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Mathematical Thinking Styles—The Advantage of Analytic Thinkers When Learning Mathematics

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Abstract: School is a space where learning mathematics should be accompanied by the student's preferences; however, its valuation in the classroom is not necessarily the same. From a quantitative approach, we ask from the mathematical thinking styles (MTS) theory about the correlations between preferences of certain MTS and mathematical performance. For this, a valid test instrument and a sample of 275 16-year-old Chilean students were used to gain insight into their preferences, beliefs and emotions when solving mathematical tasks and when learning mathematics. The results show, among other things, a clear positive correlation between mathematical performance and analytical thinking style, and also evidence the correlation between self-efficacy, analytical thinking and grades. It is concluded that students who prefer the analytical style are more advantageous in school, since the evaluation processes have a higher valuation of analytic mathematical thinking.

Keywords: mathematical thinking styles; analytic thinking; learning opportunities; cognitive perspective

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1. Introduction

The diversity of students in mathematics lessons invites us to recognize the different ways that students choose to interact with mathematical knowledge. When the teacher invites the student to perform a task or solve a mathematical problem, it is possible to identify in some students sufficient algebraic or functional answers, whilst on other occasions, visual or figurative answers, and also, it is possible to recognize mixed uses of these types of preferences in different graduations.

This range of responses is natural when there are heterogeneous groups; therefore, the multiplicity of positions raises a complex issue regarding mathematics teacher practice, since it is necessary to develop in the teacher the ability to recognize different mathematical practices, beyond a specific inclination or preference that the teacher possesses, since an acceptance of the student's personal preference should exist. This allows for maintaining a diversity of representations and mental landscapes that are people's characteristics [1], which in educational practice is a daily interaction to reflect and improve.

One situation that coexists in the classroom and threatens the heterogeneity of thinking deals with standardized tests and their effect on teaching practices. In Chile, the SIMCE test (The Spanish acronym SIMCE means "Measuring System of Education Quality") is aimed to students of the school system, being highlighted as a reliable test in measuring the quality of education [2]. This test, based on its format and interest, measures the recognition and application of mathematical concepts and properties from algebraic processes, in addition to mathematical reasoning in routine solving problems [3], whose adverse effects fall on intense preparations in schools, not professionalizing the teaching practice [2]. This situation, which promotes a mathematical formalism, limits and generates opacity in other types of valid mathematical representations or forms of the student's thinking, either from graphic or figurative representations that effectively entail correct mathematical reasoning, and that could even be determined as incorrect in practice by the teacher.

From the teacher's point of view, Entwistle [4] analyzes how the teacher's role affects the ways of understanding mathematics for the student, explaining that the teaching methods used by the teacher have a direct influence on this. Results of this nature suggest a scope for a more precise focus on teaching practice, for example, in the decision of planning and/or evaluations instruments, since the use of evaluation instruments consisting of open or closed questions are possible to cultivate mathematics for different preferences. This, from the present investigation, states that strengthening the thinking processes depends not only on how well we do it, but also on how we like or prefer to do it [5].

The present research separates itself from studies that deal with forms of learning, and highlights that the results on the relationship between learning and dominant learning paths have weakened in the last decade [6,7]. In this article, we focus on how students prefer to understand, think and present mathematics from the mathematical thinking styles [8], and how such preferences are related to the teacher valuation in order to analyze the tension between the student's and teacher's preferences. In this sense, the tension is situated in a classroom that has between 20 to 45 students, or in other words, from 20 to 45 ways of thinking and presenting mathematics, as it happens in many places in Latin America and the world.

2. Theoretical Background

Firstly, the mathematical thinking style theory (MTS) is presented, showing its central aspects; secondly, the construct of self-efficacy is stressed. Both play an essential role in the investigation presented in this paper.

2.1. Mathematical Thinking Styles Theory

The mathematical thinking styles theory (MTS) was developed by Borromeo Ferri in 2004 based on qualitative studies regarding 15- and 16-year-old learners. In further empirical studies, which also turned into quantitative investigations, including international comparative studies [9], the construct of mathematical thinking styles could be finally conceptualized and operationalized. The term mathematical thinking style is characterized as follows in [8–10]:

A mathematical thinking style is the way in which an individual prefers to present, to understand and to think through mathematical facts and connections by certain internal imaginations and/or externalized representations. Hence, a mathematical style is based on two components: (1) internal imaginations and externalized representations, and (2) on the holistic, respectively, the dissecting way of proceeding.

The reason that the MTS is so interesting for investigating learners, teachers or professors concerning mathematical teaching and learning is the fact that MTS does not focus on the performance but on the preferences of individuals. This approach is based on Sternberg's theory of thinking styles. For Sternberg, a thinking style is "a way of thinking. It is not an ability, but rather, a preferred way of using the abilities one has. (. . .) A style refers to how someone likes to do something" [5] (p. 8). In this sense, thinking styles are not associated with how well things are done, but the way one likes things to be done. Together with his colleague, Wagner created a thinking styles inventory for testing 13 different thinking styles [11]. Although Sternberg's thinking styles were not focused on a special discipline, he used this test in schools, universities, or further professional areas. Borromeo Ferri was always focused on the learning and teaching of mathematics; thus, she characterized the term mathematical thinking style and reconstructed in her qualitative studies at the beginning as analytic, visual, and integrated mathematical thinking styles with 15- and 16-year-old learners. The reason for this was that, at that time, the analysis of the typologies or classifications (e.g., [12–14]) of thinking and of how they were evolved and illustrated were not reconstructed with pupils at school. Based on an empirically grounded description, the different thinking styles are defined as follows [8,15]:

- Visual thinking style: Visual thinkers show preferences for distinctive internal pictorial imaginations and externalized pictorial representations as well as preferences for understanding mathematical facts and connections through holistic representations. The internal imaginations are mainly affected by strong associations with experienced situations.
- Analytical thinking style: Analytic thinkers show preferences for internal formal imaginations and for externalized formal representations. They are able to comprehend mathematical facts preferably through existing symbolic or verbal representations and prefer to proceed in a sequence of steps.
- Integrated thinking style: These persons combine visual and analytic ways of thinking and are able to switch flexibly between different representations or ways of proceeding.

In general, mathematical thinking styles have several principles, which are described by Borromeo Ferri [8,16] in more detail. Concerning the investigation presented here, especially the following two principles should be stressed, to which we will come back later, discussing the connection between performance, preference, and self-efficacy. Thus, a central principle is that MTS are not mathematical abilities, but preferences of how these abilities prefer to be used and a further principle is that MTS are attributes of the personality, because preferences are often connected with positive affects [8].

By reading the above characterizations of the three styles, one can see the composition of these different mathematical thinking styles. This means that there are two main components. Component (1) includes internal imaginations and externalized representations. Internal types are defined as individuals who mainly assimilate facts internally and who do not see the necessity for representations (except if they serve as means of communication). External types, however, make different external representations. If their internal imaginations then match the externalized representations (e.g., picture–picture) they are called congruent; if these do not match (e.g., picture–symbolic), they are called incongruent (see Table 1). Component (2) examines the process of solving the task which can be understood in a holistic way (the task is exploited from the whole to parts of it), in a dissecting way (the task is exploited from parts to the whole) and in ways combining these two "pure" ways. The following model shows all components:

Table 1. A model to describe the construction of mathematical thinking styles and their different kinds of mathematical thinking styles [16].

	Internally Oriented Types			**Externally Oriented Types**				
				Congruent			**Incongruent**	
(1) / **(2)**	**Picture**	**Symbolic**	**Mixed**	**Picture–Picture**	**Mixed**	**Symbolic–Symbolic**	**Picture/ Symbolic**	**Symbolic/ Picture**
Wholists								
Combiners								
Dissecters								

Due to the development of the mathematical thinking styles questionnaire through which it was possible to reconstruct the different styles on a quantitative level, further questions could be investigated. Thus, Borromeo Ferri [16] focused on the research question in a large-scale comparative study with n = 907 learners (15 years old) and n = 40 teachers to find out if there are correlations between preferences for certain mathematical thinking styles and mathematical performance. Involved in this study were not only Germany but also Japan and South Korea. The results concerning the above question showed a statistically significant result that those students with a preference for an analytic thinking style have the best grades. This situation helps to understand the evaluative processes in the teaching and learning of mathematics, implying that the best grades are related to preferences that do not possess a different valuation between them [5,16]. However, there are still many unanswered questions, in particular regarding the phenomenon that analytic thinkers outperform visual thinkers. What is the reason for this? Possible explanations could be that mathematical schoolbooks are mostly formal-oriented or that many teachers prefer the analytic thinking style, giving the analytic thinkers a greater advantage. These might be influence factors, but when looking at the MTS theory and their principles, Borromeo Ferri [8] already described that MTS are attributes of the personality. Consequently, we argue that the level of learners' self-efficacy could have a further explanation to why analytic thinkers perform better than visual and integrated thinkers. Until now, there has been no empirical evidence regarding this assumption. Thus, in the following, a brief overview concerning the self-efficacy construct is given.

2.2. Self-Efficacy Construct

The numerous works on self-efficacy are based on the socio-cognitive theory by Bandura [17], which defines the self-efficacy expectation as the subjective conviction that one's own means and abilities enable one to achieve a defined goal when there are barriers and obstacles along the way [18]. It entails the confidence in one's own competence to achieve goals under difficult conditions as well [17]. Thus, self-efficacy expectations regulate "the objectives of action and their level of aspiration, effort, and persistence, as well as [the] shield against competing intentions and the processing of success and failure" [18] (p. 315). During the action process, the competency experiences determined allow for a feedback possibility, which can lead to the modification of the self-efficacy expectation [18]. According to Bandura's socio-cognitive theory [17] not only cognitive processes are controlled, but also motivational, emotional and actional processes are influenced by subjective convictions, which in turn can be divided into:

- outcome expectation or outcome expectancies, and
- self-efficacy expectations or perceived self-efficacy.

The action–result expectation puts the result at the center and addresses the dependence of the action on the result. For example: in order to successfully pass the final school exam (result), certain skills are necessary, the acquisition of which leads to the result (successful graduation). Self-efficacy expectations focus on self-reference and the question of the personal availability of actions. The following statement illustrates this perspective: "I myself feel able to acquire this knowledge or to learn these skills." [19] (p. 36). Both expectations are significant for motivation and behavioral regulation, assuming that they complement each other rather than replace them [20]. The self-efficacy expectation is perceived to be low if the necessary knowledge of task processing is known, but due to a lack of talent or lack of concentration, it is not possible to work successfully [19]. Cognitions regarding one's own abilities therefore influence the action. A person feels self-effective when the success is internally stable; that is, when one can be associated with one's own competence. As a result, the self-efficacy expectation influences the choice of actions depending on the level of difficulty, the effort invested in the goal achievement process, the endurance of difficulties and barriers, and thus, indirectly, the degree of success [20]. The relationship of the construct's self-efficacy expectancy and performance is equated with the so-called 'egg chicken problem' ("who was there first—the chicken or the egg?"), which raises the question of which construct has the greater influence on the other construct or whether the self-efficacy expectation is actually only a reflection of past performance [21]. Among others, Honicke and Broadbent [22], in their article review, postulate a strong impact of self-efficacy expectations on performance. This creates the idea of deliberately influencing the self-efficacy expectation in learning environments, but this requires further insights into causality and the temporal pattern between performance and self-efficacy expectations. On the other hand, there is no doubt that performance also influences the existing self-efficacy expectation. The already mentioned "personal success experiences" and the resulting benefits are the strongest source of self-efficacy expectation and therefore the most influential [21]. Based on Honicke's and Broadbent's [22] results, it is clear that there is a strong impact of self-efficacy on performance, in the sense of the MTS theory. It is stated that MTS are not the same as abilities, because MTS are preferences on how we like to learn mathematics and not how good we are in learning mathematics, which focuses on the abilities. Thus, the distinction between preferences and abilities is crucial. If analytic thinkers have better grades than visual and integrated thinkers as Borromeo Ferri [16] found out, the question is now whether the analytic thinkers also have a higher level of self-efficacy than the other thinkers. Thus, the preference for learning mathematics analytically and being successful in mathematics by achieving good grades could cause a high level of self-efficacy. This, in turn, would mean that visual thinkers, who do not have such high grades, could have less self-efficacy. This has not yet been investigated and will be in the focus of the present investigation in this paper.

3. Research Questions and Methodology

3.1. Research Questions

The large-scale empirical study of Borromeo Ferri [16] compared Germany, South Korea, and Japan; no countries from South America were included. The authors of [23] focused on Chilean students with academic talent, recognizing preferences for the integrated thinking styles; in addition, they visualize the need for more extensive studies for the recognition of preference patterns in more heterogeneous Chilean groups, because the results of students with academic talent are not comparable with heterogeneous samples. Therefore, in this empirical study presented here, the focus was to know which preferences we could find among a wider sample of learners in Chile. At first, this is more than a simple replication by using the test instrument developed by Borromeo Ferri [16]. However, due to the discussion in the former section, the first research question is:

1. Do Chilean learners with a preference for analytic thinking have better grades than visual and integrated thinkers?

Hypothesis 1. *Given the results of Borromeo Ferri's study, Chilean learners who have a preference for the analytic thinking style also have better grades.*

Based on empirical evidence that good grades have an influence on self-efficacy, it was of great interest, whether, in particular the analytic thinkers, which have better grades than the other thinkers, having a higher level of self-efficacy or if self-efficacy is an influence factor. Thus, the second research question is:

2. Do Chilean learners with a preference for analytic thinkers have a higher level of self-efficacy?

Hypothesis 2. *Given that the students with better grades have greater confidence, analytical thinkers are expected to have greater self-efficacy.*

The scope of styles are defined as internal and external (described in [16]), whose styles pose the preference to work mathematical tasks individually or in groups, also expressed in alternatives about how to express mathematical development. In [24] exists relationships empirically evidenced between the visual thinking style and the external style from the thinking style framework [5], in addition, to relate the analytical and internal style; therefore, our third research question is:

3. Do Chilean learners with a preference for analytic thinking have a preference for the internal style, and do the visual thinkers have a preference for the external style?

Hypothesis 3. *Given the empirical results in [24], there exists a relationship between the analytical thinkers and the internal styles, and between the visual thinkers and the external styles.*

To deepen and discuss the understanding of students' responses, this research identified matches and mismatches of mathematical thinking styles' preferences between students and their mathematics teachers.

3.2. Instruments for Data Collection and Sample of the Study

The presented study is quantitatively oriented, because an evaluated test instrument is used which allows to answer the research questions.

For the evaluation of the preferred ways of mathematical thinking, the MaTHSCU instrument [16] was used, composed of nine general questions, each consisting of multiple questions measured with a four-staged Likert scale focused on the different dimensions of the mathematical thinking study (see Table 2). As an example, to know the preferred ways of mathematical thinking, the student will be considered to have a visual preference if they agree with statements similar to "I prefer visual explanations from my teacher or sketches on the board", or "while working on math problems, I mostly have visual images and I like to draw sketches", whereas if preferring analytical thinking, they will agree with "variables and formulae are helpful for me to understand mathematics", or "I prefer variables for solving mathematical problems". The reliability and validity have been analyzed in its version in English [16] and in Spanish [23]; the latter had a content validity by experts and was validated in a pilot study, yielding a Cronbach's alpha of 0.57 for visual preference, 0.75 for analytic preference, 0.69 for internal preference and 0.66 for external preference, under the "Mathematical Thinking Styles in School and Across Cultures, MaTHSCu-project" and its implementation in Chile.

Table 2. Distribution of dimensions in the instrument MaTHSCU (RBF) 2011/2012 [16].

Central Questions	Dimension	N° Items
Which way of mathematical thinking do you prefer the most?	visual	5
	analytic	5
How do you solve a mathematical task?	dissected	5
	holistic	5
Do you like to work on mathematical problems alone or do you prefer group work?	internal	4
	external	4
How confident do you feel about having to solve the following mathematics tasks?	self-efficacy	8

Similarly, the instrument for teachers aims to describe their preferences regarding the thinking dimensions, considering once again a content validity procedure by experts. This tool was applied to 10 10th grades' courses, whose ages averaged 16 years old (between 15 and 18 years); each course has 28 students on average (19 to 35); 5 of the 10 professors decided to participate in the study and answered the mentioned instrument. The 10 courses of the sample are part of five schools in the cities of Viña del Mar (3) and Valparaíso (2), in the Region of Valparaíso in Chile. Three schools are financed exclusively by the state of Chile from their municipalities, while the other two are subsidized by the state. According to the SIMCE standardized test performance category provided by the Chilean Ministry of Education, all the schools in the sample are at a medium (1) and high (4) level on the scale: insufficient, medium-low, medium, high. The total number of students surveyed was 275, where the gender distribution is 36.7% men and 63.3% women, and the distribution among teachers is 20% men and 80% women. In order to establish a measurement parameter with respect to the students' grades, the last grade of mathematics classes for each student (on a ascendent scale of 1 to 7 for Chilean students) was collected, serving as a way of knowing and evaluating the mathematical knowledge of each student, according to the vision of their teachers, information that is not possible to visualize with a standardized test. Finally, a mark average was generated for each course.

The instrument is composed of nine general questions that address specific questions, which are grouped by different dimensions and measures on the Likert scale, considering 1 as strongly agree, 2 as agree, 3 as disagree and 4 as strongly disagree, whose distribution is described in Table 2. For general questions 1, 2, 3, 4, 6 and 8, the dimensions were calculated obtaining the average of the answers associated with each question; the average has a range between 1 and 4, 1 being the most preferred dimension and 4 the least preferred. Regarding general question 5, the five dimensions were calculated with the sum of the Likert scales of the corresponding questions. In this case, the closer the sum to 2, the more the dimension is preferred (or 3 for the applying mathematics dimension), and the closer to 8 (or 12), the less preferred. On general question 7, a classification of each question was made by dichotomizing the Likert scale from four to two categories: 0 = agree and 1 = disagree, obtaining for each question the percentage distribution of them. The instrument was applied by the researchers and answered in person with pencil and paper, with a duration of approximately 15 to 20 min in students and teachers; all of them accepted to participate voluntarily. During the data collection, all doubts about the experimentation and the instrument were clarified before and during the data collection, in order to ensure the accuracy of the understanding of the instrument and the ethical aspects of the research.

3.3. Data Analysis Procedure

In order to establish the existence of significant differences between the different dimensions mentioned in Table 2, the Kruskal–Wallis non-parametric statistical test was used prior to the Kolmogorov–Smirnov normality test. Accordingly, it was obtained that at least one of the courses did not have a normal distribution, which justifies the calculation of the Kruskal–Wallis statistic of lack of normal distribution in the variables (dimensions), resulting in the existence of significant difference between the courses in the dimensions of thinking styles: visual, dissected, and self-efficacy. Then, it was considered to perform an analysis according to the grades obtained by the students in their last evaluation, separating them by low achievement [1.0–3.9], average achievement [4.0–6.0] and high achievement [6.1–7.0], allowing greater clarity in the analysis, in addition to representing an adequate separation according to the approval and assessment indicators used in Chile. Due to the lack of normal distribution, the Kruskal–Wallis test was used to prove the existence of a significant difference between the ranges of grades and the measurement of mathematical thinking styles. The software used for the statistical analysis procedure is ®RStudio.

For the global analysis of mathematical thinking styles, descriptive statistical analyzes for cross-dimensions with sociodemographic variables were performed; in addition, Spearman's correlation was applied (due to the lack of normality in the data).

4. Results

4.1. Global Analysis

To answer the research questions outlined above, Table 3 is presented, corresponding to all the correlations valid in the study from all sample students, without considering the separation of sample by courses.

One of the categories of interest, according to the first research question, are the grades, and how these are correlated with other categories. In this regard, it is evidenced that the highest correlation that exists is the grades with the preference of analytical thinking. This means that the best way to obtain better grades is to prefer analytical thinking, since it is better valued in the evaluation processes of the school system. In addition, it is noted that students who prefer the visual thinking style are not in the same conditions as those who prefer analytical thinking, since the study does not show a correlation between visual preference and grades, considering that there is no value dimension in the style preferences. This result complicates students' learning opportunities, since a person possesses a diversity of preferences for the treatment and development of mathematical knowledge and shows a route over others that has a greater appreciation in the educational system.

Table 3. All the correlations between categories for $n = 275$ students. All scores assume a p-value < 0.05, except those marked with *, which have a p-value < 0.01.

	Analytic	**Visual**	**Holistic**	**Dissected**	**Internal**	**External**	**Self-Efficacy**	**Grades**
analytic	1	-	-	-	-	-	-	-
visual	-	1	-	-	-	-	-	-
holistic	0.248 *	0.152	1	-	-	-	-	-
dissected	0.298 *	0.264 *	0.332 *	1	-	-	-	-
internal	0.396 *	-	0.189 *	0.222 *	1	-	-	-
external	-	0.246 *	0.229 *	0.185 *	−0.363 *	1	-	-
self-efficacy	0.418 *	0.126	0.246 *	0.340 *	0.363	-	1	-
grades	0.322 *	-	0.145	0.203 *	0.198 *	-	0.315 *	1

In terms of the preferences of thinking styles and grades, the preferences of students who have low, average, and high achievement have been analyzed, recognizing significant differences in their preferences about the analytic thinking. Initially, the preference for analytical thinking style was measured by these groups of students, which is reflected in Figure 1, showing that about three quartiles of students with high achievement grades prefer the analytical style; furthermore, this group has a low dispersion, being significantly different to students with low and average achievement grades, since both groups have an almost total dispersion. For transparent similarities, the preference of the visual style in these three groups of students was analyzed (low, average, and high achievement grades, and preferences for the visual thinking), without finding significant differences to expose, since the three medians of the groups are in the range of 2.0 to 2.4 and quartiles two and three are distributed in a similar way. In addition, the three dispersions are similarly visualized.

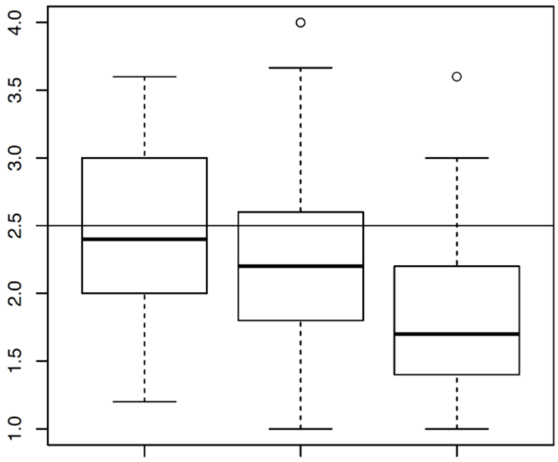


Figure 1. Preferences for analytical thinking in students with low achievement (left), average achievement (center) and high achievement (right) grades.

These results in the Chilean context strengthen and validate the hypothesis of Borromeo Ferri's study (learners who have preferences for an analytical thinking style have better grades), since in correlational terms it is possible to identify the preference of the analytical style and grades (Table 3); in addition, the group of students who have better grades coincides with having a greater preference for this style.

To address the second research question, the correlation between grades and self-efficacy as the second highest correlation in the mark's category is evidenced in Table 3. This means that students who have higher grades may have greater conviction in achieving the goals defined by their own means. However, this result should be analyzed based on the high correlation between the analytical thinking style and the student's self-efficacy—which is the highest correlation between the valid correlations of the analytical thinking style—because it is expected that students who prefer an analytical thinking style, and possibly have better grades (from the results shown above), also have a greater conviction in being able to achieve certain academic goals.

This relationship between the three categories mentioned above invites us to recognize in the data how the other categories have been correlated and what correlations stand out to understand how style preferences influence grades. In this regard, looking at the correlations with analytical thinking, the second highest after self-efficacy is the internal style, which is closer to a more individualized work when performing tasks or mathematical problems. However, the internal style is also significantly correlated with self-efficacy, being possible to describe a profile of preferences of students who have better grades: the grades are correlated with the analytical and self-efficacy as well as the internal style, which is described in Figure 2. For solving mathematical problems, there are no major differences in correlation between holistic and dissecting inclinations for analytical thinking, although it is emphasized that both have a positive correlation with respect to analytical thinking, with greater strength in dissecting preference.

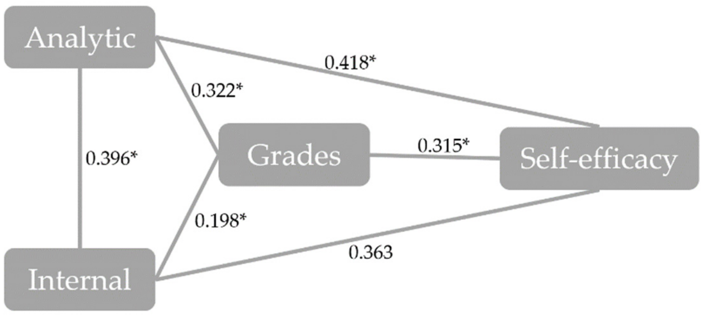


Figure 2. Preference profile for students with high achievement grades of the sample. The correlations marked with * have a p-value < 0.01.

To sum up, the high correlation between analytical thinkers and self-efficacy affirmatively answers the second research question, although it is visualized that recognizing preferences for obtaining high grades through mathematical thinking styles implies not only the preference for analytical thinking, but also the preference for the internal style. In this way, self-efficacy, although it is possible for the student to possess, is strengthened by obtaining high achievement grades.

To address the third research question, all possible two-to-two correlations between analytical, visual, internal, and external analytical styles are analyzed, which have been described in the Table 3, in order to recognize the preferred environment for the thinker's duality to work with mathematics. The results show that there are no correlations between the analytical and external preferences, and visual and internal, and on the contrary, correlations are evident between the visual and external style, and between the analytical and internal thinking, the latter being the second largest correlation between analytical thinkers. Indeed, this means that there is a correlation between these preferences in Chilean apprentices; although it is more powerful for the analytical case, the idea of establishing a relationship between the visual thinker and their preferred way of working with mathematics is emerging, which is closer to group forms of work and the joint elaboration of mathematical tasks, and at the same time, the analytical thinker has a greater preference for more individualized work in the process of mathematical work.

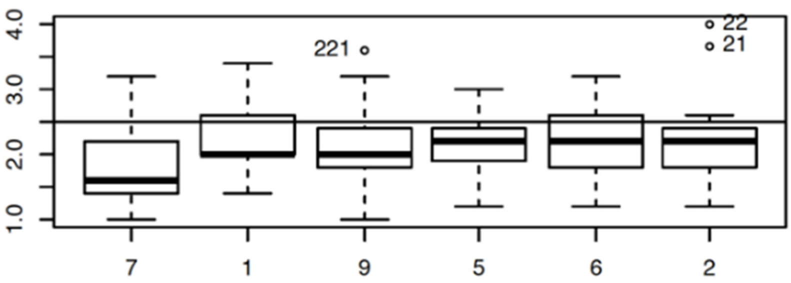
A result that corresponds to what has been said above is the high negative correlation between internal and external style, an expected situation from the definition of each of the mathematical thinking styles, and the third hypothesis of this research.

This last result establishes a clear relationship, which affirmatively answers the third research question, and which proposes to go beyond a preference, since relations are established that, from the inductive position, invite us to recognize beyond a specific style, a profile about the preferences of the learners working in mathematics.

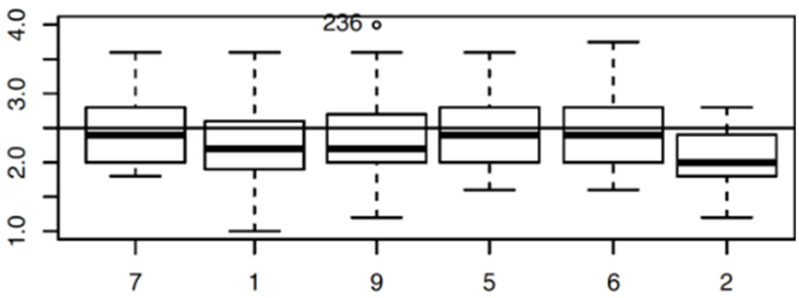
4.2. Local Analysis: Courses

By averaging the grades of the students grouped by their courses, the three courses with better grades (C7, C1, C9; whose grades are 6.3, 5.4 and 5.2, respectively) and worse grades (C5, C6, C2; whose grades are 4.4, 4.1 and 3.9, respectively) have been identified.

Regarding the MTS, Figure 3a shows that there are preferences of analytical thinking in both groups of courses, without significant differences, managing to verify that more than 50% of the students of the courses with the best grades have a preference for analytical thinking, compared to 50% of students of the courses with the worst grades. In addition, C7 stands out, presenting the highest preference. Similarly, both groups show preferences of visual thinking (Figure 3b), highlighting the low dispersion or their higher concentration of preferences in C7, and which, at the same time, is the lowest preference with respect to the comparison group, and in general, on 10 courses.



(**a**)



(**b**)

Figure 3. (**a**) Preferences for the analytical thinking style in courses with better and worse grades. (**b**) Preferences for the visual thinking style in courses with better and worse grades.

This makes us understand that the course with the highest grades (C7) has the greatest preference for analytical thinking, and at the same time, has the least preference for visual thinking. In parallel, the course with the lowest grades (C2) is one of the least preferred analytical thinking, and at the same time, the one that prefers visual thinking, identifying the first research question at the local level.

For ways to solve a mathematical problem, either holistically or dissecting a problem, most students for each course prefer both ways of proceeding with different degrees of preference and with slight differences in the dispersion. The results of courses with better and worse grades can be seen in Figure 4a,b, although there is no evidence that such preferences are related to grades.

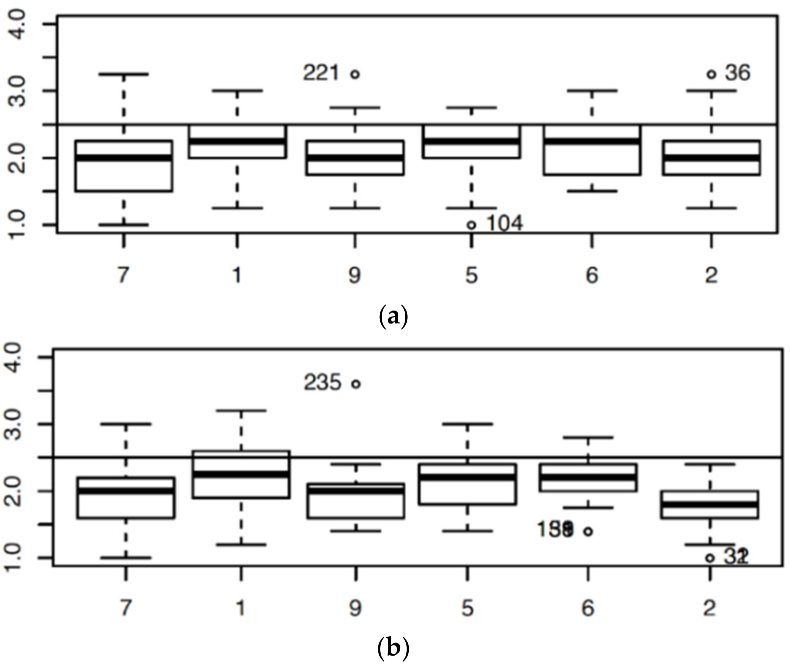


Figure 4. (**a**) Preferences for the holistic way in courses with the best and the worst grades. (**b**) Preferences for the dissected way in courses with the best and the worst grades.

Similarly, regarding the preferences of working with mathematical tasks in a group or individual (external or internal preference), there are no significant differences between the two analyzed groups (see Figure 5a,b), except for the high dispersion of courses with the worst grades in the internal preference. In this situation, the dichotomous virtue of course groups is recognized by preferring certain ways of working in mathematics, whether in ways of approaching tasks (holistic or dissected), or preferring to work or not with more people (external or internal).

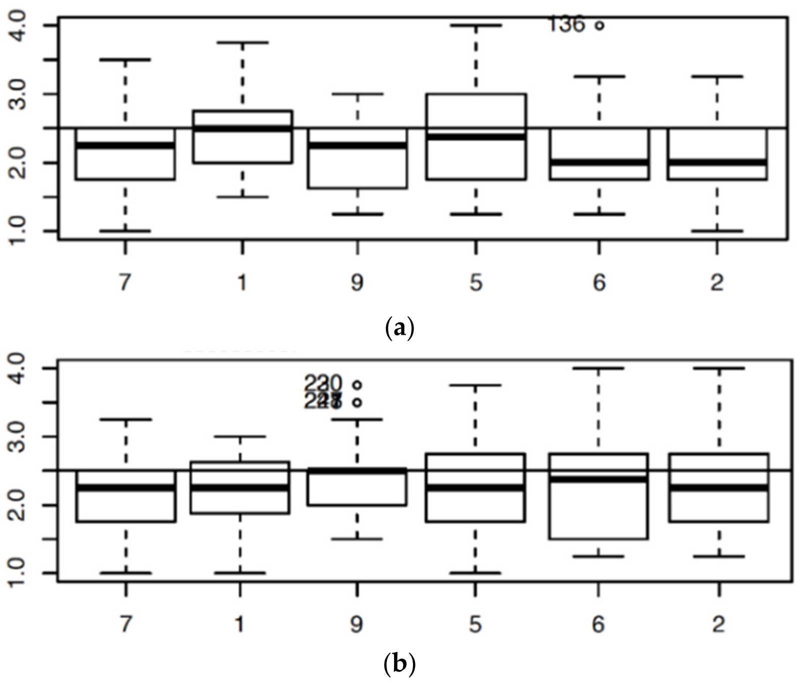


Figure 5. (**a**) Preferences for the external way in courses with the best and the worst grades. (**b**) Preferences for the internal way in courses with the best and the worst grades.

It has also measured the self-efficacy that students have, referring to how confident they feel when solving certain mathematical problems, ranging from the understanding of a percentage discount on the sale of a product, proportionality in various situations, to the resolution of second-degree equations, topics already treated according to the national curriculum. As shown in Figure 6, high confidence in courses with high and low achievement grades is recognized, highlighting that both C7 and C2 have high confidence in performing mathematical tasks, although C1, C9, C5 and C6 pose a certain similarity of preferences.

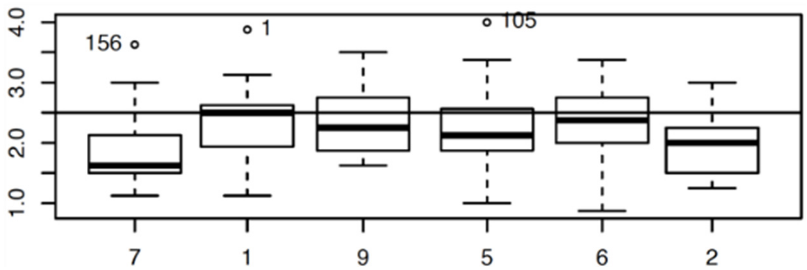


Figure 6. Self-efficacy to solve a mathematical task.

4.3. Local Analysis: Courses and Their Teachers

From the mathematical thinking styles, it is of interest to recognize the match and mismatch between the teachers' and learners' preferences since this information allows the understanding of the cases under study. As stated above, only the first five teachers have responded to the instrument, coinciding with three teachers from the local sample under study, as shown in Table 4:

Table 4. Descriptive results about some teachers of sample. All the results are on a scale of 1 to 4.

Dimensions	T1C	T5C	T2C
Analytic	2.8	2.6	2.4
Visual	1	1.4	1
holistic	2	3.25	2.25
Dissected	2	1.4	2.2
Internal	1.5	2.75	2
External	2.75	1.25	2.75

4.4. C1T

C1T (course 1 teacher) has a strong preference for visual thinking, responding with the maximum score; in addition, it is closer without the preference for the analytical thinking, which is incongruent with the preferences of the students; since there is no preference for polarized as that of his/her teacher, a greater inclination towards analytical thinking is recognized (see Figure 3a,b).

Regarding the ways to proceed when solving a mathematics task, the teacher's results show a similar preference over both possibilities, which is congruent with the preferences that the students expose (Figure 4a,b). Similarly, T1C has a greater inclination for internal style, a situation that also occurs with its students (Figure 5a,b).

4.5. C5T and C2T

Both teachers show a strong preference for visual thinking, while their students have neutrality over style; on the other hand, analytical thinking has a neutral preference on the part of teachers, while their students have a greater preference. This situation highlights the incongruency between mathematical thinking styles.

C5T shows a greater preference for dissecting mathematical problems, disagreeing on holistic approaches, while the students do not show greater polarization between holistic or dissected preferences. On the other hand, C2T shows a slight preference for both styles, while the majority of its students show a preference for the holistic approach and a strong preference for the dissection of homework. In this case, the results show incongruencies in both situations.

Finally, C5T has a strong preference for external style, disagreeing with internal style, while the students prefer both. In this case, the results are congruent with the teacher's practice and student's preferences. On the other hand, C2T shows a preference for internal style, being congruent with student's preferences.

5. Limitations of the Study

Regarding the limitations of the present study, both the teachers and their students who have decided to participate represent only a part of what happens in the classrooms of Chile. However, the assessment of mathematical preferences has been predominantly directed by standardized tests, which promote aspects of visualization and representation closer to formal mathematics; therefore, it is difficult to find results significantly different from those expressed in this research. Similarly, the associations between self-efficacy and analytical, internal-analytic, and external-visual preference are also limited towards the participants of the present study, although previous research has shown the coherence of such associations in other latitudes and contexts.

6. Discussion

A result of interest was the positive correlation between the analytical style and the grades. This may be due to the assessment made by teachers in their evaluation processes on formal and analytical mathematics, since this is often usually used in standardized tests that are applied in Chile, which prompts a greater use of formal and analytical representation of mathematics over visual. In this regard, the Chilean school context has given evidence that the treatment of the compulsory curriculum in schools is often guided by obtaining its good results by economic gratifications, for example, the SIMCE test, implying a concentration of contents and types of tasks that negatively affect teaching [3]. In this sense, the prioritization of formal and analytical mathematics in the evaluation processes predominates in the improvement of grades.

The prioritization of analytical thinking can be explained by the vision given to the mathematics teacher formation. In this regard, it is usual to funnel the image of the mathematics teacher as they were a mathematician, recognizing that the assessment of a better student is the one that is closest to the formal mathematics of the mathematician community. In this position, a democratic sense of mathematical thinking style is omitted, since the compulsory education instance does not have as a horizon the formation of mathematicians but is closer to the formation of citizens empowered by mathematics to be productive citizens of our democratic society [25], allowing the diversity of preferences on the development of mathematical knowledge.

Confidence in self-efficacy does not necessarily mean that high achievement grades are obtained. While there is a positive correlation when all students are analyzed, we also find high self-efficacy results in both the best and the worst course, specifically in C2 and C7. In short, when one has high achievement grades, one will probably have high levels of confidence in self-efficacy, but high levels of confidence in self-efficacy do not necessarily mean high grades.

Regarding the search for matches and mismatches in the courses, it is emphasized that being C1, one of the courses with better grades, only one mismatch is described: the visual thinking of the teacher, and analytical thinking of the students. An explanation could be its importance in the qualification processes since its evaluations can have a strong inclination for analytical thinking without being congruent with the teacher's preference. There are possible other explanations for this, for example, the math teachers that the students have had in previous courses, inheriting a learning of successful styles for the evaluation processes. Therefore, it is of interest for future studies to know the types of evaluations carried out by teachers, since it is possible that teachers' preferences are not related to their evaluation processes; in this case, it would be possible to explain the inconsistencies recognized between teachers, students, and their qualifications.

Finally, the significant correlation between the grades and the analytical style in this study (Chile: 0.322) is not a unique quality of Chile, since it is possible to recognize similar results in other latitudes such as "(...) German: 0.53; Japan: 0.14; South Korea: 0.29 " [16] (p. 163), emphasizing that the assessment of analytical style in classrooms and in the evaluation processes is a more general phenomenon that is observable in multiple educational environments, regardless of whether it is similar or different from them. Of course, this should be discussed in other studies that provide more evidence to the mainstreaming of the phenomenon, in such a way to expand learning and teaching opportunities.

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