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Identification of Mathematics Prospective Teachers' Conceptual Understanding in Determining Solutions of Linear Equation Systems

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Abstract: This research is motivated by a linear equations system, which is the basis for studying necessary linear algebra materials, such as rank, range, linear independent/dependent, linear transformations, characteristic values and vectors. There are still prospective mathematics teachers who have difficulty solving linear equations system and understanding the form of row echelon and reduced row echelon forms. In this study, subjects were three prospective mathematics teachers from Swadaya Gunung Jati University Cirebon who were taking matrix algebra courses. This study aims to reveal the conceptual understanding of prospective mathematics teachers in determining the solution to systems of linear equations. The results show that there are still prospective mathematics teachers who only use memory about the properties and procedures in determining whether a matrix is said to be a row echelon form or a reduced row echelon form. Then, there is still weakness in building the algorithms' relationship due to the immature knowledge of the concepts. Researchers found that many prospective mathematics teachers were more comfortable solving problems that were performed procedurally. Further research is needed to determine how the mental construction process and mathematical conceptual knowledge of prospective mathematics teachers are through meaningful learning so that conceptual understanding is maximized.

Keywords: *Conceptual understanding, conceptual knowledge, elementary row operations, linear equations systems.*

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Introduction

Mathematics is a branch of science that is often used to find solutions to everyday problems. It can be said that mathematics is a human activity (Frueenthal, 1991) that is understood through social interaction so that knowledge can be obtained. In studying mathematics, prospective mathematics teachers must understand basic concepts that include characteristics, rules, principles (Ziegler et al., 2017), and use the correct procedures to solve problems (Nathan, 2012). According to Suarsana et al. (2017), students' understanding of learning materials can develop optimally with a good understanding of concepts. Van de Walle et al. (2010) state that a curriculum is designed to deepen conceptual understanding by making meaningful relationships. In addition, the academic ability of teachers is also an essential factor (Dudley, 2013; Leong, 2013) in helping students to be able to make connections between new knowledge and the knowledge they already have (Anderson et al., 2001; Maya & Sumarmo, 2011; Rittle-Johnson, 2006; Rosita et al., 2019). The new knowledge acquired is integrated with cognitive schemes and frameworks to build a concept. This concept will form conceptual knowledge that comes from context-based examples, and then generalized into abstracts (Gilmore & Bryant, 2008) to provide a basis for understanding (Andamon & Tan, 2018) to develop into conceptual understanding. Conceptual knowledge plays an essential role in understanding mathematics concepts so that prospective mathematics teachers can explain why and how to solve a problem. Conceptual knowledge can be demonstrated by explaining definitions and procedures that involve understanding meaning, not only remembering definitions and procedures (Zuya et al., 2017). The ability to interpret by explaining the reasons for an answer, according to Rittle-Johnson and Schneider (2012), is evidence of conceptual understanding. Therefore, conceptual understanding can also be conceptual knowledge (Anderson, 2000; Rittle-Johnson et al., 2001; Crooks & Alibali, 2014) or relational understanding (Skemp, 1986).

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Prospective mathematics teachers with a conceptual understanding can see the relationship between various representations and mathematical models (Hiebert, 1986; Wong & Evans, 2007). Lack of conceptual understanding can hinder the successful transfer of newly acquired knowledge. Khashan (2014) describes that prospective mathematics teachers who are proficient in mathematics are prospective mathematics teachers who can develop conceptual knowledge, factual knowledge, and procedural knowledge. Teachers must build procedural knowledge from prospective mathematics teachers' conceptual knowledge by using their reasoning to solve problems (Zuya et al., 2017). Solving problems can be done by making a connection between visual representations and procedures that are built through conceptual understanding. Conceptually learning mathematics can enable prospective mathematics teachers to see the relationship between facts and principles, rather than just by memorizing algorithms. However, learning in Indonesia mostly only focuses on using formulas (Zakaria et al., 2010), so that the learning process becomes more difficult. Without an understanding of the basic concepts of mathematics learned at an early stage, it will result in prospective mathematics teachers facing difficulties at a later stage.

The understanding of concepts, according to Laswadi et al. (2016), is 1) able to connect mathematical concepts; 2) able to represent mathematical situations in various ways; 3) knowing which representation is most suitable for a particular situation. Kilpatrick (2001) define conceptual understanding as coherent knowledge, making it possible to learn new ideas by linking those ideas to what is already known. Gilmore and Bryant (2008) identified conceptual learning as a process where students learn how to organize information into logical mental structures, thereby enhancing conceptual understanding by strengthening thinking processes. Suh (2007) explains conceptual understanding indicators as 1) understanding the problem or task; 2) establishing a connection to a similar problem; and 3) using a model and multiple representations. Soebagyo (2017) states conceptual understanding indicators are reflected by 1) connecting the problem with other mathematical ideas; 2) representing the problem verbally, by using pictures, symbols, and also concretely; 3) solving the problem with a proper representation. This study adopts several conceptual understanding indicators to match the understanding of the concept in determining the solution to systems of linear equations. The indicators used include 1) Understanding the problem or task; 2) Ability to use, utilize, and select specific procedures or operations; 3) Ability to apply problem-solving concepts or algorithms.

The solution to linear equations is the basis for studying some necessary materials of linear algebra, such as rank, range, independent/dependent linear, linear transformations, characteristic values, and vectors. Through prospective mathematics teachers' understanding of concepts in determining system solutions of linear equations, the material in linear algebra will be easy to understand. Solving a linear equations system requires prospective mathematics teachers' ability to find connections to build relationships between mathematical ideas and algorithms. Determining the solution of a linear equations system can be solved using a matrix. Therefore, it is easy to make an analysis that includes the relationship between variables. The form of linear equations $AX = B$, for a matrix A , vectors X and B in linear algebra, can be solved by several methods such as Gauss elimination, Gauss-Jordan elimination, Gauss-Seidel elimination, and Cramer's rule. The resulting matrix can distinguish Gauss elimination and Gauss-Jordan elimination. Gauss elimination produces a matrix in row echelon form, but Gauss-Jordan elimination produces a matrix in reduced row-echelon form. Solving a system of linear equations using Gaussian elimination is a structured and efficient method and requires an understanding of the concept at every step of the solution. Before determining the solution of a system of linear equations, prospective mathematics teachers should know that systems of linear equations have some forms of solution (Borja-Tecuatl et al., 2013). The forms of solving can have exactly one solution, have no solution, and have many solutions. Prospective mathematics teachers must also understand that a linear equations system is consistent if it has one or more solutions and is said to be inconsistent if it has no solutions.

Many studies have discussed solutions to linear equations. For example, Lanya and Zayyadi (2017) explained that students who have had their understanding and high abilities, has also had good abilities in solving elementary row operation problems. This high ability, as referred to by Lanya and Zayyadi, is seen from the Cumulative Achievement Index (GPA). Ruswana (2019) analyzed students' mathematical understanding of the material in the elementary linear algebra course by looking at the percentage of students' test score achievement against the maximum score. The results were 6 students scored below 50, 5 students scored between 51 and 60, 9 students scored between 61 and 70, 5 students scored between 71 and 80, 6 students scored between 81 and 90, and 2 students scored 91 and above. Based on the results of research by Suhady et al. (2019), students are more dominant in making conceptual errors, so they need to improve their mathematical abstraction and representation skills. Maharani (2020) examined this issue by comparing the level of understanding seen from the number of students choosing to solve systems of linear equations using the Gauss elimination method and the Cramer method. The result is that students prefer the Cramer method even though it is less effective because it requires longer steps. In addition, Oktaç's (2018) research results conclude that there are difficulties faced by students at various levels of schools and universities in building the concept of a linear equation system, through synthetic and analytic geometric contexts in two and three-dimensional spaces.

So far, research on conceptual understanding in systems of linear equations has focused more on achieving scores and the level of conceptual understanding abilities. In this article, the researcher focuses more on analyzing qualitatively how students' conceptual understanding is in determining linear equation systems with elementary line operations that have not widely discussed in existing research. This study provides information that conceptual understanding in

studying linear equations system is needed, not only procedural knowledge, which is reflected in the results of the correct answers and following what the teacher has exemplified. This research is preliminary to find out how creative thinking processes can be constructing through conceptual understanding.

Methodology

Research Goal

This study aims to reveal the conceptual understanding of prospective mathematics teachers in determining solutions to systems of linear equations. To achieve this goal, prospective mathematics teachers are given linear equation system problems that require an understanding of the concept of elementary row operations.

Subject and Data Collection

This study's subjects were three prospective mathematics teachers from Swadaya Gunung Jati University Cirebon, who took a matrix algebra course.

Four problems were given to reveal the conceptual understanding of prospective mathematics teachers in determining solutions to linear equations using elementary row operations and determining research subjects.

1. Which of the following matrices is the reduced row echelon form and the row echelon form, give an explanation.

a) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, Explanation: ...

b) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$, Explanation: ...

2. Determine the set of solutions from $-3x_1 + 4x_2 - 7x_3 + 8x_4 = 5$

$$4x_1 - 8x_2 = 12$$

3. Change the system $\begin{matrix} 3x_1 - 6x_2 = 9 \\ -2x_1 + 4x_2 = -6 \end{matrix}$ into $A X = B$

4. Solve the system of equations in number 3.

Figure 1. The Problems to identify conceptual understanding (to determine research subject)

The problems given in Figure 1 are following the indicators of conceptual understanding that have been adopted by the researchers, including 1) Understanding the problem or task; 2) Ability to use, utilize, and select specific procedures or operations; and 3) Ability to apply problem-solving concepts or algorithms.

Analyzing of Data

This research method uses a qualitative approach. Miles et al. (2014) explain that qualitative data analysis activities carry out interactively to follow the research needs. After the data is collected, it is continued by analyzing data adopting the analysis conducted by Abdalla et al. (2018), including data reduction, data presentation, data interpretation, and concluding. The primary focus of this research is the subject, namely three prospective mathematics teachers. Researchers analyzed mathematical abilities according to the theory of three levels of cognitive skills (Lambert & Lambert, 2012). Three prospective mathematics teachers represent cognitive ability with criteria of high, medium, and low levels. The research instrument is in the form of a test sheet that contains conceptual understanding indicators 1) Understanding the problem or task; 2) Ability to use, utilize, and select specific procedures or operations; 3) Ability to apply problem-solving concepts or algorithms and unstructured interviews.

Researchers provide a test with questions that contain indicators of conceptual understanding. The written test results document is then reduced by examining the conceptual understanding and analyzing how the conceptual understanding process and the difficulties faced by three prospective mathematics teachers. Triangulation techniques that guarantee the data's validity were also carrying out through unstructured interviews with three prospective mathematics teachers. Interviews were conducted based on test results to confirm and determine more details about the subject's answers. The data presentation stage is carrying out to represent each conceptual understanding process, after which the researcher concludes by giving descriptive meaning to the results of data analysis.

Findings/Results

In this study, data collection began with giving a test to understand prospective mathematics teachers' concepts in determining solutions to systems of linear equations using elementary row operations. Based on the test results given to three prospective mathematics teachers, researchers had answers that needed to analyze in-depth. The three prospective mathematics teachers received learning materials on linear equations system and Gauss elimination. In analyzing the results, the researchers coded these prospective mathematics teachers as M-1, M-2, and M-3. The following sections are the results and the discussion.

In the first problem, prospective mathematics teachers are asked to explain that the given matrix is either a reduced row echelon form or row echelon form.

Subject M-1

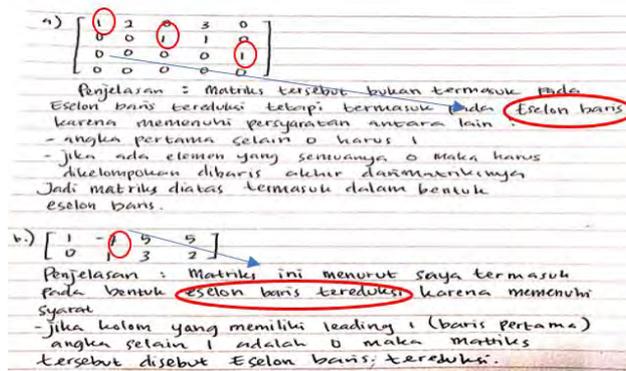


Figure 2. Written result of Subject M-1 on problem 1

In Figure 2, M-1 answers that matrix a) is not included in the reduced row echelon but belongs to the row echelon. It meets the requirements; namely, the first number other than 0 must be 1; if all 0, they must be grouping in the matrix's last row. Matrix b), according to M-1, is a reduced row echelon form because it meets the conditions if a column with one principal (first row) numbers other than 1 is 0.

The written result of M-1 in Figure 2 reveals that the row echelon form's conceptual knowledge had not been understood, so the concepts explained were switched when asked to determine whether the given matrix is a row echelon form or a reduced row echelon form. M-1 explains that the matrix a) is a row echelon form because M-1 only saw one main in each row, and rows with zero elements were placed in the last row. For the explanation on the matrix b), M-1 saw only the main one in row 1, where the column containing the main one has zero elements elsewhere. M-1 did not see any main one in row 2. Based on the interview results, M-1 claimed to know the row echelon form and the reduced row echelon form by only looking at the form of the given examples. M-1 did not understand whether matrix's properties have a row echelon form or a reduced row echelon form.

Subject M-2

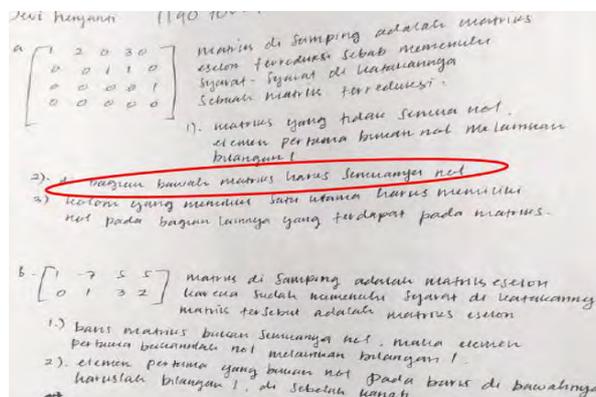


Figure 3. Written result of Subject M-2 on problem 1

In Figure 3, M-2 answers that matrix a) is a reduced echelon matrix because it fulfils the conditions, namely that the first element is not 0 but 1, at the bottom of the matrix, everything must be 0. The column that has one main must have 0 in the other part of the matrix. The b) matrix is a row echelon matrix because the row does not consist of all 0s but the first element of 1. In addition, the first nonzero element in the row below it getting to the right must be the number 1.

Subject M-2

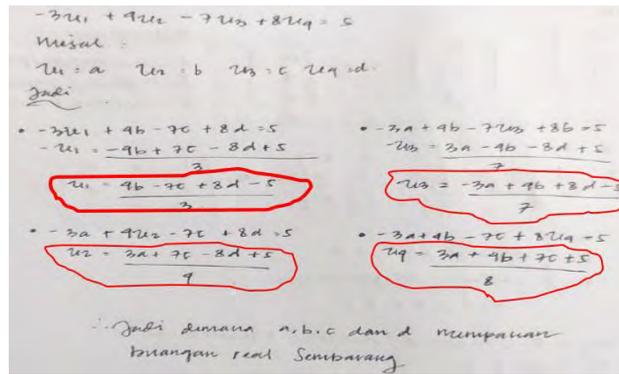


Figure 6. Written result of Subject M-2 on problem 2

In Figure 6, M-2 answers by assuming $x_1 = a, x_2 = b, x_3 = c,$ and $x_4 = d$ where a, b, c, and d are real numbers. The next step M-2 determines x_1 by substituting $x_2 = b, x_3 = c,$ and $x_4 = d$ obtained $x_1 = \frac{4b - 7c + 8d - 5}{3}$. Determining x_2 by substituting $x_1 = a, x_3 = c,$ and $x_4 = d$ obtained $x_2 = \frac{3a + 7c - 8d + 5}{4}$, determining x_3 by substituting $x_1 = a, x_2 = b,$ and $x_4 = d$ get $x_3 = \frac{-3a + 4b + 8d - 5}{7}$, and determine x_4 by substituting $x_1 = a, x_2 = b,$ and $x_3 = c$ get $x_4 = \frac{3a + 4b + 7c + 5}{8}$.

The written result of M-2 in Figure 6 reveals that M-2 subject has understood the problem to be solved, i.e. looking for the values of the variables $x_1, x_2, x_3,$ and x_4 . M-2 directly determined the solution of variables $x_1, x_2, x_3,$ and x_4 in the form of any parameter a, b, c and d, then substituted it into the equation $-3x_1 + 4x_2 - 7x_3 + 8x_4 = 5$ so that the values of $x_1, x_2, x_3,$ and x_4 were obtained in the form of an equation that has variables a, b, c, and d. It can be seen that M-2 only changed the equation variables but has not yet determined the solution of the given equation. M-2 at the interview explained that it was necessary to take each variable with certain parameters in solving a linear equation with more than one variable. When the researcher asked M-2 to substitute $x_1 = 1, x_2 = 2, x_3 = 3,$ dan $x_4 = 4$, it turned out that these values did not fulfill the four equations formed by M-2.

Subject M-3

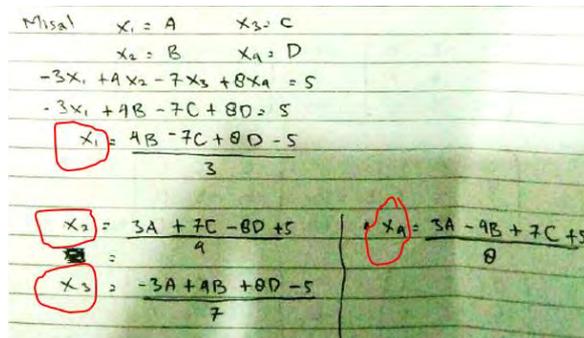


Figure 7. Written result of Subject M-3 on problem 2

Figure 7, M-3 lets $x_1 = A, x_2 = B, x_3 = C,$ and $x_4 = D$ where A, B, C, and D are any real numbers. The next step M-3 determines x_1 by substituting $x_2 = B, x_3 = C,$ and $x_4 = D$ obtained $x_1 = \frac{4B - 7C + 8D - 5}{3}$. Determining x_2 by substituting $x_1 = A, x_3 = C,$ and $x_4 = D$ is obtained $x_2 = \frac{3A + 7C - 8D + 5}{4}$, determining x_3 by substituting $x_1 = A, x_2 = B,$ and $x_4 = D$ get $x_3 = \frac{-3A + 4B + 8D - 5}{7}$, and determine x_4 by substituting $x_1 = A, x_2 = B,$ and $x_3 = C$ get $x_4 = \frac{3A + 4B + 7C + 5}{8}$. M-3 explains that the second problem has more than one solution.

The written result of M-3 in Figure 7 reveals that M-3 has the same concept of understanding as M-2. The following are the steps taken by M-3 determine the solution of the variables $x_1, x_2, x_3,$ and x_4 in the form of any parameter A, B, C, and D, then substitute it into the equation $-3x_1 + 4x_2 - 7x_3 + 8x_4 = 5$ so that the values of $x_1, x_2, x_3,$ and x_4 were obtained in the form of an equation that has variables A, B, C, and D. M-3 had the same solution as M-2. However, it turned out that M-3 also experienced confusion when being interviewed and was asked to substitute a variable with a specific value. M-3 was confused because the substituted value for the formed equation did not produce a solution.

In the third problem, subject are asked to change the system $4x_1 - 8x_2 = 12$
 $3x_1 - 6x_2 = 9$ into $AX = B$.
 $-2x_1 + 4x_2 = -6$

Subject M-1

3. Ubahlah sistem $4x_1 - 8x_2 = 12$ kedalam bentuk $AX = B$
 $3x_1 - 6x_2 = 9$
 $-2x_1 + 4x_2 = -6$

Jawab

$$\begin{bmatrix} 4 & -8 \\ 3 & -6 \\ -2 & 4 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ -6 \end{bmatrix}$$

A x B
 Matriks koefisien : A : $\begin{bmatrix} 4 & -8 \\ 3 & -6 \\ -2 & 4 \end{bmatrix}$
 Matriks Variable : X : $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
 Matriks konstanta : B : $\begin{bmatrix} 12 \\ 9 \\ -6 \end{bmatrix}$

Figure 8. Written result of Subject M-1 on problem 3

Figure 8 show M-1 answered by determining the coefficient matrix $A = \begin{bmatrix} 4 & -8 \\ 3 & -6 \\ -2 & 4 \end{bmatrix}$, the variable matrix $X = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$, and the constant matrix $B = \begin{bmatrix} 12 \\ 9 \\ -6 \end{bmatrix}$.

The written result of M-1 in Figure 8 reveals that the M-1 has understood the problem to be solved, i.e. by changing the system of linear equations to the form $AX = B$. M-1 was able to determine the coefficient matrix, variable matrix, and constant matrix even though there was a writing error in the variable matrix, i.e. $\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$ which should be $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Subject M-2

$AX = B$

$$\begin{bmatrix} 4 & -8 \\ 3 & -6 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ -6 \end{bmatrix}$$

A x B

Keterangan:
 A : matriks koefisien
 X : matriks variabel
 B : matriks konstanta

Figure 9. Written result of Subject M-2 on problem 3

In Figure 9, M-2 solves by determining the coefficient matrix $A = \begin{bmatrix} 4 & -8 \\ 3 & -6 \\ -2 & 4 \end{bmatrix}$, the variable matrix $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and the constant matrix $B = \begin{bmatrix} 12 \\ 9 \\ -6 \end{bmatrix}$.

M-2, in the written results shown in Figure 9, has correctly understood and written the system of linear equations in the form of $AX = B$.

Subject M-3

$AX = B$

$$A = \begin{bmatrix} 4 & -8 & 12 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -8 & 12 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix} = B$$

$AX = \begin{bmatrix} 4 & -8 & 12 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 4 & -8 & 12 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix}$

Jika matriks dikalikan identitas maka hasilnya matriks itu sendiri.

Figure 10. Written result of Subject M-3 on problem 3

In Figure 10, M-3 answers the coefficient matrix $A = \begin{bmatrix} 4 & -8 & 12 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix}$, the variable matrix $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the constant matrix $B = \begin{bmatrix} 4 & -8 & 12 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix}$. M-3 concludes that if a matrix is multiplying by the identity matrix, the result is the matrix itself.

The written result of M-3 in Figure 10 reveals that M-3 has understood the concept of $AX = B$ as the multiplication of matrix A in the form of an augmented matrix with an identity matrix X, and the result is matrix B. Supported by the result of the interview with M-3, it turns out M-3 concluded that $AX = B$ is the multiplication of matrix A with the X identity matrix, where the result is matrix B. M-3 understands that if there is a system of linear equation, the system is converted into an augment matrix. However, M-3 did not understand about the coefficient matrix obtained from the variable coefficients and the matrix constants obtained from the equation system constants. This is because the matrix formed has an order of 3×3 . M3 thought the matrix is multiplied by I_3 to obtain matrix B.

In the fourth problem, subject were asked to solve the system of equations in number 3.

Subject M-1

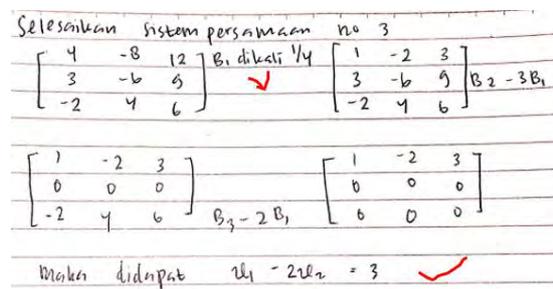


Figure 11. Written result of Subject M-1 on problem 4

M-1 solved a system of linear equations using the elementary row operation procedure shown in Figure 11. In the first row of the matrix $\begin{bmatrix} 4 & -8 & 12 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix}$ to get one main, M-1 multiplies the first-row elements where $\frac{1}{4}$ becomes

$$\begin{bmatrix} 1 & -2 & 3 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix}.$$

Then the second-row element was reduced by three times from the first-row element to $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ -2 & 4 & 6 \end{bmatrix}$. After that the third-row elements were reduced twice the first-row elements to obtain a reduced echelon matrix form $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and converted to the equation $x_1 - 2x_2 = 3$.

Based on the written results of M-1, the form of the equation $x_1 - 2x_2 = 3$ has not been a solution because the solutions for the values of x_1 and x_2 have not been determined. Then an interview was conducted to find out whether the M-1 could solve the equations it got. M-1 provided an explanation for solving the equation $x_1 - 2x_2 = 3$, by assuming $x_1 = r$ so that the value $x_2 = \frac{r-3}{2}$ was obtained.

Subject M-2

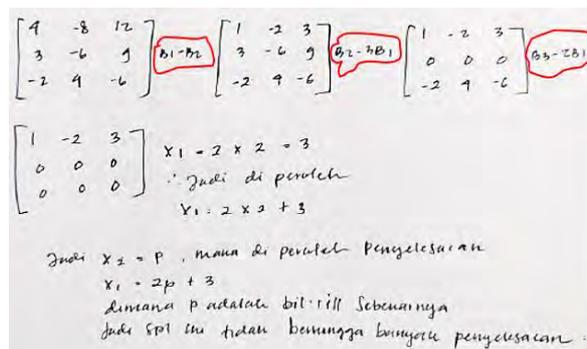


Figure 12. Written result of Subject M-2 on problem 4

Solving linear equations system using elementary row operating procedures was also carried out by M-2 as shown in Figure 12. To obtain the main one in the first row of the matrix $\begin{bmatrix} 4 & -8 & 12 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix}$ was done by reducing the first-row elements by the elements in the second row was obtained $\begin{bmatrix} 1 & -2 & 3 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix}$. Then the second-row element was reduced by three times the first-row element $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ -2 & 4 & 6 \end{bmatrix}$. The next step is two times the first-row element reduced the third-row element to become a matrix $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Obtained the equation $x_1 - 2x_2 = 3$ which corresponds to the matrix, then the M-2 replaced the variable x_2 with any value p , where p is a real number, then the value $x_1 = 2p + 3$ was obtained. M-2 concluded that the resulting linear equations system has many solutions.

Subject M-3

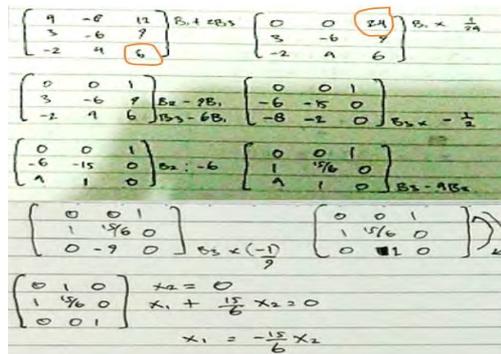


Figure 13. Written result of Subject M-3 on problem 4

M-3 adds up the first-row elements with two times the third-row elements in the matrix $\begin{bmatrix} 4 & -8 & 12 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix}$, obtained by $\begin{bmatrix} 0 & 0 & 24 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix}$ which was then the element the first-row is multiplied by $\frac{1}{24}$ to give $\begin{bmatrix} 0 & 0 & 1 \\ 3 & -6 & 9 \\ -2 & 4 & 6 \end{bmatrix}$. The next step is the second-row element is reducing by nine times the first-row element, and the third-row element is reducing by six times the first-row was obtained by the matrix $\begin{bmatrix} 0 & 0 & 1 \\ -6 & -15 & 0 \\ -8 & -2 & 0 \end{bmatrix}$. Followed by multiplying the third-row element by $(-\frac{1}{2})$ to be $\begin{bmatrix} 0 & 0 & 1 \\ -6 & -15 & 0 \\ 4 & 1 & 0 \end{bmatrix}$ then divided the second-row element by (-6) gets $\begin{bmatrix} 0 & 0 & 1 \\ 1 & \frac{15}{6} & 0 \\ 4 & 1 & 0 \end{bmatrix}$. The elements in the third row subtract four times the elements in the second-row are obtained $\begin{bmatrix} 0 & 0 & 1 \\ 1 & \frac{15}{6} & 0 \\ 0 & -9 & 0 \end{bmatrix}$. Then the third-row elements are multiplied by $(-\frac{1}{9})$ to become $\begin{bmatrix} 0 & 0 & 1 \\ 1 & \frac{15}{6} & 0 \\ 0 & 1 & 0 \end{bmatrix}$ then the first-row was replaced with the third-row obtained $\begin{bmatrix} 0 & 1 & 0 \\ 1 & \frac{15}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. From the matrix, M-3 wrote $x_2 = 0$ and $x_1 = -\frac{15}{6} x_2$.

Figure 13 shows that the solution to the linear equations system carried out by M-3 uses elementary row operating procedures. The first step performed with the operation $R_1 - 2R_3$ was correct, but the next step was wrong due to inaccuracy in changing the linear equations system to an augmented matrix. In addition, M-3 made a mistake in calculating algebraic operations. The interview results revealed that M-3 knew how to solve a system of linear equations using elementary row operations. However, according to M-3, it is challenging for M-3 to run fractional number operations, and M-3 is still confused about determining the initial step in determining the main one. This can also be seen in the written results done by the M-3 subject.

Discussion

There are many methods in solving systems of linear equations, one of which is elementary row operations. Elementary row operations can be performed by changing the matrix to an echelon row form or a reduced echelon row form. Therefore, for prospective mathematics teachers to be able to solve linear equations system with elementary row operations, it is necessary for them to know the procedure correctly (Zuya et al., 2017). The method for solving the system of linear equations can be seen in Figure 14.

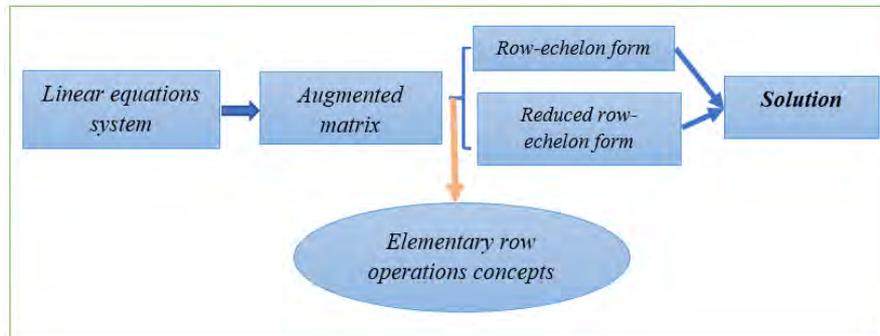


Figure 14. Method for solving the system of linear equations

The procedures carried out need to be strengthened by understanding the concept of elementary row operations to avoid prospective mathematics teachers' tendency to think practically rather than theoretically (Kazunga & Bansilal, 2018). Based on the results of tests conducted on subjects M-1, M-2, and M-3, the researchers then continue to examine the process for the test results deeper by conducting unstructured interviews and determine how each subject understands the concept of solving linear equations using elementary row operations.

In the first problem given in understanding the problem, all subjects already know what they have to do to determine whether the given matrix is a row echelon form or a reduced row echelon form. However, the M-1 subject has an inverse understanding of the concept of the row echelon form and the reduced row echelon form, and M-1 was just putting in what the subject remembers about the nature and procedures in solving problems. In addition, M1's interpretation and reasons for the answers were incorrect. It can be said that M-1 does not have conceptual knowledge. Following the statement of Britton and Henderson (2009) that solving procedural problems such as solving linear equations systems and manipulating matrices is not difficult, whereas a conceptual understanding is needed when faced with a definition (Zaslavsky & Shir, 2005). In line with Cummings (2015), a person's conceptual understanding can be measure through the ability to provide definitions, explanations, and reasons. This is different from the cases of M-2 and M-3, who have better conceptual knowledge to explain the reasons for their answers. The ability to see the relationship between the matrix elements with the properties of the matrix in the form of row echelon or reduced row echelon in different situation and procedures, indicates that the subject already has a conceptual knowledge that can then develop into conceptual understanding. Good conceptual understanding does not rule out good procedural skills (Rittle-Johnson & Schneider, 2012).

In solving the second problem, the subject must understand that each parametric equation system represents a family of functions (Oktaç, 2018). Subjects M-1, M-2, and M-3 have understood the problem given. M-1 can even use procedures and solve problems correctly so that a solution consisting of parameter values is obtained. In contrast with M-2 and M-3, they replaced the parameters a , b , c , and d with any real numbers, then they substitute the result entirely to the equation $-3x_1 + 4x_2 - 7x_3 + 8x_4 = 5$. Therefore, a , b , c , and d are not a set of solutions, because the values of a , b , c , and d only satisfy the equation $x_1 = \frac{4}{3}x_2 - \frac{7}{3}x_3 + \frac{8}{3}x_4 - \frac{5}{3}$. Mistakes in conceptual understanding can lead to incorrect procedures and calculations, as revealed in Hidayat and Iksan (2015) 's research, ultimately confusing because they think solutions are a collection of parameter values that make the system correct, Stadler (2011).

Before the system of linear equations is solved using elementary row operations, it is converted into a matrix consisting of a coefficient matrix, a variable matrix, and a constant matrix with the notation $AX = B$. Subjects M-1 and M-2 already understand the problems given and can use procedures and solve problems correctly. However, M-3, in representing the system of linear equations to a matrix form, was still inaccurate. This will have an effect when M-3 determines the solution of the system because X is seen as a solution and A as a set of coefficients (Larson & Zandieh, 2013). M-3's ability to find connections to build relationships between algorithms is still weak. This can happen because the initial knowledge possessed by M-3 is not yet strong. When the prospective mathematics teacher's experience of a material is not ready and then they were given new knowledge, the concept they built will not be as strong if the initial knowledge has been understood correctly—in line with the study of Anderson et al. (2001).

In determining the solution to a linear equations system using elementary row operations, it is necessary to understand the concept at each step of the solution. In line with Canobi (2004) opinion, conceptual understanding is close relate to

performing procedures appropriately and numeracy skills. Therefore, learning mathematics is essential by focusing on conceptual understanding without eliminating procedure-based learning (Al Mutawah et al., 2019). In the fourth problem, subjects M-1 and M-2 carried out each step by correctly applying the calculation procedure. The application of the correct calculation procedure shows that M-1 and M-2 already have good conceptual knowledge. M-1 and M-2 have found a connection from a linear equations system to form the matrix $AX = B$, which is then solved using row operations with the correct procedure to obtain a matrix in reduced row-echelon form. As seen from the works of M-1 and M-2, the form of the reduced row echelon is always unique (Bouhjar et al., 2018). Although M-1 and M-2 used different basic operations in performing procedures for elementary row operations, they still produced the same reduced row echelon form matrix. The final result is the equation $x_1 - 2x_2 = 3$ with a solution consisting of parameter values. For M-3, in contrast, errors in operating the matrix in elementary row operations prevent M-3 from producing the correct answer. Even though the procedures are appropriate, this must be an important note to understand the concept. Gilmore et al. (2017) stated that an excellent conceptual understanding could identify and select problem-solving strategies that are not too dependent on procedures.

Conclusion

This study analyzes three prospective mathematics teachers' conceptual understanding by providing four problems of linear equation systems that were arranged based on indicators of conceptual understanding. An analysis was carried out on the results of these tests carried out by prospective mathematics teachers. Then, the results of the analysis were strengthened by the results of the unstructured interviews. From the results obtained, there are still subjects who only relies on their memory of the properties and procedures in determining whether a matrix is said to be a row echelon form or a reduced row echelon form. These prospective mathematics teachers are still weak in building relationships between algorithms due to immature conceptual knowledge. Researchers found that prospective mathematics teachers were more comfortable solving problems that were done procedurally. This is in line with the study of Hannah et al. (2016). When interviewed regarding the conceptual knowledge they had worked on, there were still some who answered incorrectly. This indicates that prospective mathematics teachers better understand the material based on the examples given. Therefore, this will undoubtedly be a problem if they are faced with different mathematical questions. It is crucial to investigate further how the mental construction process and mathematical conceptual knowledge of prospective mathematics teachers through meaningful learning can maximize their conceptual understanding.

Recommendations

Based on the results obtained, to improve the quality of further research, the researcher suggests that further research be carried out for prospective teachers, teachers, lecturers, and researchers to examine the conceptual understanding process carried out experimentally in the classroom. In addition, it is necessary to research how conceptual understanding can influence the creative thinking process in mathematical problems.

Limitations

The limitation of this study is only to reveal the conceptual understanding of prospective mathematics teachers, based on the results of written tests sent via the Google Classroom application and the results of interviews via video calls. The conceptual understanding process has not been seen with direct observation in the classroom because the learning is still online. Another limitation is that this study only reveals conceptual understanding in determining the solution to systems of linear equations with elementary row operations. This study remains attractive with these limitations because the analysis results regarding the analyzed conceptual understanding are diverse and unique.

Authorship Contribution Statement

Dewi: Concept and design, data acquisition, analysis, writing. Zaenuri: critical revision of manuscript, final approval. Dwijanto: Critical revision of manuscript, editing/reviewing. Mulyono: Critical revision of manuscript, editing/reviewing

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