aspects for the development of learning results in increased opportunities for all individuals to discern the parts of a particular learning object and the structural relationships of the parts to each other and the whole object (Lo et al., 2006). Lo et al. (2006) emphasized the importance of considering the larger context in the process of making aspects discernable for learners through place-based learning entailing real situations. This context enables learners to identify contrasts among the object of learning and other phenomena, for example, by defining what mean value is by contrasting it against median, or conceptual knowledge contrasted against procedural knowledge. Thus, the theoretical framework can be used to determine the design of an instructional method for pre-service teachers to enhance their own theoretical knowledge of how to understand their own teaching; it can also be used as a research tool to analyze preservice teachers' expressed knowledge before and after an intervention.

According to Holmqvist (2004), variation theory is based on the premise that all learning requires structural pattern of variation related to a learning object. It is not about finding the right or best teaching method or variation relating to the choice of methodology; rather the focus is on variation required for discerning a specific learning object that relates to the learning goals. The pattern of variation (generalization, contrast, fusion) is used in planned instruction to visualize critical moments for learning. These critical moments that occur when individuals notice changes in their understanding of the outside world are the moments when they learn (Holmqvist, 2004).

To recognize differences in pupils' reasoning, pre-service teachers have to study additive and multiplicative relationships along with distributive reasoning. It is assumed that this knowledge will enhance pre-service teachers' focus on not only procedural but also conceptual knowledge, as they increase their own understanding of the mathematical content while simultaneously acquiring an understanding of how to teach this content to their pupils (Shulman, 1986). A second phase during which teacher educators and researchers apply these theoretical assumptions entails attempts to elicit qualitative differences in pre-service teachers' and pupils' expressed knowledge during pre- and post-tests. This can be achieved through the design of tasks, such as those used in the fourth and fifth lessons of this intervention (learning study) for pre-service teachers. The analysis presented here follows the structure developed by Holmqvist and Selin (2019). Accordingly, aspects and features to identify pre-service teachers' expressed knowledge of procedural and conceptual understanding, focusing four categories of expressions: procedural (P), procedural didactics (PD), conceptual (C), and conceptual didactics (CD).

## METHOD

The unit of analysis was pre-service teachers' expressed discernment of pupils' understanding of the mean value while solving a task. At the university, the pre-service teachers were watching video-recordings of pupils' problem-solving in group, while taking a course in mathematical didactics. The pre-service teachers were taking the course "Mathematics and learning: number sense, arithmetic and algebra", at the University. The course goal to: "assess pupils' knowledge and knowledge development in number sense, arithmetic and algebra based on the school's governing documents" was focused. To capture changes in the pre-service teachers' skills, the teacher students were video-recorded during their discussions before and after interventions, which were designed in line with Shulman's (1986) recommendation to use case-studies for enhancing teachers' teaching knowledge. In line with these findings, all pre-service teachers' pre- and post-tests included a video-recording of a group of five pupils in grade 6, who first explained, one at a time, how they solved mathematical tasks centering on the mean value. Furthermore, the grade 6 pupils subsequently described how they obtained their answers in a group discussion with other pupils. The video recording was made



**Figure 1.** The design of the teacher training intervention implemented by teacher educators

**Table 1.** Pre-service teachers who participated in the study

Group	Program and term	Women	Men	Total
RL 1	4-6 Science/Social Science	21	5	26
RL 2	4-6 Language Swe	7	2	9
RL 3	4-6 Mathematics	8	1	9
RL 4	4-6 Language Swe	6	2	8
RL 5	4-6 Science/Social Science	12	2	14
Total		54	12	66

by one of the teacher educators, who also served as a moderator in the group discussion in a regular class and classroom. At the video, the teacher asked the pupils, both individually and as a group, if they could express the concept of the mean value, and she instructed them to discuss their solutions with each other during the group session (following the procedures of previous research c.f. Bjurland et al., 2000). The video-recorded situation (11 min 26 sec) was presented in pre- and post-tests. The mean value tasks the pupils are working with are derived from the national-level maths test for grade 6 (PRIM, 2020: <https://www.su.se/primgruppen/matematik/%C3%A5rskurs-6/exempel-ur-tidigare-prov>).

This statistic task chosen is familiar for pupils in Sweden. Further, the algorithm for mean value is focused initially on procedural knowledge (c.f. Hiebert & Lefevre, 1986), but it turns a form of balance between the algorithm and the context and by that makes it possible to also assess conceptual understanding (MacCullough, 2007). The test aimed to study the pre-service teachers' focus (procedural and conceptual) when commenting on the pupils' knowledge of mean values. They watched the recording twice, just before and after the research lesson. The pre-service students were given the following two questions in both pre- and post tests before they watched the recording:

- What knowledge can be examine of the pupils' mean value knowledge (summative assessment)?
- What learning do you think the pupils have developed and what do they need to develop further (formative assessment)?

The researcher video-recorded the pre-service teachers' group discussions at pre- and post-tests, which were transcribed verbatim and analyzed by the first author. The second author conducted an independent analysis of the material to check for any incongruence. Both authors had access to all of the collected data. The pre-service teachers collaboratively answered the questions orally within smaller groups.

Learning study was used as research method to collect data to study the relation between instruction and learning, by supporting pupils' learning through teachers' enhanced professional development (Brante et al., 2015; Holmqvist, 2011; Marton & Lo, 2007). In this study the learners are pre-service teachers, and the object of learning their skills to analyze pupils' understanding of the concept taught (mean value) from conceptual and procedural perspectives. The group of teachers in this study are teacher educators working collaboratively with the researcher to develop pre-service teachers' learning. The teacher training intervention followed the design presented in **Figure 1**.

**Participants.** Two teacher educators, one researcher, and a total of 66 pre-service teachers divided into five different groups/class participated in this interventional study. In total five research lessons were designed, one per each teacher student group (G1-G5). Each research lesson lasted two hours. All of the pre-service teachers in the five groups were studying mathematics teaching for pupils in grades 4–6. **Table 1** shows the participants' details.

**Ethics.** Ethical procedures of informing participants of their rights to refuse to participate and obtaining their verbal and written consent for their voluntary participation were followed. The collected data were anonymized and coded to protect the participants' confidentiality, and the researcher had no teaching or grading role for the pre-service teachers in this study. Furthermore, the data were secured in a safe locker to avoid unauthorized access. The names of the participants and other sensitive data were removed from the stored data.

**Data collection.** The collected data comprised verbatim transcripts of the video-recorded discussions used for the pre- and post-tests, based on the video-recorded stimuli of pupils' reasoning and discussion about mean value. Data was also collected from each group's research lesson, which means in total five research lessons were video-recorded. Moreover, the data included the researchers' notes taken during their planning discussions and during video-recorded lessons as well as recorded pre- and post-test discussions (in total audio-recorded discussions and 27 video-recorded discussions from 66 pre-service teachers). **Table 2** provides descriptions of the collected data.

**Table 2.** Collected data

Data	Participants	Number
Notes from planning meetings	Teacher educator and researcher	10
Video-recorded planning meetings	Teacher educator and researcher	2
Pre-test	Pre-service teachers	66
Video-recorded lessons	Pre-service teachers	5
Post-test	Pre-service teachers	66
Video - recorded pre-test	Pre-service teachers	27
Video-recorded post-test	Pre-service teachers	27
Audio-recorded pre and posttest	Pre-service teachers	66

**Table 3.** The time-line of the research project

Date	Activity			Participants		
	Meeting	Lesson	Tests	Researcher	Teacher educator	Pre-service teachers
180119	x			x	x	
180131	x			x	x	
180215		1	pre/posttest	x	x	x
180417	x			x	x	
180423	x			x	x	
180423		2	pre/post test	x	x	x
180516	x			x	x	
181003		3	pre/post test	x	x	x
181115	x			x	x	
181122	x			x	x	
190221	x			x	x	
190221		4	pre/post test	x	x	x
190327	x			x	x	
190328	x			x	x	
190514		5	pre/post test	x	x	x

The study was implemented in Spring 2018 and ended in Spring 2019. **Table 3** presents the time schedule of the research project, encompassing all of the participants and activities that were conducted. All activities took place during the course of the regular teacher education program held at this university. The planning meetings and the analysis of the lessons that were held during the intervention were documented through notes. The meetings held to plan the design of the intervention and to analyze the instructions were video recorded. The transcripts (pre- and posttest) were produced following the intervention's implementation.

**Data analysis.** The analysis of the lessons was carried out with the support of variation theory (Holmqvist, 2006; Lo, 2013; Marton & Lo, 2017) and determined what aspects could be discerned by pre-service teachers to explore the pupils' knowledge of mean values during the lessons. Moreover, variation theory was applied in the design of hands-on tasks given to the pre-service teachers during their research lesson in groups four and five. The analysis aimed to identify pre-service teachers' expressed knowledge of procedural and conceptual understanding, focusing four categories of expressions: procedural (P), procedural didactics (PD), conceptual (C), and conceptual didactics (CD).

**Implementation of the learning study.** Each pre-service teacher student group participated in one research lesson. By that, five different iterations were enacted during the learning study intervention. The content of each lesson was analyzed and revised on the basis of the findings of the analysis of the previous research lesson outcome. The lessons were designed to facilitate the identification of differences in pre-service teachers' skills through a comparison of pre- and post-test results. Specifically, correlations between the pre-service teachers' enhanced abilities and what was offered to discern in the different lessons were examined (c.f. Holmqvist, 2004). In the extra supplemental material, the design of each of the lessons. The results of the analysis of the first three lessons had three implications for the fourth and fifth lessons. Accordingly, the following changes in inputs provided to pre-service teachers were made based on the pattern on variation used:

- Median, mean, and mode values should be presented with synchronous simultaneity to force the students to discern all forms and enhance their conceptual focus.
- Hands-on tasks should be offered, designed to simultaneously require a focus on the median, mean, and mode values, while at the same time keeping the numbers constant (**Figures 2 and 3**).
- Contrasts should be used to separate examples from generic principles, e.g., changing the number but keeping the same number for all values (median, mean and mode).

The findings of the analysis of the fourth and fifth teaching designs resulted in the development of the teacher educators' teaching strategies and a method for designing teaching tasks based on variation theory (**Figures 1 and 2**). Teacher educators require a deep understanding of mathematical content combined with a good grasp of how pupils' learning should be understood to enable them to design learning situations, materials, and tasks and to optimize the development of pre-service teachers' knowledge.

In the first task, shown in **Figure 2**, the pre-service teachers had to use the same (invariant) numbers to solve the task in three different ways while simultaneously defining the mean, median, and mode values by applying variation. The pre-service teacher



1. Calculate the mean, median and mode values.  
2, 5, 0, 8, 0, 1, 8, 5, 0, 2

**Figure 2.** The task designed for pre-service teachers to solve in Research lesson 4 and 5

- 2a. If the mean of 5 numbers is 20, what can the five different numbers be? Describe how you found the answer.  
2b. If the median of 5 numbers is 20, what can the different numbers be? Describe how you found the answer.  
2c. If the mode value of 5 numbers is 20, what can the different numbers be? Describe how you found the answer.

**Figure 3.** The task designed for pre-service teachers to solve in research lessons 4 and 5

All pupils in class 6a sold tickets. They did not sell the same amount of lottery tickets each. This is how many tickets the girls in the class sold:

Name	Number of tickets
Maja	11
Samira	8
Nadja	8
Sara	7
Eila	7
Laura	5
Alva	5
Fatima	4
Nellie	3
Wilma	2

- a. What is the mean value of how many lottery tickets the girls sold?  
Show how you solve the task. ( $60/10=6$ )
- b. There are 30 pupils in class 6A. The mean value of sold tickets in the class is five. How many lottery tickets did the pupils (both boys and girls) sell altogether?  
Show how you solve the task. ( $30 \times 5 = 150$ )
- c. What is the mean value of how many lottery tickets the boys sold?  
Show how you solve the task. ( $150 - 60 = 90$ )

**Figure 4.** The individual and group task on mean values that pupils had to solve

students discuss different context related to the different numbers in the tasks. The task was constructed from a variation theory perspective through the different ways while simultaneously defining the averages (Holmqvist, 2004), and related to the need of a context for developing a conceptual knowledge (Cobb & Moore, 1997).

The second task (Figure 3) also entailed variations, but the pattern of variation in this task differed from that in the previous one. Here, the solution (20) was invariant, but the pre-service teachers had to identify the numbers (five in each task) to determine what made 20 the mean, median, and mode values. In contrast to the first task, the numbers in this task varied. In this tasks the focus is on the algorithm, based on MacCullough, (2007) arguments for understanding of mean value include both the algorithm for the mathematic arithmetic mean value and the arithmetic mean value as a mathematical point of balance. The latter was contributing by using the three averages median, type and mode with the solution as invariant simultaneity.

## RESULTS

In line with our aim of exploring skill development among pre-service teachers for engaging with pupils' mathematical knowledge, an interventional study was conducted from a didactic perspective, focusing on the mean value before and after testing. The results are presented in the same order as the research questions. The analysis revealed what pre-service teachers discerned regarding the pupils' knowledge expressed in video-recorded discussions that were used for the pre-test and post-test. Subsequently, the pre-service teachers' statements on what they discerned during the pre- and post-test evaluations were analyzed. Lastly, the differences in their discernment before and after the intervention were analyzed.

**The pupils' expressed knowledge about mean values.** The first question was aimed at identifying aspects of CK expressed by the pupils in the video-recorded sequence that was analyzed by the pre-service teachers during the pre- and post-tests. These aspects, presented in a video clip (a duration of 11 min 26 sec) could be discerned by the pre-service teachers when they analyzed the pupils' reasoning during the next phase of the intervention. During this phase, the analysis centered on pupils solving tasks relating to mean values in the video clip. The pupils ( $n = 5$ ) were given a problem entailing three tasks to solve individually (one pupil video-recorded at a time) and during group work conducted by the same five pupils. Figure 4 depicts the assigned task.

**Table 4.** Pupils' solutions and discerned aspects relating to the object of learning (mean value)

Pupil	Solutions	Discerned aspects of mean value	Not yet discerned aspects	Irrelevant aspects misleading understanding
1	a. 6	Sum divided by sellers	-	-
	b.			-
	c.	20 boys x 5 = 100		Mean value x n (boys)
2	a. 60	-	Divison of the sum by the number of sellers	Sum
	b. 150	-		
	c.	150 - 60 = 90		
3	a. 6	Sum divided by sellers	-	-
	b. -			
	c. 4.5	60 x 2 (twice as many boys than girls) = 120 Change to 90/20		
4	a. 6	Sum divided by sellers	-	-
	b.			
	c.	20 boys x 5		
5	a. 6	Sum divided by sellers	-	-
	b.			
	c.			

Different kinds of knowledge were unveiled in the pupils' statements on their reasons and in their responses that could be discerned by the pre-service teachers and explored during their pre- and post-test discussions. **Table 4** shows the content in pupils' video-recorded explanations and group discussions that could be explored.

The data shown in **Table 4** reveals that the pupils discerned aspects for mean value differently. Pupil 3, for example, was able to discern aspects needed to determine the mean value and had no difficulty solving the three tasks, even though the third solution was not presented in writing on the test sheet.

**The pre-service teachers' discernment of pupils' knowledge.** The second research question focused on aspects of the pupils' expressed knowledge that were discerned by the pre-service teachers before and after the intervention. A detailed analysis of the teachers' pre- and post-test responses conducted after the third lesson (held in October 2018) revealed weak knowledge development. After the analysis, the designs of the fourth and fifth lessons were changed through the introduction of a task informed by variation theory for the pre-service teachers to solve. In this task, the mean value was contrasted with other types of averages to enable the student teachers to develop their CK (see **Figures 2** and **3**). Accordingly, there was a time gap after the first three lessons, which ended in October 2018. Their content was collectively analyzed by the teacher educators, leading to the redesigning of the fourth and fifth lessons (held in February and March 2019, respectively). The qualitative analysis performed on the pre-service teachers' discussions during the pre-test in lesson 1 revealed that while the teachers talked about the mode average, they did not use this word to express it. Moreover, they did not recognize the mode average, confusing it with the mean in their discussion. Their own difficulties were clearly expressed as they watched the pupils' video-recorded discussions.

Excerpt from the pre-test for lesson 1:

PST4: It is difficult to put the mean value into words.

PST2: Yes, it is.

PST4: I had difficulties with it too.

PST2: Yes.

PST3: Yes.

PST4: Difficult

Three evident knowledge-related challenges were identified in the analysis. First, the pre-service students' discussions focused on the pupils' use of *correct terms and methods* (procedures). This tendency was especially marked in the discussion about the use of the wrong word for addition. Second, the discussions focused on the pupils' use of reading and problem-solving strategies (procedures), as revealed by their use of the term "reading strategies" as well as by their implicit allusions to *more information* connect to the strategies. Third, the pre-service teachers' discussion focused on the methods employed by the teacher (procedures): how the teacher asked the questions and how he or she organized the group discussion. **Table 5** provides examples of the pre-service students' assessment statements.

**Table 5.** Examples of pre-service teachers' responses in pre- and post-tests relating to lessons one to three

Lesson	Pre-test	Post-test
1	"It bothers me when the pupils <i>use wrong words</i> , such as <i>plus</i> instead of the word <i>addition</i> ."	"One option that he does not know the mean value concept. He <i>adds</i> the numbers and gets the <i>total sum but not the mean value</i> ."
2	"It's only about <i>procedures</i> . It is so overall (in school)."	"There is a big different between a and c. A has more <i>procedures</i> . In c, the procedure is the same, but you need to <i>have more information</i> ."
	"It is difficult to move on. It depends on <i>how the teacher asks the questions</i> ."	"(not heard)... through the discussions <i>with the teacher they</i> (pupils in the video) <i>start the reasoning</i> ."
3	"It's about how they interpret and <i>process the information</i> ."	"There is an <i>understanding</i> , [...] it does not show that he can and if they not had the discussion it would not have come out"
	" <i>Reading strategies</i> really get you in, and that tells you little about the mathematical competence..."	"First I thought that she did not understand, but after I had had the opportunity to think more [about this], I had a <b>different understanding</b> about her thinking."

**Table 6.** Examples of responses obtained during the pre- and post-tests conducted for the fourth lesson

Pre-test	Post-test
"I sat for a long time thinking about what I was counting."	"But there is more. Yes they know and have <i>learned</i> the formula okay. [They will] count it then divide it but they do not know why they count [to] 10."
"Did you also got 4.5 or?"	" <i>Most people have learned how to calculate the mean value</i> if you get all the information. But when information is left out, as mentioned by you, it immediately becomes more difficult."
"Yes, but no one there calculated the mean value; [they] did not read the data properly. They only <i>calculated</i> how many lots the boys sold. No one had calculated the mean value."	

**Table 7.** Examples of pre-service teachers' responses in the pre- and post-tests relating to the fifth lesson

Pre-test	Post-test
"I think they know how to calculate the mean value; they have typed in a formula as well, but they do not know what they are doing. None of them could explain why they divide with 10; none of them could explain how they calculated, I think."	"For the calculation, everyone can—or almost everyone can— calculate the mean value."
"I don't really know what I'm doing."	"Because he was really good here (solved task c but not a and b). So there are <i>different qualities (abilities)</i> ... even if you understand what you are doing but can't figure it out, then you also have no knowledge. I can understand what the mean value is, but if I have no idea how to calculate it. What does it matter? This is hard!"

The results indicated that there were significant changes in the pre-service students' understanding of the implemented procedures after the third lesson. There was a greater focus on content in the post-tests, especially in the third lesson. The teaching designs that followed the first three lessons incorporated their insights and focused on the development of teaching strategies and the design of powerful teaching tasks. These tasks (shown in **Figures 2** and **3**) were important for deepening the pre-service teachers' understanding of the mathematical content (shown in **Figures 2** and **3**).

There were three notable findings of the analysis of the pre-service teachers' discussions during the pre-test. First, these discussions focused on the pupils' use of *correct terms and methods* (procedures). This emphasis was clearly revealed in the discussion about the application of the wrong word for addition. Second, they focused on the pupils' use of reading and problem-solving's strategies (procedures), as revealed by their use of the term "reading strategies" but also through indirect allusions to highlight the need of *more information*. Third, they focused on the methods employed by the teacher (procedures), notably how the teacher asked questions and guided the reasoning. The results of the analysis revealed significant differences in what the teachers discerned from the pupils' expressed understanding in the post-test discussions.

In light of the modified designs of the fourth and fifth lessons (see extra supplemental material) the teacher educator hypothesized that the pre-service teachers should be offered to discern by design tasks of average value simultaneous (**Figures 2** and **3**) increasingly conceptual dimension to a greater extent, as indicated by excerpts from the fourth lesson shown in **Table 6**.

During lesson four, the pre-service teachers discussed the pupils' understanding of the mean value in the post-tests. They discerned pupils' learning that entailed procedural as well as conceptual understanding. Conceptual understanding was unveiled in comments about what they did not know and why and when information was left out. This finding is notable, as the pre-service teachers also found this task difficult to solve when information was left out during the intervention. One pre-service teacher made the following comment on the last task: "I do not know the numbers of boys" and this lead to no solution. Thus, the pre-service teachers expressed some of the difficulties that the pupils had expressed in the video- recorded discussions in the pre-test and the post-test.

After revising the teaching design of the fifth lesson, including the follow-up questions used (see extra supplemental material), some further changes between the pre-service teachers' discussions in pre- and post-tests were observed (**Table 7**). Specifically, discussions about the pupils' learning regarding the mean value now reflected a combination of procedural and conceptual understanding.

There were still comments about the calculation during the pre-test. The pre-service teachers related their own understanding to the pupils' understanding. They continued to discuss the formula in relation to procedures. However, there is a change regarding the pre-service students' expressions after the research lesson in G4. The teacher students in this group focus on pupils'



**Table 8.** Identified focus areas in pre-service teachers' expressions of pupils' learning: Procedural (P), procedural didactics (PD), conceptual (C), and conceptual didactics (CD)

Definitions of pupils' knowledge by pre-service teachers	RL 1 (n=10)		RL 2 (n=7)		RL 3 (n=7)		RL 4 (n=7)		RL 5 (n=11)	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
	7 m 2 s	7 m 56 s	5 m 41 s	5 m 6 s	7 m 16 s	7 m 22 s	3 m 54 s	4 m 27 s	6 m 23 s	3 m 51 s
Procedural knowledge	92 P 6 PD	56 P 13 PD	38 P 9 PD	23 P 19 PD	33 P 33 PD	17 P 11 PD	65P 39PD	9 P 7 PD	36 P 51 PD	3 P 9 PD
Conceptual knowledge	0 C 0 CD	0 C 0 CD	1 C 0 CD	2 C 0 CD	2 C 1 CD	18 C 1 CD	4 C 0 CD	11 C 16 CD	12 C 4 CD	10 C 19 CD

understanding of being able to represent the content of the assignment with the algorithm for mean. After the research lesson in G5, one pre-service student expresses the need for a balance with both mastering the algorithm but also distinguishing the algorithm as representation in the task. These changes could be connected to how the pre-service teachers talked about the different qualities of the knowledge expressed by the pupils. This shift can be interpreted as reflecting a deeper understanding of the mathematical content. The pre-service teachers talked about necessary abilities for doing the calculation associated with a procedural understanding, but they also referred to the need for a deeper understanding at a conceptual level.

**Differences in discernment before and after interventions.** The third and last research question was about differences, if any, in pre-service teachers' understanding of pupils' knowledge before and after the intervention. The analysis focused on four distinct categories that were used to differentiate the ways in which pre-service teachers discerned pupils' knowledge. These categories were derived from a theoretical framework used to investigate procedural and conceptual understanding. The P category focused on how pupils solved task general, and the C category applied to their knowledge of the mathematical content in relation to the task and pupils in the video.

Excerpt relating to procedural knowledge:

You are unclear, not just show how you think. I think many times as we have grown up to show how you think, you do it with numbers. Three different calculations.

Excerpt relating to conceptual knowledge:

Then I thought there was a pupil who understood why she should divide.

These categories were divided into two sub-groups to incorporate a didactic perspective on how and whether expressions associated with the categories relate to teaching methods (PD, CD). PD focuses on how to teach in a way that fosters a procedural understanding and CD focuses on how to teach in a way that fosters a conceptual understanding.

Excerpt relating to PD:

One should be able to practice more [to understand] what it means [and] to be able to develop communication and conceptual abilities.

Excerpt relating to CD:

I thought for a while about what it felt like when...their reasoning. It felt like they realized what each speech represents, that is, the number of pupils ... they got a more relational understanding then.

**Table 8** presents a summary of the differences between pre-service teachers' expressions relating to their descriptive statements on procedural and conceptual understanding in their pre- and post-tests. The results were qualitatively analyzed in relation to the design of the lessons to develop a deeper understanding of what impacts may have inspired the changes in the pre-service teachers' expressions.

Although there were still comments in the pre-test about the calculation, a clear shift from procedural to conceptual discernment occurred between the third and fifth lessons, with the latter being prominent in the fourth and fifth lessons. The pre-service teachers discussed their own understanding and the pupils' understanding and explored the design of instructional methods for enhancing pupils' knowledge. They also continued to discuss the formula in relation to procedures.

The final lesson (5) revealed changes in the way pre-service teachers talked about different qualities, which can be interpreted as reflecting a deeper understanding of the mathematical content that enabled a formative assessment. The pre-service teachers talked about abilities relating to a procedural understanding that were required for the calculation while also noting the need for a relational understanding. The types of averages that the pre-service teachers focused on were identified, revealing that the discussion primarily centered on mean values. The pre-service teachers had the opportunity to discern the pupils' reasoning about the mean form of averages through their observation of the sequence of video recordings used in the intervention.

As **Table 9** shows, this content the mean of averages was also prominent in the analysis of the content of the teachers' discussion. However, some pre-service teachers still talked about the median and the mode during the fourth and fifth lessons. This finding was analyzed from the perspective of the type of knowledge (conceptual or procedural) they were focusing on (**Table 8**). As shown in **Table 8**, during the two last lessons (fourth and fifth) the student teachers did focus more on C and especially on CD, and their implications for instruction. This finding does not indicate that the pre-service teachers' focus shifted in terms of the widening of knowledge in the different types of averages; rather, their focus shifted from procedural to conceptual knowledge. This intensified focus on what it takes to learn, rather than how pupils solve tasks matched the aim of the intervention.

**Table 9.** Word counts of expressions relating to different types of averages

Frequency of averages	RL 1 (n=10)		RL 2 (n=7)		RL 3 (n=7)		RL 4 (n=7)		RL 5 (n=11)	
	Pre	Post	Pre	Post	Pre	Post	Post	Pre	Post	Pre
	7 m 2 s	7 m 56 s	5 m 41 s	5 m 6 s	7 m 16 s	7 m 2 s	7 m 56 s	5 m 41 s	5 m 6 s	7 m 16 s
Mean	14	10	3	3	17	6	29	15	36	19
Median	0	0	0	0	0	0	4	0	1	0
Mode	0	0	0	0	0	0	5	0	6	0
Total amount of words	1546	1063	598	695	1660	1260	2922	1965	3080	2612
		+31%		+16%		-24%		-33%		-15%
Mean words/student	154	106	85	99	237	180	417	280	280	237

**Tables 8 and 9**, which depict the results of the analysis of word counts and types of knowledge, reveal a decline in the use of words for all types of averages in all of the lessons, with the exception of the third lesson. Very few words of this type were uttered during this lesson compared with recordings with the same time in the other groups. In the group that participated in lesson five, the same time resulted in 3,080 words, whereas the figure for the group in lesson three was only 695. There was an association between the decreased use of words for averages and the focus of the discussions. As the discussions became more focused on the content and how it was understood by the pupils, references to whether or not the answers on the mean value were correct decreased in the post-tests. The discussion on how pupils' learning could be enhanced through teaching instruction may have resulted in the increased use of words communicating didactical issues.

## DISCUSSION

The aim of this study was to explore pre-service teachers' procedural and conceptual understanding of pupils' understanding of mean value, before and after an intervention. To determine how they reasoned about the pupils' learning, a pre-test and a post-test that included pupils' individual and group explanations of how they have solved mathematical tasks, focusing on the mean value, were implemented. The pre-service teachers were offered to discern the confusion expressed by the pupils. However, they found it difficult to determine why the pupils were unable to solve the tasks and why they confused the different kinds of averages in their reasoning during the first three lessons. The offered discernment in the fourth and fifth research lessons subsequently changed through the incorporation of a task in which all of the averages were calculated using the same numbers.

The analysis of the pre-service teachers' expressions in the test situations revealed that they were unsure of how to use the concept of mean value. Even if they understood how to solve the tasks assigned to the pupils, they stated that they did not have a clear idea of what they were doing. The pre-service students' challenges presented during the three first lessons were to distinguish the pupils' difficulties through their reasoning about the algorithm solutions connected to mean value in the task. The pre-service students expressed procedures in general about the concept (addition terms) and reading strategies. The pre-service students did not distinguish the pupils' expressed learning during their analysis of the video-recorded material from pupils' task solving (c.f. Chen & Herbst, 2012), which can be due to the pre-service teachers' own lack of mathematical knowledge. Therefore, in research lessons four and five, hands-on tasks were constructed which offered the pre-service students to develop their own conceptual knowledge about the object of learning (c.f. Maker, 2014). Based on the assumptions of variation theory (Holmqvist, 2004; Marton, 2015), the tasks were constructed for explicit knowledge about the mode, median, and mean value simultaneity, to help the pre-service teachers to separate these modes from each other. In the fourth and fifth research lessons, the pre-service students expressed an explored understanding of the pupil's reasoning about their solutions about the mean value. This could be compared to Sullivan et al.'s (2013) arguments for representational tasks that enhance conceptual development and understanding of mathematics regarding the students. Furthermore, even if tasks are claimed to need to be about a context and contribute to statistical reasoning for developing conceptual knowledge to distinguish the algorithm in a context (Cobb & Moore, 1997; MacCullough, 2007), our results point out that the context might not be needed to be in the focus to make the students discern a concept. Instead, in this study, the pre-service students were needed to explore their understanding of the more precise mathematical knowledge of mean value decontextualized. This included both the algorithm for the mathematic arithmetic mean value and the arithmetic mean value as a mathematical point of balance (cf. MacCullough, 2007).

In their study of primary school teachers, Goudling, Rowland, and Barbers (2002) similarly found that the teachers had inadequate mathematical knowledge. Moreover, this has been found to be the case even for very elementary content (Hill, Rowland, & Ball, 2005). The results of this study confirm this finding, indicating that this factor undermined the pre-service teachers' skills for understanding and exploring what the pupils were doing. Their difficulties in clearly expressing what they saw also made it more difficult for them to share their thoughts with their colleagues within the group. Consequently, their focus was more directed toward the procedures and pupils' activities that they could describe. This finding accords with what we described as procedural knowledge in this study, and by giving the teacher students hands-on tasks to develop their own understanding, this also resulted in a conceptual focus when they studied the pupils' task-solving.

Shulman (1986) recommended the use of case studies, which were applied in the pre- and post-tests in this study. However, because case studies were not used in the lesson, this did not impact on the pre-service teachers' understanding. Instead, a mathematical task on averages was added in the fourth and fifth research lessons, which appeared to have influenced a shift among the pre-service teachers towards a focus on the pupils' conceptual understanding. The desing of the mathematical task that was grounded in variation theoretical assumptions (Marton, 2015), requiring the teachers to work on the three types of averages simultaneously, enabled them to discern what it takes to understand the mean value in contrast to other types of

averages. Related to the video where the five pupils deal with mean value, this resulted in a shift from reasoning about pupils learning in general, a procedural knowledge to a conceptual focus. The pre-service teachers' increased conceptual knowledge enhanced by the lesson design with hands-on tasks helped them to identify the pupils' informal and formal expressed understanding through analyzing the pupils' reasoning. As the tasks were constructed to offer enhanced conceptual knowledge based on rich information-relationship between algorithm, averages, and context (as promoted by c.f. Hiebert & Lefevre, 1986) the pre-service students could identify the pupils' conceptual understanding to a higher degree, in line with previous research (c.f. Champman, 2013).

Although this study was not aimed at identifying what the teacher educators had learned, it illuminates how the learning study model, with its iterative cycles, was applied in the intervention (Holmqvist, 2011), providing the teacher educators with guiding inputs to produce a more effective teaching design. After conducting three lessons, the team did not observe any differences in the pre-service teachers' skills during their reasoning. However, by practically applying the theoretical framework to construct a task for the pre-service teachers, they contributed to bridging the gap between theory and practice within teacher education. Even if the pre-service teachers understood the theoretical framework, transforming this knowledge into practice, which entailed working with children, did not appear to be possible until they began to think about how to solve tasks relating to averages. The design emphasized the act of contrasting the averages in relation to each other to enable the teachers to discern differences among them. The teachers' confusion encountered in the first three lessons decreased as they developed a clearer understanding of the content and used the correct words for the concepts that they were exploring. That resulted in an increased focus on how to teach to develop conceptual understanding. In other words, the student teachers gained insights into how to teach to develop specific CK beyond merely knowing what to do or what should be known, extending their skills to teaching that enables specific CK to be developed.

## LIMITATIONS

Because this study was conducted in one university offering teacher education, the number of participants was limited as the impact on what is offered in the pre-service students' courses is high. Another limitation of the study was that no control group was used to assess whether, and in what way, smaller differences of discernment of CK may have existed between the interventional groups and the controls. Moreover, the choice made to use only the mean value at the inception of the study, instead of averages, was found to be a limitation in the teaching design for the pre-service teachers to understand the development of the pupils' knowledge regarding the mean value. Their understanding of the mean value, in contrast to other averages, was found to enhance their learning. Finally, the theoretical perspective applied in the study may have had a greater impact over a longer duration. However, regular teaching was resumed after the intervention.

**Author contributions:** All authors have sufficiently contributed to the study, and agreed with the results and conclusions.

**Funding:** The project has no external funding.

**Declaration of interest:** The authors do not have any conflicts of interests.

**Ethical Declaration:** Ethical standards have followed the guidelines from the Swedish Research Council, as well as the principles of Committee on Publication Ethics (COPE). All participants have got information about the aim of the project, how data will be saved, analyzed and presented, as well as that participation is voluntary and that they can cancel their participation whenever they wish.. They have also signed an informed consent.

## REFERENCES

- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407. <https://doi.org/10.1177/0022487108324554>
- Brante, G., Holmqvist Olander, M., Holmqvist, P. O., & Palla, M. (2015). Theorising teaching and learning: pre-service teachers' theoretical awareness of learning. *European Journal of Teacher Education*, 38(1), 102-118. <https://doi.org/10.1080/02619768.2014.902437>
- Chapman, O. (2013). Investigating teachers' knowledge for teaching mathematics. *Journal of Mathematics Teacher Education*, 16(4), 237-243. <https://doi.org/10.1007/s10857-013-9247-2>
- Cobb, P., & Yackel, E. (2011). Chapter 4. Introduction. *A journey in mathematics education research—Insights from the work of Paul Cobb*, 33-40. [https://doi.org/10.1007/978-90-481-9729-3\\_4](https://doi.org/10.1007/978-90-481-9729-3_4)
- Goulding, M., Rowland, T., & Barber, P. (2002). Does it matter? Primary teacher trainees' subject knowledge in mathematics. *British Educational Research Journal*, 28(5), 689-704. <https://doi.org/10.1080/0141192022000015543a>
- Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale: Erlbaum.
- Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American educational research journal*, 42(2), 371-406. <https://doi.org/10.3102/00028312042002371>
- Holmqvist, M. (2004). *En främmande värld: om lärande och autism* [A foreign world: about learning and autism]. Studentlitteratur.
- Holmqvist, M. (2011). Teachers' learning in a learning study. *Instructional science*, 39(4), 497-511. <https://search.ebscohost.com/login.aspx?direct=true&db=edsj&AN=edsj.23882877&lang=sv&site=eds-live&scope=site>

- Holmqvist, M. (2017). Models for collaborative professional development for teachers in mathematics. *International Journal for Lesson and Learning Studies*, 6(3), 190-201. <https://doi.org/10.1108/IJLLS-12-2016-0051>
- Holmqvist, M. (Ed.). (2006). Att teoretisera lärandet. [To theorize learning]. In M. Holmqvist (Ed.), *Lärande i skolan: Learning study som skolutvecklingsmodell* [Learning in School: Learning Study as School Development Model] (pp. 9-24). Studentlitteratur.
- Holmqvist, M., & Selin, P. (2019). What makes the difference?: An empirical comparison of critical aspects identified in phenomenographic and variation theory analyses. *Palgrave Communications*, 5, 1-8. <https://doi.org/10.1057/s41599-019-0284-z>
- Kullberg, A., Runesson, U., Marton, F., Vikström, A., Nilsson, P., Mårtensson, P., & Häggström, J. (2016). Teaching one thing at a time or several things together?—Teachers changing their way of handling the object of learning by being engaged in a theory-based professional learning community in mathematics and science. *Teachers and teaching*, 22(6), 745-759. <https://doi.org/10.1080/13540602.2016.1158957>
- Lewis, C., & Tsuchida, I. (1997). Planned educational change in Japan: The case of elementary science instruction. *Journal of Education Policy*, 12(5), 313-331. <https://doi.org/10.1080/0268093970120502>
- Liljekvist, Y. (2014). *Lärande i matematik: Om resonemang och matematikuppgifters egenskaper*. [About reasoning and the characteristics of mathematical problems] (Doctoral dissertation, Karlstads universitet).
- Lo, L. M., Chik, P., & Pang, M. F. (2006). Patterns of variation in teaching the colour of light to Primary 3 students. *Instructional Science*, 34(1), 1-19. <https://doi.org/10.1007/s11251-005-3348-y>
- Lo, M. L. (2012). *Variation theory and the improvement of teaching and learning*. Acta Universitatis Gothoburgensis.
- Marton, F., & Booth, S. (1997). *Learning and awareness*. Routledge.
- Marton, F., & Lo, M. L. (2007). Learning from “The Learning Study”. *Tidskrift för Laerutbildning och Forskning [Journal of Research in Teacher Education]*, 2007(1), 31-44.
- Marton, F., Runesson, U., & Tsui, A. B. M. (2004). The space of learning. In F. Marton (Ed.), *Classroom discourse and the space of learning* (pp. 3-36). Lawrence Erlbaum Associates Publisher. <https://doi.org/10.4324/9781410609762>
- Pang, M. F., & Marton, F. (2003). Beyond “lesson study”: Comparing two ways of facilitating the grasp of some economic concepts. *Instructional Science*, 31(3), 175-194. <https://doi.org/10.1023/A:1023280619632>
- PRIM (2020). Examples of tasks for assessment in mathematics Grade 6. Stockholm University, Department of Mathematics and Didactics. <https://www.su.se/primgruppen/matematik/%C3%A5rskurs-6/exempel-ur-tidigare-prov>
- Radford, L. (2013). Three key concepts of the theory of objectification: Knowledge, knowing, and learning. *REDIMAT-Journal of Research in Mathematics Education*, 2(1), 7-44. <https://search.ebscohost.com/login.aspx?direct=true&db=eric&AN=EJ1112191&lang=sv&site=eds-live&scope=site>
- Rowland, T. (2014) The Knowledge Quartet: the genesis and application of a framework for analysing mathematics teaching and deepening teachers’ mathematics knowledge. *SISYPHUS Journal of Education*, 1(3), 15-43. <https://doi.org/10.25749/sis.3705>
- Rowland, T., Martyn, S., Barber, P., & Heal, C. (2000). Primary teacher trainees’ mathematics subject knowledge and classroom performance. *Research in Mathematics Education*, 2(1), 3-18. <https://doi.org/10.1080/14794800008520064>
- Shulman, L. (1986) Those who understand, knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14. <https://doi.org/10.3102/0013189X015002004>
- Skott, J., Mosvold, R., & Sakonidis, C. (2018): Classroom practice and teachers’ knowledge beliefs, and identity. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developing research in mathematics education. Twenty years of communication, cooperation, and collaboration in Europe* (pp. 162-180). Routledge. <https://doi.org/10.4324/9781315113562-13>
- Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for research in mathematics education*, 36(5), 404-411. <https://search.ebscohost.com/login.aspx?direct=true&db=afh&AN=18830665&lang=sv&site=eds-live&scope=site>
- Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world’s teachers for improving education in the classroom*. Free Press.
- Tutak, F. A., & Adams, T. L. (2017). A study of geometry content knowledge of elementary preservice teachers. *International Electronic Journal of Elementary Education*, 7(3), 301-318. <https://search.ebscohost.com/login.aspx?direct=true&db=edsdoj&AN=edsdoj.50746d225684271b5eb0766e964675e&lang=sv&site=eds-live&scope=site>