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Teaching Classroom Mathematics: Linking Two Pedagogical Models for Promoting Student Engagement and Conceptual Connections

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Abstract: This paper explains how an original conceptual framework model for mathematics pedagogy, the Australian Curriculum Conceptual Rubric (ACCR), has continued to be used successfully by the author in pre-service and in-service teacher education programs over the past ten years or more. Now further enhanced by a deeper reflection upon Peter Sullivan's Six Principles (2011) for the effective teaching of classroom mathematics, the ACCR is based on four preparatory "big questions" that the teachers may ask of themselves and their students. The model is also a sequenced system of conceptual "rubrics" whose aim is to encourage, in new teachers especially, a beginning sense of hierarchical mathematics concept building and connectedness. Using Sullivan's Principles for corroboration, the ACCR presents some useful ideas for helping teachers to keep track of the important elements of practical, effective teaching, and to use engaging and meaningful language in their classrooms.

Introduction

Mathematics as a discipline is usually agreed to be some kind of negotiated, organised *construct* of knowledge – and the exact nature of this construct has been a source of debate and fascination for a very long time. Yet whether the structure of mathematics is primarily envisaged as hierarchically fixed, or as organically evolving, or as something in between, less experienced mathematics teachers often have difficulty in accessing any kind of clear visual representation of a connected mathematics. This paper continues the author's interrogation of a more practical model for looking at mathematical *connections for teaching* that is consistent with the Australian Curriculum for mathematics (AC, ACARA, 2013).

The framework described in this paper, the Australian Curriculum Conceptual Rubric (ACCR), was first introduced five years ago in a paper that explored the usefulness of the mapping resources provided to some pre-service mathematics teachers (Ormond, 2016). Four "key" questions were developed into a suite of tabular rubrics and maps, which consisted of visual organisers at both macro- and micro-levels, and were based upon four "conceptual areas". In the ACCR documents provided to students these four questions are consistently colour-coded thus in order to facilitate teachers' understanding of the associated framework:

- What meaningful **patterns** are there in this maths content?
- What sort of **fluency** do students need in this maths?
- What **relationships and connections** can be seen in this maths?
- What **practical relevance** does this maths have to students' everyday lives?

The model continues to offer teachers a practical supplementary tool for planning

effective teaching across the three AC strands, *Number and Algebra*, *Measurement and Geometry*, and *Statistics and Probability*. In the intervening years the author has also reflected more deeply on the fundamental ideas that underpin its rationale. In very recent times, the lens provided by the National Numeracy Learning Progression (AC, ACARA, 2020) has further illuminated the important conceptual trajectories that underlie the scope and sequence of the Curriculum, also supporting the notion of iterative conceptual growth so fundamental to the framework. Even more importantly, a pivotal template for the effective teaching of mathematics (Sullivan, 2011) has acted as a conduit in this reflective process.

This paper discusses recent attempts to link the design of the ACCR more openly to important *mathematical language* for the classroom. While the ACCR model was first presented using the *Number and Algebra* strand for illustration (Ormond, 2016), and while this strand is again used as the general exemplar for much of the discussion, this paper now examines the four themes in relation to each of the three AC strands. The ACCR model again offers a conceptual teaching framework that maps explicitly to relevant AC content descriptions, but it also now more clearly prompts some important questions for establishing more successful and interesting mathematics classrooms. These are such questions as: *How does the maths fit together and how as a teacher can I make sense of this? How are conceptual relationships built up in the maths? How can the maths help students to solve problems and to understand their world?* In summary, the ACCR conceptual framework resource aims to encourage in teachers two “dispositions” – namely, a willingness to seek out and use the relevant *scope and sequence* of the mathematics content; and a desire to develop a sense of the *connectedness and correspondences* in all mathematics across the three key strands.

In 2011, Peter Sullivan presented his six Principles for successful mathematics teaching and learning (Sullivan, 2011, pp. 24-30). These tenets summarise the contributors to effective practice in a mathematics classroom:

Principle 1: Articulating goals

Identify key ideas that underpin the concepts you are seeking to teach, communicate to students that these are the goals of the teaching, and explain to them how you hope they will learn.

Principle 2: Making connections

Build on what students know, mathematically and experientially, including creating, and connecting, students with stories that both contextualise and establish a rationale for the learning.

Principle 3: Fostering engagement

Engage students by utilising a variety of rich and challenging tasks that allow students time and opportunities to make decisions, and which use a variety of forms of representation.

Principle 4: Differentiating challenges

Interact with students while they engage in the experiences, encourage students to interact with each other, including asking and answering questions, and specifically plan to support students who need it and challenge those who are ready.

Principle 5: Structuring lessons

Adopt pedagogies that foster communication and both individual and group responsibilities, use students’ reports to the class as learning opportunities, with teacher summaries of key mathematical ideas.

Principle 6: Promoting fluency and transfer

Fluency is important, and it can be developed in two ways: by short everyday practice of mental processes; and by practice, reinforcement and prompting transfer of learnt skills.

Sullivan's synthesis of effective practice strongly aligns with the ACCR conceptual framework. The Principles have similar intentions, promoting as they do the importance in classrooms of recognising clear mathematical signposts and participating in collaborative sense-making. Reflection upon the Principles has indeed led to that added emphasis in more recent years upon the negotiated, directed *language* of the mathematics classroom that best leads to student comprehension; or, upon how teachers may articulate and demonstrate the conceptual connections in an engaged, exploratory manner that nevertheless leads to a consensus of understanding. The author also saw the practical advantage in linking Sullivan's ideas with the original "big questions" of the ACCR by distilling their meaning further. The final section of the paper demonstrates in more detail the "distillation process" used to enhance the model further¹.

These four deceptively simple questions of the ACCR may serve as a kind of mnemonic or practical checklist for teachers as they prepare classroom lessons and tasks. A more detailed description of the ACCR rubrics is now provided, and this is related to suggested ways to *represent and talk about* the mathematics in classrooms. The process in this extra thinking around the ACCR model can be summarised as:

- The conceptual areas have been re-considered or "audited", using a mapping exercise to Sullivan's six Principles (see Tabs. 2 and 3).
- The conceptual areas have in turn been unpacked into 12 linked components for the three strands of the Australian Curriculum (see Fig. 2).
- This group of visual organisers has been elaborated upon for the three separate AC strands themselves (see Fig. 4); and been developed further for three phases of schooling, denoted as Early Primary, Primary, and Secondary (see Fig. 8).

If used concurrently the rubrics may lead teachers to a more cohesive understanding of the scope and sequence *within* each strand, and *between* each strand – and may help them to develop effective questioning and discussion techniques.

Conceptions of Mathematics and the Australian Curriculum

Mathematics educators have for a long time interrogated how notions of mathematical structure impact the conceptual understanding of both teachers and their students. This is desirable because of the hierarchical building of ideas that is the nature of mathematics itself: we cannot learn some concepts until others have first been established. Whether one teaches in more traditional ways – as Davis (2012) describes it, with "a view of mathematical knowledge as manifesting in compressed and relatively static forms" – or in more fluid, investigative, student-centred ways, some sort of hierarchy needs to be acknowledged. More contemporary images of mathematical knowledge building use words such as "evolving", "emergent" and "systemic" (Davis, 2012; Lakoff & Núñez, 2000); and the recent mapping of conceptual trajectories seen in documents such as the National Numeracy Learning Progression (NNLP) (ACARA, 2020, Version 3) is now more iterative and self-evolving and far less linear (Siemon et al., 2019). This indeed can be seen in the partly or fully shaded segments of the NNLP conceptual framework

¹ A slightly shortened and paraphrased version of these is often used by the author for the sake of simplicity when teaching. The full, unabridged version of Sullivan's Principles, and the abridged version, can be seen together in Appendix C, Table 1.

circle diagram (see Appendix A). In the last few years “project based learning” in collaborative cross-disciplinary STEM (Science, Technology, Engineering and Mathematics) classrooms has elevated student inquiry into even more practical, practice-based approaches, which in turn require real expertise in mathematics teachers as they strive in more complex scenarios to draw out key mathematical ideas and formalise their students’ understanding (Edmunds et al., 2017).

These strategies seem far removed from adherence to Jean Piaget’s original step-like, sequential stages of learning (Piaget, 1952) – yet the history of mathematical knowledge building has always focused, and still does, on some sort of construction. Social constructivism itself has consistently relied upon “scaffolding” up through a hierarchy of “negotiated” learning (Bauersfeld, 1995; Vygotsky, 1962).

Mathematics is of course about more than its building blocks: it is just as much about the processes of joining these together. Warren (2003) defined “knowledge of mathematical structure” as “knowledge of mathematical objects and the *relationship* between the objects and the properties of those objects” (italics added). Commentators have long advised that mathematical learning is best carried out in a social context that encourages observation of the *connections and relationships* between mathematical ideas (Kilpatrick, 2010; Sfard, 1991, 2008; Skemp, 1986). Whether such understanding systemically evolves out of itself, or travels in a more predictably linear fashion, the process of establishing such links (operational or relational learning) must still precede (structural) understanding. Askew et al. (1997) reminded us of the importance of *connectionist* teaching and learning, and of classroom teachers finding a sort of balance between *transmission* and *discovery* approaches. Askew et al. found in their 1997 study that teachers who appeared to establish stronger mathematical connections for their students were also generally happier to employ strategies of open-ended problem solving and risk-taking. They found that “the challenge of an application”, even when students have not “learnt a skill in advance of being able to apply it” could in fact result in very meaningful learning. Anna Sfard also describes “operational” and “structural” ways of learning and understanding as interlinking with each other in a “backwards-forwards” fashion, gradually moving students towards a solid comprehension of an idea, a “new, and then permanent, awareness of that object” (Ormond, 2000).

Sfard named this final moment of realization as “reification” (1991), and later as “commognition” (2008); and argued that getting right that balance of student inquiry and conceptual accomplishment in a mathematics lesson is the most important pedagogical skill for teachers. As Siemon et al. (2019) explain, students’ deep mathematical understanding in a classroom is the result an amalgam of teaching strategies that are dependent on

“fine-grained research that is focused on individual student learning trajectories and intimate analyses of classroom pedagogical practices, as well as large-scale research that explores how student populations typically engage with the big ideas of mathematics over time” (p.1)

The ACCR model here discussed is likewise based upon the belief that the recognition of mathematical *ideas and relationships* provides the most lasting kind of reification. And just as helping students to reach such moments in the mathematical journey should be a chief aim of the connectionist teacher, much more *overt* information about mathematical connections needs to be offered to teachers, especially inexperienced ones. This paper argues that the “big ideas” of mathematics need to be broken into some clear conceptual markers that guide new teachers through the pedagogical waters.

The Australian Curriculum (AC) is currently presented in near-equivalent forms in Australian classrooms across the states, and offers teachers a hierarchical framework for teaching mathematics. The three “strands”, or content areas, are supported by clear and consistent “working mathematically” themes running across all strands and years: the four proficiency strands of *understanding, fluency, problem solving* and *reasoning*. The sequential presentation of

the AC, with the Achievement Standards, and the mathematics content for each year laid out with the strands placed side-by-side, is a necessary tool for classroom teachers. The Curriculum outlines a consistency of content inclusion and a multi-strand approach for a particular year group.

The AC does indeed help teachers to make some mathematical connections. Yet a basic hurdle is still faced by many practising teachers, the everyday challenge of translating it into engaging and meaningful mathematics lessons that also create conceptual links for their students. This is because the links between the mathematical ideas still need first to be *made by the teachers themselves*, before they can work successfully with their students. A national curriculum might offer an organized planner for mathematics teaching, but the AC is still often not quite enough for the fundamental, early support of some teachers. Thomas (2015) criticised the AC for its relative failure to “focus on the development of conceptual skills to enhance capacity to analyse, evaluate, and create”, something needed “to ensure that learning is contextualised” (p. 96). Mulligan et al. (2012) claimed that while the AC might offer “on the surface ... the opportunity for greater conceptual and connected knowledge and the development of teaching practices that focus on relational thinking”, it does relatively little to help new teachers to understand the nature of that “relational thinking” (p. 65). More than a curriculum content sequence, even populated with associated “proficiencies”, is needed for that.

Many new teachers, especially primary teachers, often also need assistance to acquire their own deeper understanding of mathematics (Hohensee, 2017; Tanisli & Kose, 2013), and sometimes even with their own actual “reification” of fundamental ideas – in other words, help with soundly constructing their own Pedagogical Content Knowledge (PCK) (Shulman, 1986). Teachers need continuing support in developing their professional expertise. In an analysis of the offerings of mathematics educators over the past fifty years, Schoenfeld stated that, “We have made great progress in understanding teacher thinking and learning, but there is much to be done in supporting teacher growth.” (Schoenfeld, 2016, p. 518). Brent Davis’s 2014 research analysed some teachers’ understanding of the mathematics they bring to students, and focused on “the tacit dimensions of teachers’ mathematics knowledge” and how these may surface and coalesce to form their idiosyncratic understandings. Davis (2012, 2015) talks of his observations of teachers’ “emergent mathematical knowledge” and defines their “realisations” – and those of their students – as the collection of “all manner of associations that a learner might draw on and connect in efforts to make sense of a mathematical construct grouping” (Davis, 2012, p. 3). Davis’s work has focused on “the complexity of teacher knowledge” and upon on “strategies to uncover it, analyse it, and re-synthesise it in ways that will make it more available for teaching” (Davis & Renert, 2014).

The ACCR also attempts to bring teachers’ half-forgotten understandings and knowledges to the surface. Such realisations as described by Davis may be promoted in the form of complementary verbal-symbolic and visual-spatial-kinaesthetic presentations of the concepts (Skemp, 1986), which a teacher can facilitate by covering the multiple representations that connect with both the left- and right-brain aspects of complex mathematical ideas. Visual concept mapping of such multi-representational strategies via the ACCR model has been used extensively in the author’s mathematics education teaching of ITE and, more recently, in-service teacher students; and its success as an approach was explored in research concerning its specific usefulness for presenting *Number and Algebra* concepts (Ormond, 2016). It was stated then:

Just as a sense of balance is needed in the strategic use of those well-researched classroom teaching orientations, the same is no doubt true for mathematics education programs for pre-service teachers. Teaching how to teach needs to be modelled at every point. As structured as the discipline of mathematics may be described to be, it has been argued in this paper that even more structure and

guidance is needed to support improved teacher PCK. ...The guidance needed should be one that attempts to enable or unearth teachers' sense of mathematical connections... Perhaps there is something about a minute analysis of the scoping and sequencing of a mathematics strand that can liberate novice teachers to experiment with inquiry processes – knowing that a sound understanding of some sort of master plan will mean that they can diverge a little without getting lost. (p. 28-29)

It needs to be said also that the author's notion of four "key pedagogical ideas" is something more (and less) than the consideration of the ontological meaning and structure of mathematical thought discussed so often in the literature. The four ACCR conceptual areas are in fact pedagogical prompts for practical classroom teaching (how to talk about the mathematics), with two of the areas concerned with the "big picture" nature of all mathematics – patterns and relationships – and two more grounded in procedures to establish numerical skills – fluency with facts and everyday use of mathematics. The teaching students in the earlier study "consistently demonstrated a marked improvement in both their pedagogical and their content understandings", and each "made at least one reference to such experiences" as ...

- a *sudden awareness* or realisation;
- a brand new *big picture overview* of the content and the pedagogy;
- a new sense of the *deep connections* in the mathematics; and/or
- a sense of *continuity* and *scope* in the mathematics. (p.17)

In more recent years since the study reported on in 2016, a higher proportion of students had been teaching mathematics, either in-field or out-of-field, for several years – yet similar student affirmations as to a new sense of "connectedness" in their PCK have been frequently offered. (Some of these can be seen in Appendix B.) Additionally, some new observations and reflection have now value-added to the author's teaching approach. It was felt that, at the same time as more clearly "seeing" the mathematics, many new teachers also need more practical access to a *language* about it, in which they ask important mathematical questions of themselves and their students, questions that relate to its very nature as a discipline. In doing this, they are also adhering to the first two of Sullivan's 2011 Principles, those of "articulating goals" and "making connections". This is clearly seen in his references to "communicating to students ... the key ideas that underpin the concepts, ... the goals of the teaching," and "explaining to them how [the teacher] hopes they will learn" (*Principle 1*); and "creating and connecting students with stories that both contextualise and establish a rationale for the learning" (*Principle 2*).

In short, as well as *finding the links in the mathematics*, many mathematics teachers need assistance with *developing ways to talk about them in clear language* to their students (O'Halloran, 2005). The ACCR has continued to provide one more strategy for ensuring a more comprehensive presentation and "articulation" of mathematical ideas in a classroom, encouraging teachers to define as clearly as possible the desirable range and coverage of these ideas. To use the colour coding again: each time they prepare to teach, they should be considering the **necessary fluencies** and **everyday uses** of the mathematics, and balancing these with an examination of its **meaningful patterns** and **deeper mathematical relationships**. In the next sections the various tiers of the ACCR tabular rubrics are explained, and, by providing some examples of their use in the author's teacher education classes, they are linked to this improved strategy for facilitating more effective classroom language.

The Tiers of the Australian Curriculum Conceptual Rubric (ACCR)

Two “dispositions” that the ACCR attempts to encourage in mathematics teachers were mentioned earlier –a willingness to develop an understanding both of the importance of locating and using *scope and sequence* content information; and of developing a sense of the *relationships* in mathematics within and across the three key strands. The ACCR was originally written to offer practical support for pre-service and early career mathematics teachers in relation to “big picture thinking”. This is especially important for primary classrooms, where many new teachers fall into the habit of presenting mathematics in isolated snapshots that do not connect together sensibly for their students. As explained, the four “big questions” of the ACCR were operationalised as four “conceptual areas” for all school mathematics teaching. These are seen on the left-hand-side in Figure 1, which associates the conceptual areas with the questions a teacher could be thinking about when preparing classwork, and also what he or she could pose as “good questions” (Sullivan & Clarke, 1991) in the classroom.

Figure 2 illustrates the 12 overarching, cross-strand components of the ACCR, across all years of schooling. David Clarke used the analogy of “grain-size” in teachers’ pedagogical decision-making about classroom assessment types (Clarke, 1997). In a useful analogy for assessment types he contrasted fine-grained micro-level “sand” with the macro-level of the “boulder”. If the actual Curriculum itself is seen as the sand, Figure 2 is essentially the boulder of the ACCR, as each of the 12 components of this rubric attempts to capture a highly complex idea with as few words as possible. This figure therefore explains the first “tier” of the model, showing how the conceptual areas can be seen to be *strand-specific*, while at the same time inviting more explicit unpacking of depth and detail.

**Australian Curriculum Conceptual Rubric (ACCR), Mathematics
Conceptual Areas for ALL MATHEMATICS, ALL YEARS (F-10)**

Conceptual areas of all strands ↓	Big questions for the teacher’s classroom preparation	Big questions for the students in the classroom
RECOGNISING PATTERNS in MATHS	<i>What meaningful patterns are there in this maths content?</i>	“What patterns can we see here in this maths?”
MATHS FACTS and FLUENCIES	<i>What sort of fluency do students need in this maths?</i>	“What maths do we need to really learn here, and be able to remember?”
EQUIVALENCE and RELATIONSHIPS in MATHS	<i>What relationships and connections can be seen in this maths?</i>	“How does this maths fit together ...? Say the same thing as ...?”
NUMERACY for EVERYDAY LIFE	<i>What practical relevance does this maths have to students’ everyday lives?</i>	“How do we use this maths in our lives?”

Figure 1. The conceptual areas of the ACCR, linked to the pedagogical questions.

Australian Curriculum Conceptual Rubric, ALL STRANDS, ALL YEARS (F-10)

Conceptual areas of all strands ↓	Number and Algebra	Measurement and Geometry	Statistics and Probability
RECOGNISING PATTERNS in MATHS	Number patterning and graphs	Patterns in space, shape and time	Patterns in chance and data
MATHS FACTS and FLUENCIES	Operations with number and algebra	Space, shape and time measurement facts	Chance processes and measuring data
EQUIVALENCE and RELATIONSHIPS in MATHS	Reasoning about equations, equality and inequality	Reasoning about space, shape and time	Reasoning about chance and data
NUMERACY for EVERYDAY LIFE	Practical numeracy (estimation, calculation and financial maths)	Practical numeracy (estimation, orientation, and measurement)	Practical numeracy (estimation of chance, collecting and interpreting data)

Figure 2. The first “tier” of the mapping of the four “conceptual areas” of mathematics teaching, across all three AC strands and all school ages.

It could also be noted here that Figure 2 appears to be conceptually consistent with the general NNP conceptual framework (ACARA, 2020, Version 3, p. 5). The NNP refers to student “outcomes” of understanding that a student “says, does, or produces” and their “developmental levels” are envisaged as partial or full segments of a circle, the “elements”. It is interesting that all of the *full* or *almost full* segments emphasise **Recognising Patterns** and **Equivalence and Relationships**, and *most* of them fit comfortably with **Facts and Fluencies** and **Numeracy for Everyday Life**. (See Appendix A for a more detailed comparison.)

As with any attempt to summarise a complex system in a few words, these first rubrics are of course deceptively simple, and each actually implies a much deeper meaning. Yet it is argued here that in their apparent simplicity lies their strength, in terms of clear visual organisers with which less experienced teachers may at least start. In the next section, the objective of helping teachers to “locate and use *scope and sequence* content information” will be considered more closely, further explaining the machinery of the ACCR resource.

The ACCR and Understanding Mathematical Scope and Sequence

It was explained earlier that the ACCR resource aims to encourage in teachers two habits of mind. The first of these, that willingness to seek out and use the relevant *scope and sequence* of the mathematics content, is now discussed and exemplified further.

The encouragement of these “dispositions” has been incorporated into the author’s mathematics education programs for pre-service and in-service teachers with some success, according to their feedback (Ormond, 2016). In asking the pre-service teachers and re-training teachers to step outside an often more limited view of mathematics and to look at it both holistically and structurally, many found a sudden metacognitive clarity as to what they were actually trying to do in a mathematics classroom.

Student 7 (2014): I hadn’t looked at the curriculum in this depth and compared each year in mathematics in such depth before. So that really did help. You have to read the curriculum and analyse it to reach that level of understanding. One

thing [the lecturer] talked about was how important it is to look at the year before and the year after and see where they have come from and where they are going. (Ormond, 2016, p. 16)

As stated, similar comments have been voluntarily offered by students in the years since 2016.

Student 2 (email November 2017), secondary mathematics teacher: ... I know that I will keep going back to the presentations, documents and all the additional materials ... provided. (Also see Appendix B.)

Student 3 (email October 2018), secondary mathematics teacher: Thank you so much for the course content. This unit has really put an understanding to what I have been doing in the classroom (Also see Appendix B.)

The two “visual strategies” for pedagogical understanding were defined in this earlier paper as “the vertical perspective (scoping and sequencing the Australian Curriculum up through the years)” and “the horizontal perspective (introducing the ... conceptual areas across content for connecting the mathematics)” (Ormond, 2016, p. 16). These perspectives are exemplified in unit resources, a PowerPoint teaching slide example of which can be seen in Figure 3.

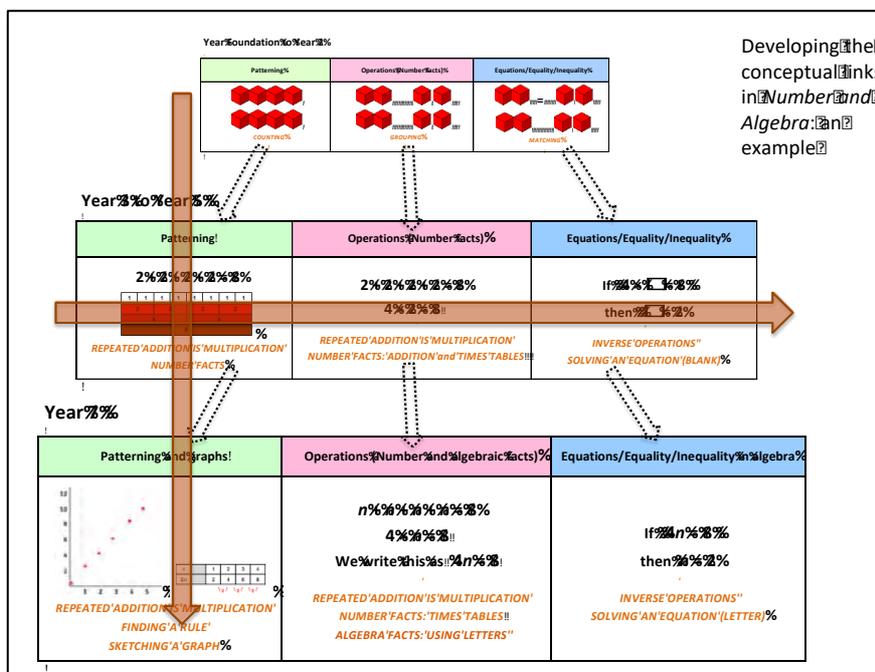


Figure 3. “An example of a PowerPoint slide from the introductory materials [for the unit]: showing the horizontal and vertical perspectives for thinking about the mathematics, to be used as tools for developing stronger pedagogical content knowledge” (Figure 4, Ormond, 2016).

The various rubrics offered in the ACCR may also be seen, as in Figure 3, in terms of “reading down” and “reading across”. To achieve a more holistic view of scope and sequence than presented by the AC, one may choose a strand and examine in its rubric the 12 conceptual framework components across three schooling phases. Figure 4, a more detailed, second-tier rubric than Figure 2 (second-tier in that it focuses only on one strand) provides an example using *Measurement and Geometry*. (The generic “all-phase” conceptual areas are seen highlighted grey in the last column, and can be compared with the first-tier Figure 2.)

The **MATHS FACTS and FLUENCIES** components are seen to move from fluencies around drawing, describing and manipulating (Years Foundation to 2); through to

an understanding of measurements of all attributes, and of simple geometric categories (Years 3 to 6); to the skills of carrying out indirect measurement, of transformations and of the generalisations of geometric properties in two- and three-dimensions (Years 7 to 10). The **NUMERACY for EVERYDAY LIFE** components involve estimation skills in all the attribute areas, and practical uses of measurement and scale. The **RECOGNISING PATTERNS** components again move students from early experimentation with the attributes through to more and more complex generalisations of properties (Years Foundation to 6); and then to geometric generalities based on algebra and geometric proof (Years 7 to 10). Pattern recognition can be used to develop and consolidate the fluencies and to highlight commonalities, differences and congruencies (Smith et al., 2007).

Australian Curriculum Conceptual Rubric, *MEASUREMENT and GEOMETRY*

“Conceptual areas” ↓	Years F-2 (Early Primary)	Years 3-6 (Primary)	Years 7-10 (Secondary)	YEARS F-10
RECOGNISING PATTERNS in MATHS	Patterns in space, shape, and time. (How large am I compared to other things? How do we measure length, area, capacity, volume, mass and time?)	Orientation in space; patterns in time measurement; categories, properties and transformations of 2D and simple 3D shapes	Orientation in space; categories, properties and transformations of 2D and 3D shapes and their composites; linking geometry with algebra (trigonometry and Pythagoras)	Patterns in space, shape and time
MATHS FACTS and FLUENCIES	Naming, drawing, describing and manipulating shapes; interpreting simple elapsed time and directions	Space and shapes measurement facts; telling the time and calculating elapsed time; reading scales, maps and clocks	Geometric and measurement properties; calculating direction; linking geometry with algebra (xy coordinate system)	Space, shape and time measurement facts
EQUIVALENCE and RELATIONSHIPS in MATHS	Shapes and time that are the same and not the same	Shapes, measurements and time that are the same and not the same	Congruence and similarity; circle and angle properties; geometric proof; linking geometry with algebra (trigonometry and Pythagoras)	Reasoning about space, shape and time
NUMERACY for EVERYDAY LIFE	Practical numeracy (Where am I in space? Estimation of length, area, capacity, volume, mass; drawing; telling simple time; naming months and seasons)	Practical numeracy (orientation, estimation and measurement of space and shape size, reading maps, the time and timetables)	Practical numeracy (orientation, geometric estimation and measurement, compass bearings; scheduling)	Practical numeracy (estimation, orientation, and measurement)

Figure 4. Mapping of the four “conceptual areas” of mathematics teaching, across all the *Measurement and Geometry* strand and all school ages: an example of the second tier of the ACCR rubrics.

The **EQUIVALENCE and RELATIONSHIPS** component is always the most predominant in terms of interrogating underlying mathematical connections, both within and between the strands. More will be said about this in later sections of the paper which examine that “sense of the *connectedness and correspondences*”. Yet as far as scoping and sequencing in this strand is concerned, the ever-present mathematical theme of “what is the same and

what is not the same” – invariance and variance, as described by Goos et al. (2017, p. 228) – can clearly be seen. The conceptual area focused on equality and inequality is seen to move from an evolving system of comparisons and categorisations of attributes (Years Foundation to 6); to such notions as congruence and similarity, or Pythagorean triad theory, or transformations on trigonometric functions (Years 7 to 10).

A final explanatory point to be made here concerns the source of these 12 “components” and these are of course the AC Content Descriptions themselves (supported by the associated proficiencies and Achievement Standards). The most detailed rubric offered in the ACCR suite is seen in Fig. 5, where for the strand in question a list of relevant content descriptions by code is provided for each conceptual area and student learning phase. With content codes included, this is a third-tier rubric. Again, brevity is a limitation of such a skeleton framework, and further detail is required by teachers for planning tasks and lessons – but this is amply provided in the AC itself, and the links to the Content Descriptions have been provided to start this process. If a teacher wishes to emphasise fluency or everyday numeracy, he or she can use this rubric to quickly identify some relevant AC descriptions. Interestingly, because of the complexity of the mathematics and the richness of the Descriptions themselves (intended by Peter Sullivan and the other ACARA mathematics writers to provide *breadth*, with implied but not always highly elaborated *depth*), some of these codes are seen repeatedly appearing in more than one of the conceptual area categories. (For an example of this see the third slide in Fig. 6, where codes are repeated in several conceptual areas in both the Year 3 and Year 4 lists.) If a Content Description code is repeated, it is hoped that the teacher will reflect on it through the lens of the different “big questions” so as to establish the focus of the lesson.

AUSTRALIAN CURRICULUM CONCEPTUAL RUBRIC, by STRAND			
MEASUREMENT and GEOMETRY			
Conceptual area	Years F-2 (EC after 4 years)	Years 3-6 (Primary)	Years 7-10 (Secondary)
RECOGNISING PATTERNS IN MATHEMATICS	Patterns in space, shape, and time. (How large am I compared to other things? How do we measure length, area, capacity, volume, mass and time?) ACMMG006 ACMMG037 ACMMG007 ACMMG043 ACMMG028 ACMMG019 ACMMG021	Orientation in space, patterns in time measurement, categories, properties and transformations of 2D and simple 3D shapes ACMMG061 ACMMG108 ACMMG063 ACMMG114 ACMMG065 ACMMG135 ACMMG066 ACMMG140 ACMMG091 ACMMG143	Orientation in space; categories, properties and transformations of 2D and 3D shapes and their compasses; linking geometry with algebra (trigonometry and Pythagoras) ACMMG159 ACMMG220 ACMMG160 ACMMG221 ACMMG161 ACMMG232 ACMMG164 ACMMG224 ACMMG165 ACMMG242 ACMMG181 ACMMG245 ACMMG198 ACMMG200
MAKING FACTS AND FLUENCIES	Naming, drawing, describing and manipulating shapes; interpreting simple elapsed time and directions ACMMG007 ACMMG042 ACMMG008 ACMMG043 ACMMG020 ACMMG044 ACMMG021 ACMMG045 ACMMG022 ACMMG046 ACMMG023	Space and shapes measurement facts; telling the time and calculating elapsed time; reading scales, maps and clocks ACMMG061 ACMMG111 ACMMG062 ACMMG112 ACMMG066 ACMMG113 ACMMG084 ACMMG114 ACMMG084 ACMMG114 ACMMG084 ACMMG114 ACMMG085 ACMMG135 ACMMG088 ACMMG136 ACMMG090 ACMMG138 ACMMG091 ACMMG140 ACMMG108 ACMMG141 ACMMG109 ACMMG142 ACMMG143	Geometric and measurement properties; calculating directions; linking geometry with algebra (x-y coordinate system) ACMMG159 ACMMG216 ACMMG160 ACMMG217 ACMMG161 ACMMG218 ACMMG163 ACMMG219 ACMMG222 ACMMG224 ACMMG197 ACMMG222 ACMMG198 ACMMG244 ACMMG202
IDENTIFYING AND RELATIONSHIPS IN MATHEMATICS	Shapes and time that are the same and not the same ACMMG006 ACMMG037 ACMMG007 ACMMG038 ACMMG009 ACMMG044 ACMMG019 ACMMG046 ACMMG022	Shapes, measurements and time that are the same and not the same ACMMG061 ACMMG111 ACMMG064 ACMMG112 ACMMG087 ACMMG114 ACMMG088 ACMMG115 ACMMG089 ACMMG141 ACMMG108 ACMMG142	Congruence and similarity; circle and angle properties; geometric proof; linking geometry with algebra (trigonometry and Pythagoras) ACMMG163 ACMMG220 ACMMG164 ACMMG221 ACMMG166 ACMMG222 ACMMG197 ACMMG243 ACMMG199 ACMMG222 ACMMG200 ACMMG244 ACMMG201 ACMMG202
NUMERACY FOR EVERYDAY LIFE	Practical numeracy (Where am I in space? Estimation of length, area, capacity, volume, mass; telling simple time; drawing; naming months and seasons) ACMMG009 ACMMG037 ACMMG010 ACMMG038 ACMMG020 ACMMG039 ACMMG022 ACMMG040 ACMMG041	Practical numeracy (orientation, estimation and measurement of space and shape size, reading maps, the time and timetables) ACMMG062 ACMMG108 ACMMG065 ACMMG109 ACMMG085 ACMMG110 ACMMG086 ACMMG112 ACMMG087 ACMMG113 ACMMG090 ACMMG136 ACMMG139	Practical numeracy (orientation, geometric estimation and measurement, compass bearings, scheduling) ACMMG159 ACMMG216 ACMMG161 ACMMG217 ACMMG195 ACMMG218 ACMMG219 ACMMG221

Figure 5. Mapping of the four “conceptual areas” of mathematics teaching, across all of the *Measurement and Geometry* strand and all school ages, including articulation to relevant content descriptions in the Australian Curriculum.

A more specific example of how these ideas have been used in a pre-service mathematics education classroom is offered here, this time in relation to primary level *Number and Algebra*. Seen below in the first two slide shots in Figure 6 are examples from a selection of PowerPoint slides which tasked the teaching students with planning a lesson based on the number fact $3 \times 7 = 21$. They were asked to think about content and teaching approaches for Years 3, 4, and 5, also considering the scope and sequence of the ideas. They were then required to organise their “brain-stormed” ideas into the four conceptual areas and to match them with appropriate AC content descriptions, using the AC website and the ACCR document together. The next part of the lecture, exemplified in the last two slide shots in Figure 6, involved listening to a description of how, with some deeper thought, the four conceptual areas could be mapped across Years 3 to 5 so as to gain a sense of the concept-building involved. The pre-service teachers could then use this fairly simplistic example to model and develop other lessons based around times tables or number facts. The four pedagogical questions were a constant theme in the explanation.

Brainstorm: $3 \times 7 = 21$

number of days in a week ... number of days in 3 weeks $3 \times \square = 21$

$3 \times 7 = 7 \times 3$ Commutative Law

Seven lots of a number is 21. What is the number?

$3, 6, 9, 12, 15, 18, 21, \dots$

$3 \times 7 = 21$

$0, 7, \dots, 21, \dots$

$0, 7, 14, 21, \dots$

If I divide 21 by a number and get 3, what did I divide by?

$7 \times 3 = 21$

$21 \div 3 = 7$

"Seven lots of 3 is 21 and this needs to be learned as $7 \times 3 = 21$."

interpreting a calendar "Every Tuesday in February"

$3 \times 7 = 7 \times 3$

$(3 \times 3) + (3 \times 4) = 3 \times (3 + 4) = 21$ Distributive Law

Rearranging: the four "aspects" of $3 \times 7 = 21$

Patterning

$3, 6, 9, 12, 15, 18, 21, \dots$

$0, 7, 14, 21, \dots$

Relationships and connections

"I start with 0 and add 3 each time and I see a pattern where each number goes up by 3 each time."

"I start with 21 and subtract 3 each time and I see a pattern where each number goes down by 3 each time."

If I divide 21 by a number and get 3, what did I divide by?

Seven lots of a number is 21. What is the number?

$3 \times \square = 21$

$3 \times \square = 7 \times \triangle = 21$

What are the values of the two blanks, \square and \triangle ?

Operations

$3 \times 7 = 21$

$21 \div 3 = 7$ $7 \times 3 = 21$

$3 \times 7 = 7 \times 3$ Commutative Law

$(3 \times 3) + (3 \times 4) = 3 \times (3 + 4) = 21$ Distributive Law

"Seven lots of 3 is 21 and this needs to be learned as $7 \times 3 = 21$."

Everyday numeracy

number of days in a week ... number of days in 3 weeks

"Every Tuesday in February"

interpreting a calendar

Conceptual areas

Conceptual area	Year 3	Year 4
RECOGNISING PATTERNS IN MATHS	ACMNA051, ACMNA052, ACMNA053, ACMNA054, ACMNA055, ACMNA056, ACMNA057, ACMNA058, ACMNA059	ACMNA071, ACMNA072, ACMNA073, ACMNA074, ACMNA075, ACMNA076, ACMNA077, ACMNA078, ACMNA079, ACMNA080
MATHS FACTS AND FLUENCIES	ACMNA051, ACMNA052, ACMNA053, ACMNA054, ACMNA055, ACMNA056, ACMNA057, ACMNA058, ACMNA059	ACMNA071, ACMNA072, ACMNA073, ACMNA074, ACMNA075, ACMNA076, ACMNA077, ACMNA078, ACMNA079, ACMNA080
EQUIVALENCE AND RELATIONSHIPS IN MATHS	ACMNA054, ACMNA055, ACMNA056, ACMNA057, ACMNA058, ACMNA059	ACMNA077, ACMNA078, ACMNA079, ACMNA080
NUMERACY FOR EVERYDAY LIFE	ACMNA055, ACMNA056, ACMNA057, ACMNA058, ACMNA059	ACMNA077, ACMNA078, ACMNA079, ACMNA080

Year 3: Recall multiplication facts for two, three, five and ten and related division facts; (ACMNA056) establishing multiplication facts using number sequences

Year 4: Describe, continue, and create number patterns resulting from the forming, addition, subtraction and writing the rules for number patterns; describing a rule for number patterns; then creating the pattern

Year 3: Recall multiplication facts up to 10 × 10 and related division facts; (ACMNA075) using known multiplication facts to calculate related division facts

Year 4: Recall multiplication facts up to 10 × 10 and related division facts; (ACMNA075) using known multiplication facts to calculate related division facts

Year 3: Represent and solve problems involving multiplication using efficient mental and written strategies; (ACMNA057) writing simple word problems numerically form and vice versa

Year 4: Represent and solve problems involving multiplication using efficient mental and written strategies; (ACMNA077) writing simple word problems numerically form and vice versa

Year 3: Solve word problems involving multiplication and division where there is no remainder; (ACMNA082) representing words problems using number sentences

Year 4: Solve word problems involving multiplication and division where there is no remainder; (ACMNA082) representing words problems using number sentences

Again, to summarise the last three slides ...

The ACCR document allows teachers to isolate quickly the AC content descriptions and conceptual areas for one year level, or a small number of consecutive years ... for example, Year 3.

It also allows them to consider in depth a particular "conceptual area" (or "big question") across several consecutive years (reading down and across) ... for example, Year 3 Patterning: what is before and after?

What patterns are ...

What sort of fluency ...

What relationship ...

What practical rel ...

Figure 6. Four selected slides from a primary mathematics education audio-lecture, concerning *Number and Algebra*, and illustrating the relevant scope and sequence of ideas in *Number and Algebra*, for **Patterning**, **Operations**, **Everyday numeracy**, and **Equivalence and Relationships**.

The ACCR and Understanding Mathematical Connectedness: *Within* the Strands

The next two sections of the paper now focus more upon the second of these teacher competencies, that is, developing a “sense of the connectedness and correspondences in mathematics”, both *within* and *across* the strands. To return again to the visual organisers, the first-tier, most large-scale ACCR rubric is used, first seen in Figure 2. Figure 7 below is essentially a repetition of Figure 2, but with the notion of conceptual “direction” highlighted. So just what else can such “rubrics” offer teachers for their preparation of classroom experiences, and the *language* of those classrooms? This is now considered.

Firstly, as seen in Figure 7, teachers should be able to see at a glance the consistent nature of mathematics content, and how all of the conceptual areas apply “down” all of the topic concepts covered. This also corresponds neatly with the notion of the four AC proficiency strands² applying equally across all strands and year levels. That is, in all three strands it is necessary to *problem-solve* and to *reason* (mostly **RECOGNISING PATTERNS** and reasoning about **EQUIVALENCE and RELATIONSHIPS**); to achieve personal *fluency* (mostly **MATHS FACTS and FLUENCIES** and **NUMERACY for EVERYDAY LIFE**); and to gain a connected *understanding* (a synthesis of all four conceptual areas).

Australian Curriculum Conceptual Rubric, ALL STRANDS
Years F-10 Mathematics

"Conceptual areas" of all strands ↓	Number and Algebra	Measurement and Geometry	Statistics and Probability
RECOGNISING PATTERNS in MATHS	Number patterning and graphs	Patterns in space, shape and time	Patterns in chance and data
MATHS FACTS and FLUENCIES	Operations with number and algebra	Space, shape and time measurement facts	Chance processes and measuring data
EQUIVALENCE and RELATIONSHIPS in MATHS	Reasoning about equations, equality and inequality	Reasoning about space, shape and time	Reasoning about chance and data
NUMERACY for EVERYDAY LIFE	Practical numeracy (estimation, calculation and financial maths)	Practical numeracy (estimation, orientation, and measurement)	Practical numeracy (estimation of chance, collecting and interpreting data)

Figure 7. Reading “down” a conceptual rubric. (Also see Fig. 2.)

In Figure 8 is seen a more detailed conceptual rubric for Years 7 to 10/10A, across all three strands. As it also includes AC Content Description codes it is working at the third-tier level of detail. (It should be contrasted with Figure 4, which was also third-tier but showed a *scope and sequence* across years for just *one strand*.)

² It should be noted here that the author does not consider the four AC proficiencies to directly map one-to-one with the four ACCR conceptual areas. After extensive mapping it was decided that the proficiencies all apply in differing degrees and combinations across all conceptual areas, with **EQUIVALENCE and RELATIONSHIPS** often relating to all four at once.

CONCEPTUAL RUBRIC for the AUSTRALIAN CURRICULUM, by MATHEMATICS STRANDS
Years 7-10 (Secondary)

Conceptual areas of the strand ↓	NUMBER and ALGEBRA Equations Functions		MEASUREMENT and GEOMETRY Shape Location Transformations		STATISTICS and PROBABILITY Chance Data			
	AC Content Descriptions		Conceptual areas of the strand ↓	AC Content Descriptions		Conceptual areas of the strand ↓	AC Content Descriptions	
	Years 7-8	Years 9-10		Years 7-8	Years 9-10		Years 7-8	Years 9-10
RECOGNISING PATTERNS in MATHS	Patterning and algebraic functions and graphs	ACMA149, ACMA150, ACMA152, ACMA151, ACMA178, ACMA180, ACMA280, ACMA182, ACMA184, ACMA186, ACMA193, ACMA194	ACMA208, ACMA209, ACMA212, ACMA213, ACMA214, ACMA215, ACMA294, ACMA295, ACMA237, ACMA238, ACMA239	Orientation in space; patterns in time measurement; categories, properties and transformations of 2D and simple 3D shapes	ACMM159, ACMM160, ACMM161, ACMM164, ACMM165, ACMM181, ACMM198, ACMM200	ACMM220, ACMM221, ACMM222, ACMM224, ACMM242, ACMM245	Summarising data; describing and predicting simple chance	ACMP167, ACMP168, ACMP169, ACMP170, ACMP171, ACMP172, ACMP204, ACMP205, ACMP202, ACMP203, ACMP207, ACMP225, ACMP226, ACMP282, ACMP283, ACMP284, ACMP285, ACMP246, ACMP249, ACMP250, ACMP251, ACMP252
MATHS FACTS and FLUENCIES	Operations (number properties and algebraic meaning and manipulation)	ACMA149, ACMA150, ACMA151, ACMA152, ACMA153, ACMA154, ACMA155, ACMA156, ACMA157, ACMA158, ACMA159, ACMA175, ACMA176, ACMA177, ACMA179, ACMA280	ACMA177, ACMA230, ACMA231, ACMA232, ACMA233, ACMA234, ACMA235, ACMA236, ACMA237, ACMA238, ACMA239, ACMA240, ACMA241	Space and shapes measurement facts; telling the time and calculating elapsed time; reading scales, maps and clocks	ACMA159, ACMA160, ACMA161, ACMA163, ACMA195, ACMA196, ACMA197, ACMA198, ACMA199, ACMA200	ACMM216, ACMM217, ACMM218, ACMM219, ACMM221, ACMM224, ACMM222, ACMM244	Collecting, arranging, representing and interpreting data; exploring chance processes	ACMP167, ACMP168, ACMP169, ACMP170, ACMP171, ACMP172, ACMP204, ACMP205, ACMP202, ACMP203, ACMP207, ACMP225, ACMP226, ACMP282, ACMP283, ACMP284, ACMP285, ACMP246, ACMP249, ACMP250, ACMP251, ACMP252
EQUIVALENCE and RELATIONSHIPS in MATHS	Equations, equality and inequality (number and algebra)	ACMA149, ACMA150, ACMA152, ACMA175, ACMA176, ACMA179, ACMA280, ACMA182, ACMA191, ACMA192, ACMA194	ACMA215, ACMA296, ACMA234, ACMA235, ACMA236, ACMA237, ACMA238, ACMA239, ACMA240, ACMA241	Shapes, measurements and time that are the same and not the same	ACMM163, ACMM164, ACMM166, ACMM197, ACMM199, ACMM200, ACMM201, ACMM202	ACMM220, ACMM221, ACMM222, ACMA243, ACMM222, ACMM244	Equal and unequal data; equal and unequal chance	ACMP168, ACMP170, ACMP204, ACMP202, ACMP203, ACMP204, ACMP205, ACMP207, ACMP225, ACMP226, ACMP282, ACMP283, ACMP246, ACMP249, ACMP250
NUMERACY for EVERYDAY LIFE	Practical numeracy (financial maths, calculator and spreadsheet use, numerical estimation and sense; scientific notation)	ACMA157, ACMA158, ACMA173, ACMA174, ACMA180, ACMA187, ACMA188, ACMA189	ACMA210, ACMA215, ACMA229	Practical numeracy (orientation, estimation and measurement of space and shape size, reading maps, the time and timetables)	ACMM159, ACMM161	ACMM216, ACMM217, ACMM218, ACMM221, ACMM221	Practical numeracy (collecting and correctly interpreting data)	ACMP169, ACMP170, ACMP171, ACMP172, ACMP284, ACMP206, ACMP227, ACMP228, ACMP283, ACMP253

Figure 8. The ACCR for Years 7 to 10/10A across all three strands: mapping at the third-tier level.

Two more specific examples, using PowerPoint slides from postgraduate pedagogy enrichment units concerning transition (Years 6 and 7) mathematics, in two different strands, consider similar ideas to those in Figure 6. In Figure 9 a geometric task had been chosen and teachers were asked to explain the use or relevance of three of the conceptual areas to the creation of this task. In Figure 10, teachers were asked to consider the links within a topic in pre-algebra/algebra, where patterning concepts are used to develop the notion of function.

Figure 9. Two selected slides from a Postgraduate Certificate mathematics education lecture, illustrating with a geometric example the conceptual areas of **Patterning**, **Operations**, and **Equivalence and Relationships** (task from *Geometry Mathematics Book for Year 6*)

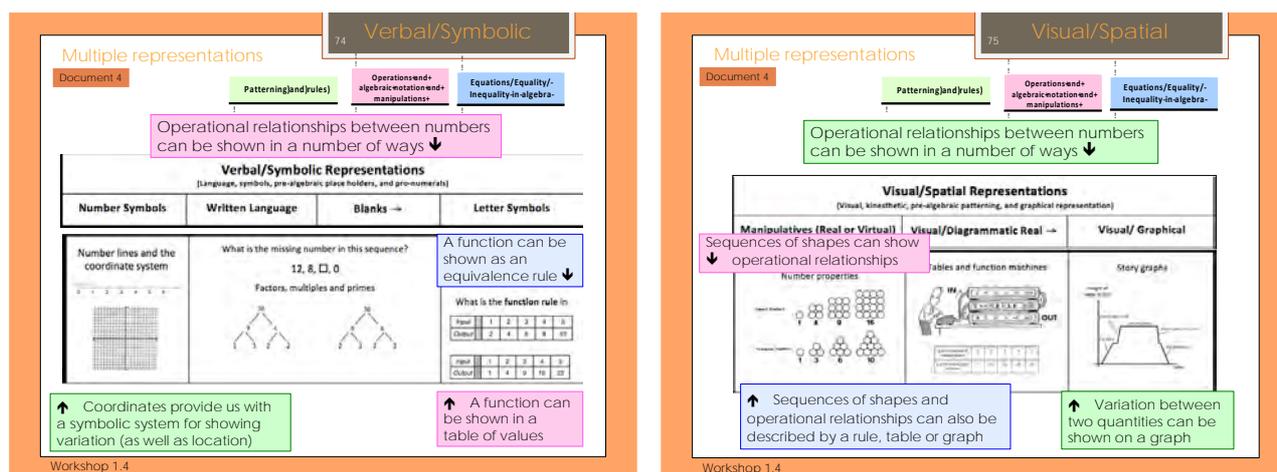


Figure 10. Two selected slides from a Postgraduate Certificate mathematics education lecture, illustrating the use of verbal-symbolic, visually diagrammatic and graphical representations of some pre-algebra and algebra. The conceptual areas of **Patterning**, **Operations**, and **Equivalence and Relationships** are signposted in the mapping. (Unit mapping document, Year 6 and 7, *Number and Algebra*.)

In both cases above, the “connections” being examined are those between the conceptual areas *within* particular strands, and by topic. However, deeper relationships – exploring that “deep ontological gap between operational and structural conceptions” (Sfard, 1991, p.4) – can also be considered using the ACCR. These are seen in the links *between* the strands that illustrate the coherence of mathematics as a whole. It could also be argued that in this kind of reflective practice lies the richest source of engaging ideas for conversation in a mathematics classroom, especially in regards to “good questioning” (Sullivan & Clarke, 1991). This is discussed next.

The ACCR and Understanding Mathematical Connectedness: *Across* or *Between* the Strands

Looking at the different (“across”) perspective in Figure 11, the cohesive nature of mathematics and just what an attempted disclosure of this cohesion may do for the learner is suggested. It can be observed that teaching in all three strands of mathematics needs attention to the same four conceptual areas:

- **RECOGNISING PATTERNS** applies to topics involving numbers, algebra, graphs, measurement, space, shape, time, chance and data.
- **MATHS FACTS and FLUENCIES** involves learning and recalling important facts about numbers, algebra, graphs, and measurement of space, shape, time, chance and data;
- **NUMERACY for EVERYDAY LIFE** involves choosing tools, instruments and mental strategies on a regular basis to solve daily problems occurring with numbers, algebra, graphs, and the measurement of space, shape, time, chance and data; and, perhaps most importantly of all,
- all mathematical reasoning depends on a deeper understanding of the notion of mathematical **EQUIVALENCE and RELATIONSHIPS**: what is the same, and not the same, in representations of numbers, algebra, graphs, measurement, space, shape, time, chance and data.

In other words, mathematics should not be “taught in isolated pockets” (Ormond, 2016) according to topics set for the day or the week: it is both a science and an art in that it links logically and cohesively across topics (eventually, as we know, merging conceptually

more and more between the strands themselves, as calculus does for statistical analysis, for example). Such a rubric contributes to that common *language* that can be used to talk about all topics, one that helps students to associate mathematical ideas from the different strands.

Australian Curriculum Conceptual Rubric, ALL STRANDS
Years F-10 Mathematics

"Conceptual areas" of all strands ↓	Number and Algebra	Measurement and Geometry	Statistics and Probability
RECOGNISING PATTERNS in MATHS	Number patterning and graphs	Patterns in space, shape and time	Patterns in chance and data
MATHS FACTS and FLUENCIES	Operations with number and algebra	Space, shape and time measurement facts	Chance processes and measuring data
EQUIVALENCE and RELATIONSHIPS in MATHS	Reasoning about equations, equality and inequality	Reasoning about space, shape and time	Reasoning about chance and data
NUMERACY for EVERYDAY LIFE	Practical numeracy (estimation, calculation and financial maths)	Practical numeracy (estimation, orientation, and measurement)	Practical numeracy (estimation of chance, collecting and interpreting data)

Figure 11. Reading “across” a first-tier conceptual rubric. (Also see Figs. 2 and 7.)

The four questions themselves, spoken aloud as they are or paraphrased for students, can steer these conversations (also see Fig. 1):

- “What patterns can we see here in this maths?”
- “What maths do we need to really learn and be able to remember?”
- “How does this maths fit together ...? Say the same thing as ...?”
- “How do we use this maths in our lives?”

Goos et al. (2017) remind us that “*invariance* – that is, a relationship or property that stays the same and does not change when some change is permitted – is a key aspect of geometric thinking (Johnston-Wilder & Mason, 2005)” (p. 228). The conceptual area of **EQUIVALENCE and RELATIONSHIPS in MATHS** is based here on a very similar sentiment concerning *all* mathematical reasoning; and it is therefore this conceptual area of the ACCR that has the strongest connection with that second disposition for cultivating a “sense of the *connectedness and correspondences*” in the discipline.

It is argued here that the discipline of mathematics is largely built on noticing, using and proving when entities are the “same” or “not the same”. Equivalence can also be thought of, semantically, as direct “correspondence”. From solving simple linear equations in one unknown variable to integrating a definite integral to find a quantity equal to the area under a curve, the process of *preserving operational and functional equivalences* (balancing sides of the linear equation, using a trigonometric identity to re-cast an integrand in a form that can be more easily integrated) is maintained. The constraints in a linear inequality problem which can be understood by graphing lines to show the region *corresponding* to the solutions, as opposed to that region outside *not* corresponding to the solutions, is another example. Factorising a fifth-degree polynomial expression in x is about breaking it down into an *equivalent* product of algebraic expressions that are in the end as close as possible to all being first-degree ones – this in turn *corresponds* to finding the possible roots of the polynomial when expressed as a function, which in turn *correspond* to the x -intercept values of that curve. Hypothesis testing in statistics problems is based on finding, within set limits of confidence, the likelihood of two population means or proportions *not* being equal to each other – and this can be represented graphically with variously overlapping curves (“how

much the *same* are they?") that illustrate the population distributions.

Sometimes the "equivalence" is in the form of *exact equality of quantities*, as in solving equations or evaluating definite integrals or factorising a polynomial expression to achieve an equivalent expression – sometimes the equivalence is by nature more of a *correspondence*, one that links between a *verbal-symbolic representation* of a mathematical idea with its *visual-spatial-graphical one*. An example is in the sketching of a parabola $y = f(x)$ which, through its corresponding non-factorised, factorised, or "completed square" forms shows any or all x -intercepts, a y -intercept, and maximum or minimum points. Sometimes the equivalence is a kind of "shorthand", as in the expression of a consistent set of linear equations in matrix form, one which is more easily "reduced" to a set of solutions for the unknowns.

Table 1 provides some other examples of this, across all three strands. Example 1 shows "correspondences" across the three strands, in terms of a *common representation* (a divided circle). Example 1 therefore also displays a shared relationship *between* the strands: proportional thinking can be used in fractions, degree relationships in a circle, and chance concepts. Example 2 shows how the notion of equality and inequality is important, albeit in different ways, in all three strands. And in both examples, the inherent *relationships within* are also described, seen in the second row of the examples, each time.

Whichever way one looks at it, equivalence is everywhere in mathematics, and truths about equivalence and non-equivalence invariably indicate important conceptual relationships. This idea is intrinsic to mathematical sense-making of all kinds, and is therefore worthy of a "conceptual area" that is far deeper than just noting patterns or obeying operational rules; thus, **EQUIVALENCE and RELATIONSHIPS** is the most complex of the four pedagogical areas in the ACCR, and also the most embedded in the relational learning that can result in Sfard's "reification".

Such a complex idea needs to be carefully scaffolded for many new teachers, who in turn must do the same, if in a simpler way, for their students in classrooms. And this can be looked at from two limiting perspectives, depending on the expertise of the mathematics teachers. Those with well-developed mathematics understanding may still need help with, as Freudenthal expressed it in 1983, "unclogging" their "automatisms" or "original sources of insight". (Davis would call this "substructuring.") When one "sees" the function simultaneously as an algebraic equation and as a sketched curve without even thinking about it, one has to work very hard to appreciate the very different perspective of a lower secondary classroom student. On the other hand, pre-service teachers with less mathematical confidence may have never actually had to contemplate mathematics from an ontological or structural view at all: they may still need to build a sense of its connectedness before sharing it with their students.

		NUMBER and ALGEBRA	MEASUREMENT and GEOMETRY	STATISTICS and PROBABILITY
		Reasoning about equations, equality and inequality	Reasoning about space, shape and time	Reasoning about chance and data
Example 1	EQUIVALENCE	A fraction of a whole can be shown with a shaded sector of a circle.	The area of a shaded sector in a circle can be found using the angle its connecting rays makes at the centre of the circle.	A shaded sector in a circle can be used to describe the probability of an event occurring.
	↓ RELATIONSHIPS	This depends on the area of the circle being the "whole".	This depends on there being 360 degrees in one full turn of the circle, and the area of the sector being the angle/360 times the area of the circle.	This depends on the area of the circle being equal to the probability of 1 (certainty).
	→ RELATIONSHIPS → Proportional thinking is correspondingly used <i>across strands</i> , in fractions concepts, degree relationships in a circle, and chance concepts			
Example 2	EQUIVALENCE	Solving an equation involves correctly balancing and maintaining <u>equality</u> across both sides of the working.	Categorising types of quadrilaterals is about recognising <u>equal</u> or <u>unequal</u> sides or angles.	Calculating that an <u>equal</u> chance of something occurring is equal to 0.5, and impossibility is 0.
	↓ RELATIONSHIPS	This is based on an understanding of <u>inverse operational relationships</u> in arithmetic.	This is based on recognising <u>correspondences</u> in geometric properties that are able to be generalised.	This is based on an understanding that probabilities <u>are related to</u> fractional numbers between and including 0 and 1.
	→ RELATIONSHIPS → <i>Within all three strands</i> , the concept of exact equality is a key mathematical idea.			

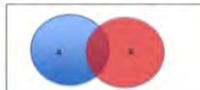
Table 1. Using common representations of concepts, and mathematical “correspondences”, across the three strands.

73 Appendix 1

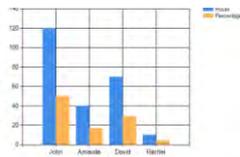


Pause for thought: “equivalence” in mathematics

It could be argued that ideas of equivalence – and non-equivalence – occur right throughout mathematics, and are indeed essential to the very nature of all mathematics.







$32 > 28$
 $28 < 15 \times 2$

Also briefly consider the next slide, which offers a simple “conceptual framework” for primary mathematics and its teaching, aligned to the Australian Curriculum.

Workshop 1.3

Figure 13. Selected slide from a Postgraduate Certificate mathematics education lecture, prompting the notion of **Equivalence and Relationships** applying across mathematics, across the three strands.

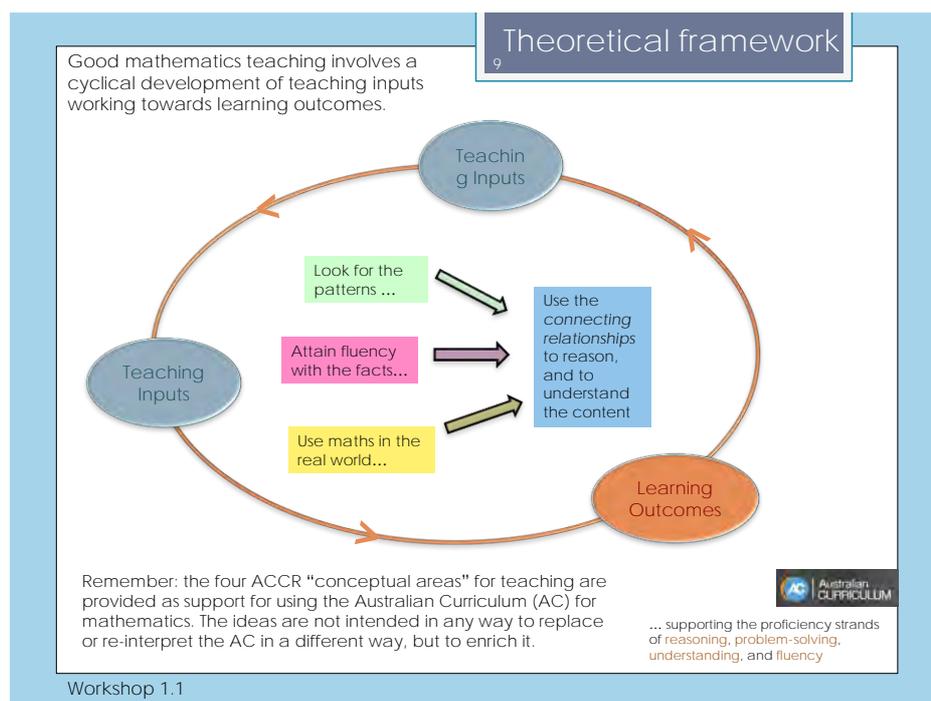


Figure 14. Selected slide from a Postgraduate Certificate mathematics education lecture, illustrating the links between the four ACCR conceptual areas.

Understanding mathematical connectedness both *within* and *between* the strands is a very challenging task for novice teachers, needing self-discipline and reflection. The concept has been introduced by the author to pre-service and in-service teachers in several ways in mathematics education scenarios (see Figs. 13 and 14). While the representations in Figure 13 comment on the notion of the blue conceptual area being evident *across* the mathematical strands, Figure 14 emphasises that *within* each strand, “teaching inputs” that involve **pattern exploration**, **fluency and practice** and **application in real life contexts** all lead to the “learning outcomes” that are dependent upon these processes, and that can formalise or reify for students the “**key ideas that underpin the concepts**” (Sullivan, 2011). That is what is meant by “relational learning”.

It was explained earlier that Sullivan’s six fundamental Principles for effective teaching provided the impetus for this re-invigorated thinking around the ACCR model. Of its four “big” pedagogical questions, two involve a great deal more thought and effort from teachers. While **MATHS FACTS and FLUENCIES** and **NUMERACY for EVERYDAY LIFE** correspond to important features of *Principle 2: Making connections* and *Principle 6. Promoting fluency and transfer*, the other two conceptual areas, **RECOGNISING PATTERNS** and **EQUIVALENCE and RELATIONSHIPS**, each relate to at least five of the six Principles³. Table 2 represents the author’s exploration of the links between the Sullivan Principles and the conceptual areas. The first impression upon looking at it is its multi-coloured nature – and this has confirmed the author’s sense that the conceptual areas weave through *all* of the Principles. It also appears to indicate that the “four big questions” of

³ It needs to be mentioned here also that *Principle 4: Differentiating challenges*, with its sensitivity to student differences, underpins all successful mathematics teaching and learning, and so is assumed to be an all-encompassing aim. It is in fact inherent in all four of the pedagogical questions, and also relates strongly to the AITSL Graduate Teacher Standard 1, *Knowing how students learn*, and Standard 4, *Create and maintain supportive and safe learning environments* (AITSL, 2011). This Principle relies on careful preparation of lessons that consider the likely prior knowledge of students and employ differentiation of tasks for both less and more mathematically capable students. The other five Principles are all embedded by this aim, so mapping of this Principle has not been attempted in Table 2.

the ACCR are indeed a “distillation” of the most important features of effective classroom teaching.

The **MATHS FACTS and FLUENCIES** and **NUMERACY for EVERYDAY LIFE** conceptual areas are of course important aspects of a mathematics classroom (something testified to by the prominence many less confident mathematics teachers place on pure drill and memorisation). The first row of Table 2 illustrates the significance of Principles 2 and 6 to these conceptual areas. Interestingly, *Principles 2* and *6* still correspond to the other two conceptual areas also, as seen in the blue and green highlights. This is not surprising, as “stories” and “contexts” are so important to the **exploration of mathematical patterns**, as much as they are to a sense that the mathematics is relevant to everyday life and needs to be learned and internalised. Similarly, building on prior knowledge to create **new understandings and connections** is unavoidable for effective work on fluency and numeracy.

Table 2 supports the argument that the conceptual areas of **RECOGNISING PATTERNS** and **EQUIVALENCE and RELATIONSHIPS** much more closely encapsulate the essence of Sullivan’s Principles. (See Tab. 2 and Fig. 15.) Earlier in the paper it was suggested (and illustrated in Figs. 3 and 4) that the four conceptual areas can be used to develop a sense of the scope and sequence of a topic’s concepts. It was also suggested earlier that recognising mathematical relationships and speaking a coherent mathematical language link best with both *Principle 1: Articulating goals* and *Principle 2: Making connections*. These Principles relate to **RECOGNISING PATTERNS** in the sense that “key ideas that underpin the concepts” and “contextualising and creating a rationale for the learning” are intrinsic to pattern recognition. A familiarisation with **EQUIVALENCE and RELATIONSHIPS** “creates and connects students with stories” and again pulls out “key ideas”. These two Principles can be applied both *within* the strands (for example, making patterns with matches to establish a linear relationship function), and *between* the strands (for example, as already discussed, using the idea of the area of a sector of a circle to show both fractions of a “whole”, or probabilities of an event occurring when the circle area is 1).

The conceptual areas for teaching mathematics, based on the four big questions for teaching	Corresponding Sullivan Principles (2011)
<p>MATHS FACTS and FLUENCIES <i>What sort of fluency do students need in this mathematics?</i></p> <p>NUMERACY for EVERYDAY LIFE <i>What practical relevance does this mathematics have to our everyday lives?</i></p>	<p>Principle 2: Making connections Build on what students know, mathematically and experientially, including creating, and connecting, students with stories that both contextualise and establish a rationale for the learning.</p> <p>Principle 6: Promoting fluency and transfer Fluency is important, and it can be developed in two ways: by short everyday practice of mental processes; and by practice, reinforcement and prompting transfer of learnt skills.</p>
<p>RECOGNISING PATTERNS in MATHS <i>What meaningful patterns are there in this mathematics content?</i></p> <p>EQUIVALENCE and RELATIONSHIPS in MATHS <i>What relationships and connections can be seen in this mathematics?</i></p>	<p>Principle 1: Articulating goals Identify key ideas that underpin the concepts you are seeking to teach, communicate to students that these are the goals of the teaching, and explain to them how you hope they will learn.</p> <p>Principle 2: Making connections Build on what students know, mathematically and experientially, including creating, and connecting, students with stories that both contextualise and establish a rationale for the learning.</p> <p>Principle 3: Fostering engagement Engage students by utilising a variety of rich and challenging tasks that allow students time and opportunities to make decisions, and which use a variety of forms of representation.</p> <p>Principle 5: Structuring lessons Adopt pedagogies that foster communication and both individual and group responsibilities, use students' reports to the class as learning opportunities, with teacher summaries of key mathematical ideas.</p> <p>Principle 6: Promoting fluency and transfer Fluency is important, and it can be developed in two ways: by short everyday practice of mental processes; and by practice, reinforcement and prompting transfer of learnt skills.</p>

*Relevant to all strategies: 4: Differentiating challenges: Socially interact with and support students of all mathematical abilities

Table 2. The four “conceptual areas” of mathematics teaching, represented also as pedagogical questions, related using the ACCR colour codes to five of Sullivan’s six Principles for effective mathematics teaching.

Other Sullivan Principles are also here briefly reflected upon. *Principle 3: Fostering engagement* encourages teachers to “use a variety of forms of representation”, “challenging” students with “rich tasks” that depend on them having “time and opportunities to make decisions”. Such tasks are at the heart of exercises that employ looking for patterns and finding links and relationships. The effective practice of *Principle 5: Structuring lessons* depends on attention to the scope and sequence of concepts; to the prior knowledge of students; to the desirable range of difficulty; to cooperative group work on noticing and recording mathematical trends; and to the teacher’s subsequent formalisation of the key relationships between the concepts. *Principle 6: Promoting fluency and transfer* contributes both to procedural fluency and to students’ own formalisation (reification) of the key concepts.

Conclusion

One final piece of analysis is seen in Table 3, which represents the culmination of this work and is now being shared with the author’s current mathematics education students. It summarises in a practical way the links between the two pedagogical models discussed in this paper. Using this table allows teachers, pre-service and in-service, to frame their understanding of the four conceptual areas in terms of the foundational principles for all classroom mathematics teaching that have been devised by Sullivan.

<i>Big questions for the teacher’s classroom preparation/ “Big questions for the students“</i>	The key points of five of Sullivan’s Principles, linked specifically to the four ACCR “big questions” (distilled from Figure 1 and the analysis in Table 2)
<p><i>What sort of fluency do students need in this maths?</i></p> <p>“What maths do we need to really learn here, and be able to remember?”</p>	<ul style="list-style-type: none"> • Build on what students know, mathematically and experientially (2) • Use students’ reports to the class as learning opportunities (5) • Fluency is important, and it can be developed ... by short everyday practice of mental processes
<p><i>What practical relevance does this maths have to students’ everyday lives?</i></p> <p>“How do we use this maths in our lives?”</p>	<ul style="list-style-type: none"> • Build on what students know, mathematically and experientially (2) • Create, and connect students with, stories that both contextualise and establish a rationale for the learning. (2) • Fluency is important, and it can be developed ... by practice and reinforcement (6)
<p><i>What meaningful patterns are there in this maths content?</i></p> <p>“What patterns can we see here in this maths?”</p>	<ul style="list-style-type: none"> • Identify key ideas that underpin the concepts (1) • Create, and connect students with, stories that ... contextualise (2) • Engage students by utilising a variety of rich and challenging tasks that allow students time and opportunities to make decisions (3) • Foster communication and both individual and group responsibilities (5)
<p><i>What relationships and connections can be seen in this maths?</i></p> <p>“How does this maths fit together ...? Say the same thing as ...?”</p>	<ul style="list-style-type: none"> • Identify key ideas that underpin the concepts (1) • Communicate to students that these are the goals of the teaching (1) • Explain to them how you hope they will learn (1) • Build on what students know, mathematically and experientially (2) • Create, and connect students with, stories that both contextualise and establish a rationale for the learning (2) • Engage students by utilising a variety of rich and challenging tasks which use a variety of forms of representation (3) • Use teacher summaries of key mathematical ideas (5) • Fluency is important, and it can be developed ... by prompting transfer of learnt skills (6)

Table 3. The four ACCR pedagogical questions and the key corresponding Sullivan descriptors, related using the ACCR colour codes. (Also see Fig.1.)

The relationship between the two frameworks is also seen in Figure 15, where Figure 14 has been enhanced with extra links to Sullivan’s Principles. The relative predominance of those two conceptual areas, green and blue, can be seen, now also representing **RECOGNISING PATTERNS** as the primary source of *teaching inputs* and **EQUIVALENCE** and **RELATIONSHIPS** as that of student *learning outcomes* related to these inputs. Indeed, Sullivan’s “key ideas that underpin the concepts” (*Principle 1*, 2011) provide the central goal of effective mathematics teaching.

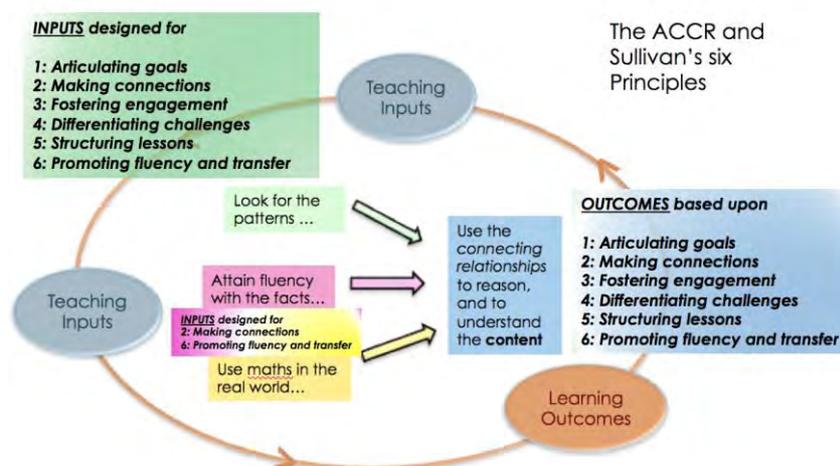


Figure 15. Connecting the ACCR model with Sullivan's six Principles for effective teaching of mathematics. PowerPoint slide developed in 2019.

This paper has interpreted the body of knowledge that is mathematics from the author's perspective and understanding of the needs of the new classroom teacher, who must convey that knowledge, in a logical way, to students. It has suggested the value of initially using a fairly spare "structure" of conceptual keystones and themes – one that nevertheless implies a complex internal organisation of ideas – in a first venture into the deeper ontological questions that accompany any study of mathematics.

The author further believes that any organized and sequenced Primary/Secondary mathematics curriculum can be interpreted in this way, and has argued that such a visual and thematic framework may help teachers to keep track of some powerful tools for practical, effective teaching. By keeping the important elements of big picture thinking for PCK in the forefront of their minds as they prepare tasks and plan potential classroom conversations, the focus may remain on student-centred, connected mathematics teaching. The paper has also argued that Sullivan's six Principles for the effective classroom delivery of mathematics underpin the very fundamental questions of the ACCR framework, questions that teachers may use to enhance both their own, and their students', understanding of the mathematics.

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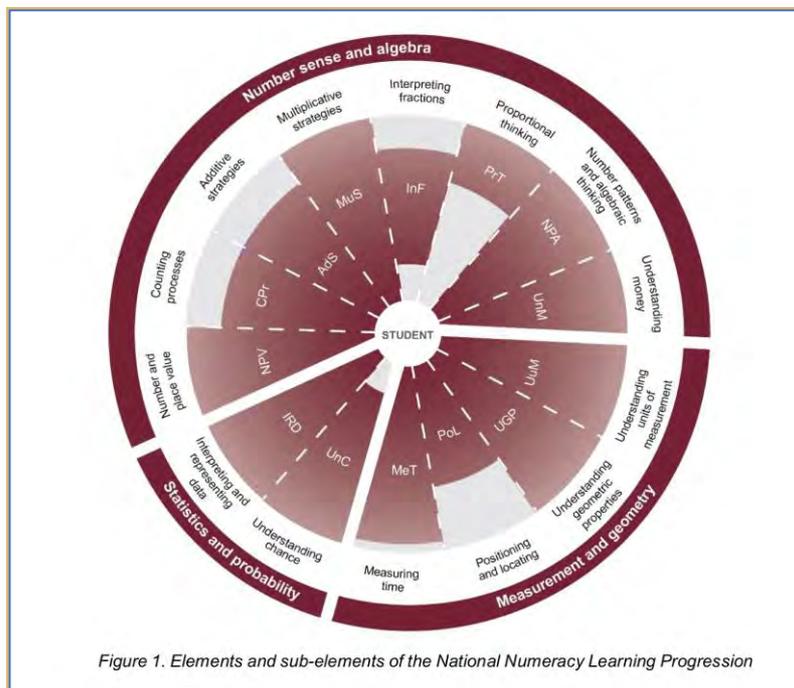
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APPENDIX A

Conceptual framework for the National Numeracy Learning Progression (ACARA, 2020, p. 5).



Australian Curriculum Conceptual Rubric, ALL STRANDS

Conceptual areas of all strands ↓	Number and Algebra	Measurement and Geometry	Statistics and Probability
RECOGNISING PATTERNS in MATHS	Number patterning and graphs	Patterns in space, shape and time	Patterns in chance and data
EQUIVALENCE and RELATIONSHIPS in MATHS	Reasoning about equations, equality and inequality	Reasoning about space, shape and time	Reasoning about chance and data
NNLP <i>(all full or almost fully shaded segments)</i>	<i>Number patterns and algebraic thinking</i>	<i>Understanding geometric properties</i> <i>Measuring time</i>	<i>Interpreting and representing data</i> <i>Understanding chance</i>
MATHS FACTS and FLUENCIES	Operations with number and algebra	Space, shape and time measurement facts	Chance processes and measuring data
NUMERACY for EVERYDAY LIFE	Practical numeracy (estimation, calculation and financial maths)	Practical numeracy (estimation, orientation, and measurement)	Practical numeracy (estimation of chance, collecting and interpreting data)
NNLP <i>(various shaded segments)</i>	<i>Number and place value</i> <i>Additive strategies</i> <i>Multiplicative strategies</i> <i>Proportional thinking</i> <i>Interpreting fractions</i> <i>Counting processes</i> <i>Understanding money</i>	<i>Understanding geometric properties</i> <i>Positioning and locating</i> <i>Measuring time</i>	<i>Interpreting and representing data</i> <i>Understanding chance</i>

Appendix A Figure 1. The first tier of the four ACCR “conceptual areas” of mathematics teaching, across all three AC strands, mapped against the NNLP elements of its conceptual framework.

APPENDIX B

Voluntary course-end comments from students concerning support of conceptual mapping materials.

Student 1, email October 2017, teaching secondary mathematics

Thank you again for your wonderful workshop materials which I will definitely look over again even after the Course is over.

Student 2, email November 2017, teaching secondary mathematics

I really want to thank you for the interesting and extremely valuable units. I know that I will keep going back to the presentations, documents and all the additional materials you have so kindly provided.

Student 3, email October 2018, teaching secondary mathematics

Thank you so much for the course content. This unit has really put an understanding to what I have been doing in the classroom. It has given me content knowledge that was missing in my CV.

Student 4, email November 2018, teaching secondary mathematics

I just wanted to say thank you for the learning opportunity this course has allowed me and for all the work you have put into creating this course. I have gained so much from undertaking this course and am becoming more confident every day that I will be able to effectively support future maths students in their journey through this amazing subject. I am excited about how much there is to learn from this course and although the assessments are completed and the course is coming to an end for 2018, I am looking forward to further breaking down the 2018 content over the summer holidays in my journey towards mastery.

Student 5, email June 2019, teaching secondary mathematics

Thanks so much for the wonderful delivery of your course ... The course outline, reading material, schedule, lecture notes etc were very clear and I feel that the course has prepared me well for teaching maths in my school. It has opened a new world to me and given possible new directions.

Student 6, email June 2020, teaching secondary mathematics

I would also like to thank you because I have learnt so much doing this course and your slides and audio presentation are very clear, well-sequenced and very informative. I now know the importance of the conceptual areas in mathematics, i can navigate confidently with the ACARA and appreciate the importance of giving students as much opportunities to do problem solving task. I now understand the importance of developing conceptual areas using different mode of presentation to children - especially in the primary years, as it will give them a strong foundation for high school. My favourite topic was learning about Numbers and Algebra. I had so many misconceptions that I never knew I had, but you clarified the many misconceptions I had. Thank you once again for being an excellent lecturer and your passion is evident in your work.

APPENDIX C

More detailed version of Table 2 in the body of the text.

The conceptual areas for teaching mathematics, based on the four big questions for teaching	Corresponding Sullivan Principles (2011, in full)	Corresponding Sullivan Principles (paraphrased)*
<p>MATHS FACTS and FLUENCIES <i>What sort of fluency do students need in this mathematics?</i></p> <p>NUMERACY for EVERYDAY LIFE <i>What practical relevance does this mathematics have to our everyday lives?</i></p>	<p>Principle 2: Making connections Build on what students know, mathematically and experientially, including creating, and connecting, students with stories that both contextualise and establish a rationale for the learning.</p> <p>Principle 6: Promoting fluency and transfer Fluency is important, and it can be developed in two ways: by short everyday practice of mental processes; and by practice, reinforcement and prompting transfer of learnt skills.</p>	<p>2: Making connections Using prior knowledge, recognition of relationships, and contextual links to build a negotiated understanding.</p> <p>6: Promoting fluency and transfer Establishing fluency with practice, reinforcement and shared communication of learnt skills.</p>
<p>RECOGNISING PATTERNS in MATHS <i>What meaningful patterns are there in this mathematics content?</i></p> <p>EQUIVALENCE and RELATIONSHIPS in MATHS <i>What relationships and connections can be seen in this mathematics?</i></p>	<p>Principle 1: Articulating goals Identify key ideas that underpin the concepts you are seeking to teach, communicate to students that these are the goals of the teaching, and explain to them how you hope they will learn.</p> <p>Principle 2: Making connections Build on what students know, mathematically and experientially, including creating, and connecting, students with stories that both contextualise and establish a rationale for the learning.</p> <p>Principle 3: Fostering engagement Engage students by utilising a variety of rich and challenging tasks that allow students time and opportunities to make decisions, and which use a variety of forms of representation.</p> <p>Principle 5: Structuring lessons Adopt pedagogies that foster communication and both individual and group responsibilities, use students' reports to the class as learning opportunities, with teacher summaries of key mathematical ideas.</p> <p>Principle 6: Promoting fluency and transfer Fluency is important, and it can be developed in two ways: by short everyday practice of mental processes; and by practice, reinforcement and prompting transfer of learnt skills.</p>	<p>1: Articulating goals Identifying and communicating the underpinning key ideas and how students will learn.</p> <p>2: Making connections Using prior knowledge, recognition of relationships, and contextual links to build a negotiated understanding.</p> <p>3: Fostering engagement Challenging and engaging students with rich tasks that use a variety of forms of representation.</p> <p>5: Structuring lessons Using pedagogical strategies that result in conversation, group exploration, and then formalization of the key mathematical ideas.</p> <p>6: Promoting fluency and transfer Establishing fluency with practice, reinforcement and shared communication of learnt skills.</p>

*Relevant to all strategies: 4: Differentiating challenges: Socially interact with and support students of all mathematical abilities

Appendix C, Table 1. The four “conceptual areas” of mathematics teaching, represented also as pedagogical questions, related using the ACCR colour codes to five of Sullivan’s six Principles for effective mathematics teaching. Included here is a third column listing an abbreviated version of each of the Principles.