From the Quantum Mechanical State Space to the Position and Momentum Spaces through a Simple Relation

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Abstract

In an attempt to address the need for an alternative presentation of the quantum mechanical position and momentum spaces, we provide a presentation that is more constructive and less calculative than those found in literature. Our approach is based on a simple, intuitively understood relation that expresses the physical equivalence of the quantum mechanical state space to the position and momentum spaces. With this work, we hope to offer a perspective complementary to those found in standard quantum mechanics textbooks.

Keywords: Position space, momentum space, state space, position operator, momentum operator.

INTRODUCTION

In 1926, Schrödinger developed wave mechanics, a formulation or representation of quantum mechanics which is based on the idea that the quantum systems are described by wave functions satisfying a wave equation, which is known as the Schrödinger equation (Aspect & Villain, 2017; Gieres, 2000). Schrödinger also demonstrated the physical equivalence of wave mechanics to matrix mechanics, the other known at that time formulation of quantum mechanics, that Heisenberg, Born, and Jordan had developed (Aspect & Villain, 2017; Gieres, 2000). In the following years until 1931, Dirac, Jordan, and von Neumann developed a representation-free or invariant formalism of quantum mechanics, according to which each quantum system is associated with a separable, infinite-dimensional, complex Hilbert space, which is known as state space (Gieres, 2000; Van Hove, 1958). The elements of the state space are Dirac kets of finite norm representing possible bound states of the examined system (Gieres, 2000).

In the framework of the Hilbert space formulation of quantum mechanics, the wave functions of Schrödinger's wave mechanics are square-integrable functions belonging to a Hilbert space that is isomorphic[1], thus physically equivalent, to the state space, and it is known as position or momentum space, depending on whether the wave functions are expressed in terms of position or momentum, respectively. The position and momentum spaces are also referred to as position and momentum representations, respectively.

Apart from their significance in the historical development of quantum mechanics, the position and momentum spaces play important role in the process of teaching – thus of learning too – and also of applying the quantum theory to physical systems, as it can be seen by referring to standard quantum mechanics textbooks (Griffiths, 2005; Merzbacher, 1998; Sakurai & Napolitano, 2011).

However, as demonstrated in (Marshman & Singh, 2013 & 2015), students face difficulties when practicing quantum mechanics in different spaces. Therefore, a need exists for an alternative presentation of the quantum mechanical position and momentum spaces. To this end, we reformulate the physical equivalence of the state space to the position and momentum spaces in terms of a simple relation and provide an intuitively geometric presentation of the latter spaces starting from the former space. For simplicity and clarity, we examine a one-particle system, and the emphasis is given on the one dimensional case[2], i.e. the case where the particle moves on the real line, as the generalization to three dimensions is straightforward.

From the State Space to the Position Space through a Simple Relation

We consider a particle moving, under the influence of a potential, on the real line. The state space of our system is an abstract Hilbert space of Dirac kets. The position space is then realized by taking projections of ket states on the directions – on the axes – defined by the position eigenstates of the particle. Each position eigenstate $|x\rangle$, $x \in \mathbb{R}$, defines a direction – an axis – by means of the projection operator $|x\rangle\langle x|$. We call this axis the x-axis. We note that the x-axis does not belong either to the state space or to the position space. It belongs to a hyperspace, i.e. a bigger space, of the state space (see below). In physical space, the x-axis corresponds to a point x on the real line, along which the particle moves. Another position eigenstate, say $|x'\rangle$, $x' \neq x$, defines, by means of the projection operator $|x'\rangle\langle x'|$, another axis, the x'-axis,

[1] *Two Hilbert spaces H_1, H_2 are isomorphic if there exists an isomorphism relating them, i.e. a linear mapping U from H_1 to H_2 that it is everywhere defined on H_1, it is onto on H_2, and it preserves the scalar product, i.e. U is a unitary operator from H_1 to H_2. Two isomorphic Hilbert spaces are considered as physically equivalent.*

[2] *We note that the position and momentum spaces of a particle moving in one dimension are often referred to as one-dimensional position and momentum spaces, respectively, and similarly for a particle moving in three dimensions, but this does not mean that the two spaces are one dimensional (or three dimensional, respectively). They are infinite-dimensional, as is the state space.*

which, in physical space, corresponds to the point $x' \neq x$ on the real line, along which the particle moves. The position of the particle is an observable quantity, thus the position operator for the particle is Hermitian, and then its eigenstates belonging to different eigenvalues are orthogonal (Dirac, 1947; Griffiths, 2005; Merzbacher, 1998). As a result, the x'-axis is orthogonal to the x-axis.

We consider an element of the state space, say the state $|\psi\rangle$, and the state $\hat{A}|\psi\rangle$, where $\hat{A}$ is an operator acting on the state space and $|\psi\rangle$ belongs to the domain of $\hat{A}$. The states $|\psi\rangle$ and $\hat{A}|\psi\rangle$ are respectively projected on the x-axis as $\langle x|\psi\rangle|x\rangle$ and $\langle x|\hat{A}|\psi\rangle|x\rangle$. The signed magnitudes of the two projections are, respectively, $\langle x|\psi\rangle$ and $\langle x|\hat{A}|\psi\rangle$. We identify the element $\langle x|\psi\rangle$ as the value of the position-space wave function at the point x, i.e.

$$\psi(x) \overset{\text{def.}}{=} \langle x|\psi\rangle \tag{1}$$

The element $\langle x|\hat{A}|\psi\rangle$ is the projection of the state $\hat{A}|\psi\rangle$ on the x-axis. The state space is isomorphic to the position space, thus the description of the particle in state space is equivalent to its description in position space.

We express the equivalence of the state space to the position space by the following relation

$$\langle x|\hat{A}|\psi\rangle = \hat{A}(x)\langle x|\psi\rangle \tag{2}$$

where the operator $\hat{A}(x)$ is the expression of the operator $\hat{A}$ in position space. Since the element $\langle x|\psi\rangle$ is the position-space wave function (1), (2) is also written as

$$\langle x|\hat{A}|\psi\rangle = \hat{A}(x)\psi(x) \tag{3}$$

Thus, if the state-space operator $\hat{A}$ is expressed, in position space, by the operator $\hat{A}(x)$, then the state $\hat{A}|\psi\rangle$ is described, in position space, by the wave function $\hat{A}(x)\psi(x)$, where $\psi(x)$ is the wave function describing the state $|\psi\rangle$.

Position and Momentum in Position Space

The expression $\hat{x}(x)$ of the position operator $\hat{x}$ in position space can be derived by considering the element $\langle x|\hat{x}|\psi\rangle$, which, by means of (3), is written as

$$\langle x|\hat{x}|\psi\rangle = \hat{x}(x)\psi(x) \tag{4}$$

Since the position is an observable quantity, the position operator has a complete set of eigenstates (Dirac, 1947; Griffiths, 2005; Merzbacher, 1998). Thus, the set $\{|x'\rangle\}_{x'\in\mathbb{R}}$ spans the state space, and also consists of orthogonal eigenstates, because the position operator is Hermitian. As a result, it holds the closure relation

$$\int_{-\infty}^{\infty} dx' |x'\rangle\langle x'| = \hat{1},$$

i.e. the sum of all projection operators $|x'\rangle\langle x'|$, $x'\in\mathbb{R}$, is equal to the identity operator. Besides, the orthogonality of two arbitrary position eigenstates $|x\rangle$ and $|x'\rangle$ is expressed by the relation[3]

$$\langle x'|x\rangle = \delta(x'-x),$$

where $\delta(x'-x)$ is the delta function with support at $x'=x$.

The arbitrary position eigenstate $|x'\rangle$ corresponds, in position space, to the wave function $\langle x|x'\rangle$, which is the delta function $\delta(x-x')$. Since the state and position spaces are isomorphic, the norms of $|x'\rangle$ and $\delta(x-x')$ are equal. The norm of $\delta(x-x')$ is infinite, since

$$\int_{-\infty}^{\infty} dx|\delta(x-x')|^2 = \int_{-\infty}^{\infty} dx\delta(x-x')\delta(x-x') = \delta(x'-x') = \infty$$

Then, the position eigenstates have infinite norm and thus they do not belong to the state space, because, as we have mentioned, the state space contains states of finite norm, which correspond to square-integrable position-space wave functions describing physical states. This means that the position eigenstates are not physical states. They are meant as generalized kets that span the state space as elements of a hyperbasis, i.e. a basis belonging to a bigger space that contains the state space and is accommodated in a construction called rigged Hilbert space (de la Madrid, 2005).

By means of the closure relation of the position eigenstates, the element $\langle x|\hat{x}|\psi\rangle$ is written as

$$\langle x|\hat{x}|\psi\rangle = \langle x|\hat{x}\left(\int_{-\infty}^{\infty} dx'|x'\rangle\langle x'|\right)|\psi\rangle = \int_{-\infty}^{\infty} dx'\langle x|\hat{x}|x'\rangle\langle x'|\psi\rangle,$$

3 The spectrum of the position operator is continuous, thus the place of the Kronecker delta, which is used to express the orthogonality of discrete-basis states, has been taken by the delta function.

i.e.

$$\langle x|\hat{x}|\psi\rangle = \int_{-\infty}^{\infty} dx' \langle x|\hat{x}|x'\rangle\langle x'|\psi\rangle \tag{5}$$

Since $|x'\rangle$ is a position eigenstate with eigenvalue x', $\hat{x}|x'\rangle = x'|x'\rangle$, and thus

$$\langle x|\hat{x}|x'\rangle = \langle x|x'|x'\rangle = x'\langle x|x'\rangle = x'\delta(x-x')$$

Also, the element $\langle x'|\psi\rangle$ is the wave function $\psi(x')$ at x'. Thus, (5) reads

$$\langle x|\hat{x}|\psi\rangle = \int_{-\infty}^{\infty} dx'x'\delta(x-x')\psi(x') = x\psi(x),$$

i.e.

$$\langle x|\hat{x}|\psi\rangle = x\psi(x)$$

Comparing the last equation with (4) yields

$$\hat{x}(x)\psi(x) = x\psi(x),$$

and since the wave function is arbitrarily chosen,

$$\hat{x}(x) = x$$

Thus, in position space, the position operator is the position coordinate. The expression $\hat{p}(x)$ of the momentum operator $\hat{p}$ in position space can then be derived by the canonical commutation relation for position and momentum

$$[\hat{x}, \hat{p}] = i\hbar \tag{6}$$

which, in position space, reads

$$\left[\hat{x}(x), \hat{p}(x)\right] = i\hbar \tag{7}$$

The momentum operator $\hat{p}(x)$ is linear and Hermitian. We observe that a “solution” for $\hat{p}(x)$ to (7) is

$$\hat{p}(x) = -i\hbar \frac{d}{dx} \tag{8}$$

If there exists another solution to (7), say $\hat{P}(x)$, then

$$\left[\hat{x}(x), \hat{p}(x)\right] = \left[\hat{x}(x), \hat{P}(x)\right]$$

or

$$\left[\hat{x}(x), \hat{p}(x) - \hat{P}(x)\right] = 0,$$

i.e. the operator $\hat{p}(x) - \hat{P}(x)$ commutes with the position operator. Then, since $\hat{x}(x) = x$, the operator $\hat{p}(x) - \hat{P}(x)$ must be a function of x, i.e.

$$\hat{p}(x) - \hat{P}(x) = f(x)$$

Besides, if $p(x)$ is a momentum eigenfunction in position space, with momentum p, then

$$\hat{p}(x)\, p(x) = pp(x)$$

and

$$\hat{P}(x)\, p(x) = pp(x)$$

and thus, subtracting the two previous equations,

$$f(x)\,p(x)=0$$

As an eigenfunction, $p(x)$ cannot be the zero function, and then from the last equation we derive that $f(x)$ is the zero function, and thus $\hat{P}(x)=\hat{p}(x)$, which means that the solution (8) is unique.

To summarize, in position space, the position operator is the position coordinate, while the momentum operator is the differential operator (8).

The momentum eigenfunctions $p(x)$ in position space are then derived by solving the momentum eigenvalue equation in position space, which, by means of (8), reads

$$-i\hbar p'(x)=pp(x) \tag{9}$$

and it is easily solved by separation of variables and integration, to give

$$p(x)=A\exp\left(\frac{ipx}{\hbar}\right) \tag{10}$$

where A is a complex constant that does not depend either on x or on p [4]. Since $p(x)$ is not square-integrable, as $|p(x)|$ is constant, the constant A cannot be calculated by a normalization condition.

To calculate the constant A, we proceed as follows: similarly to the wave function (1), we write the momentum eigenfunction $p(x)$ in position space as

$$p(x)\overset{\text{def.}}{=}\langle x|p\rangle \tag{11}$$

where $|p\rangle$ is a momentum eigenstate. Since the wave function $p(x)$ is not square integrable, the ket $|p\rangle$ has infinite norm. As it happens with the position eigenstates, the momentum eigenstates are also meant as generalized kets not belonging to the state space, but since the momentum operator represents an observable quantity, i.e. the momentum, the momentum eigenstates span the state space and also, they are orthogonal, which means that $\langle p'|p\rangle=\delta(p'-p)$.

[4] We note that, in position space, the position x is a variable, while the momentum p is a parameter. In momentum space, the opposite holds.

Using the closure relation of the position eigenstates, the element $\langle p' | p \rangle$ is written as

$$\langle p' | p \rangle = \langle p' | \left(\int_{-\infty}^{\infty} dx |x\rangle\langle x| \right) | p \rangle = \int_{-\infty}^{\infty} dx \langle p' | x \rangle \langle x | p \rangle = \int_{-\infty}^{\infty} dx \langle x | p' \rangle^* \langle x | p \rangle ,$$

where, in the last equality, we used the conjugate symmetry of the scalar product (the asterisk denotes complex conjugation). Thus,

$$\langle p' | p \rangle = \int_{-\infty}^{\infty} dx \langle x | p' \rangle^* \langle x | p \rangle ,$$

By means of (11) and then (10), the last equality reads

$$\langle p' | p \rangle = |A|^2 \int_{-\infty}^{\infty} dx \exp\left(\frac{i(p-p')x}{\hbar} \right),$$

and then by the orthogonality of the eigenstates $|p\rangle$ and $|p'\rangle$, we obtain

$$|A|^2 \int_{-\infty}^{\infty} dx \exp\left(\frac{i(p-p')x}{\hbar} \right) = \delta(p'-p) \tag{12}$$

To proceed, we'll use the following integral representation of the delta function

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} du \exp(ivu) = \delta(v),$$

where v is a real parameter. Setting $v = p - p'$, the previous equation reads

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} du \exp(i(p-p')u) = \delta(p-p') \tag{13}$$

The delta function is even, i.e. $\delta(p-p') = \delta(p'-p)$, thus comparing (12) and (13) yields

$$|A|^2 \int_{-\infty}^{\infty} dx \exp\left(\frac{i(p-p')x}{\hbar} \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \exp(i(p-p')u) \tag{14}$$

Finally, changing the variable x to $y = x/\hbar$, the integral on the left-hand side reads

$$\int_{-\infty}^{\infty} \hbar dy \exp\left(i\left(p-p'\right)y\right),$$

which is equal to the integral on the right-hand side of (14) times the reduced Planck constant, as the integration variable is dummy and can be renamed to u. Thus, (14) finally gives

$$|A| = \frac{1}{\sqrt{2\pi\hbar}},$$

and up to a constant phase, we end up to

$$A = \frac{1}{\sqrt{2\pi\hbar}}$$

The momentum eigenfunction (10) then reads

$$p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right) \tag{15}$$

or, using the definition (11),

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right) \tag{16}$$

Besides, from the conjugate symmetry of the scalar product, we have $\langle p|x\rangle = \langle x|p\rangle^*$, and then, by means of (16), we obtain

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\frac{ipx}{\hbar}\right) \tag{17}$$

The element $\langle p|x\rangle$ is the projection of the arbitrary position eigenstate $|x\rangle$ on the arbitrary momentum eigenstate $|p\rangle$. Then, similarly to (11), we identify the element $\langle p|x\rangle$ as the position eigenfunction $x(p)$ in momentum space, i.e.

$$x(p) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\frac{ipx}{\hbar}\right) \tag{18}$$

The Momentum Space

Similarly to the position space, the momentum space of our particle is realized by taking projections of kets on the directions – on the axes – defined by the momentum eigenstates $|p\rangle$, $p \in \mathbb{R}$, by means of the projection operators $|p\rangle\langle p|$. Then, similarly to (1), the momentum-space wave function $\tilde{\psi}(p)$ describing a state $|\psi\rangle$ in momentum space is defined as

$$\tilde{\psi}(p) \overset{\text{def.}}{=} \langle p|\psi\rangle \tag{19}$$

As the position space, the momentum space is also physically equivalent to the state space. Thus, similarly to (3), if $\hat{A}$ is an operator acting on state space, then the physical equivalence of the state space to the momentum space implies that

$$\langle p|\hat{A}|\psi\rangle = \hat{A}(p)\tilde{\psi}(p) \tag{20}$$

where the operator $\hat{A}(p)$ is the expression of the operator $\hat{A}$ in momentum space.

We can now express the momentum-space wave function in terms of the position-space wave function. Indeed, inserting the closure relation of the position eigenstates on the right-hand side of (19), between the bra $\langle p|$ and the ket $|\psi\rangle$, we obtain

$$\tilde{\psi}(p) = \langle p|\left(\int_{-\infty}^{\infty} dx|x\rangle\langle x|\right)|\psi\rangle = \int_{-\infty}^{\infty} dx\langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\exp\left(-\frac{ipx}{\hbar}\right)\psi(x),$$

where, in the last equality, we used (1) and (17). Thus, the momentum-space wave function is the Fourier transform of the position-space wave function, i.e.

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\exp\left(-\frac{ipx}{\hbar}\right)\psi(x)$$

In the same way, inserting the closure relation of the momentum eigenstates[5] on the right-hand side of (1), and using (16) and (19), we write the position-space wave function as the inverse Fourier transform of the momentum-space wave function, i.e.

[5] As the position eigenstates, the momentum eigenstates also satisfy a closure relation.

$$\psi(x)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}dp\exp\left(\frac{ipx}{\hbar}\right)\tilde{\psi}(p)$$

As we did to find the expression of the position operator in position space, the expression $\hat{p}(p)$ of the momentum operator in momentum space can be derived by considering the element $\langle p|\hat{p}|\psi\rangle$, where $|\psi\rangle$ is an arbitrary state of our particle. Using (20), the previous element is written as

$$\langle p|\hat{p}|\psi\rangle=\hat{p}(p)\tilde{\psi}(p) \tag{21}$$

Also, using the closure relation of the momentum eigenstates, the element $\langle p|\hat{p}|\psi\rangle$ is written as

$$\langle p|\hat{p}|\psi\rangle=\langle p|\hat{p}\left(\int_{-\infty}^{\infty}dp'|p'\rangle\langle p'|\right)|\psi\rangle=\int_{-\infty}^{\infty}dp'\langle p|\hat{p}|p'\rangle\langle p'|\psi\rangle$$

The ket $|p'\rangle$ represents a momentum eigenstate, with momentum p', thus $\hat{p}|p'\rangle=p'|p'\rangle$, and the element $\langle p|\hat{p}|p'\rangle$ reads $p'\langle p|p'\rangle$, and since the momentum eigenstates are orthogonal, the last expression reads $p'\delta(p-p')$. Also, from (19), the element $\langle p'|\psi\rangle$ is the momentum-space wave function $\tilde{\psi}(p')$. Then, performing the delta-function integration, we end up to

$$\langle p|\hat{p}|\psi\rangle=p\tilde{\psi}(p)$$

Comparing the last equation with (21) and taking into account that the wave function $\tilde{\psi}(p)$ is arbitrary, we obtain

$$\hat{p}(p)=p\,,$$

i.e. in momentum space, the momentum operator is the momentum coordinate.

To derive the expression $\hat{x}(p)$ of the position operator in momentum space, we can use the reasoning we employed to derive the momentum operator in position space. Alternatively, we can use the position eigenfunctions we have already calculated in (18). Thus, the position eigenvalue equation in momentum space reads

$$\hat{x}(p)\exp\left(-\frac{ipx}{\hbar}\right)=x\exp\left(-\frac{ipx}{\hbar}\right),$$

with a solution for $\hat{x}(p)$

$$\hat{x}(p) = i\hbar \frac{d}{dp}$$

It can be easily seen that the operators $i\hbar\, d/dp$ and p satisfy the canonical commutation relation (6) in momentum space. Therefore, in momentum space, the position operator is the differential operator $i\hbar\, d/dp$, while the momentum operator is the momentum coordinate.

Example

As an example, we'll show that the stationary Schrödinger equation is the energy eigenvalue equation in position space.

Considering a particle of mass m in a one-dimensional potential, its Hamiltonian in state space reads

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}),$$

where $\hat{x}, \hat{p}$ are, respectively, the position and momentum operators. Then, the energy eigenvalue equation for the particle reads

$$\hat{H}|\psi_E\rangle = E|\psi_E\rangle,$$

where $|\psi_E\rangle$ is an eigenstate of energy E. Projecting both sides of the previous equation on the axis defined by the position eigenstate $|x\rangle$ yields

$$\langle x|\hat{H}|\psi_E\rangle = \langle x|E|\psi_E\rangle = E\langle x|\psi_E\rangle = E\psi_E(x),$$

where, in the last equality, we used the definition of the position-space wave function (see eq. (1)). Thus

$$\langle x|\hat{H}|\psi_E\rangle = E\psi_E(x)$$

By means of (2), the element $\langle x|\hat{H}|\psi_E\rangle$ reads

$$\hat{H}(x)\langle x|\psi_E\rangle = \hat{H}(x)\psi_E(x),$$

where $\hat{H}(x)$ is the expression of the previous Hamiltonian in position space. Thus

$$\hat{H}(x)\psi_E(x) = E\psi_E(x) \tag{22}$$

In position space, the position operator is the position coordinate, while the momentum operator is the differential operator $-i\hbar\, d/dx$. Thus, the Hamiltonian of the particle in position space is

$$\hat{H}(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x),$$

and (22) is then written as

$$-\frac{\hbar^2}{2m}\psi_E''(x) + V(x)\psi_E(x) = E\psi_E(x)$$

or

$$\psi_E''(x) + \frac{2m}{\hbar^2}\left(E - V(x)\right)\psi_E(x) = 0,$$

which is the stationary Schrödinger equation (also known as the time-independent Schrödinger equation).

The Three-Dimensional Case

For a particle moving in three dimensions, using the same reasoning as in the one-dimensional case, the position space is realized by projecting ket states $|\psi\rangle$ on the particle's position eigenstates $|\vec{r}\rangle$, where $\vec{r}$ is a vector in $\mathbb{R}^3$. The position-space wave function is then defined as

$$\psi(\vec{r}) \overset{\text{def.}}{=} \langle \vec{r}|\psi\rangle$$

Similarly to (3), if $\hat{A}$ is a state-space operator and the state $|\psi\rangle$ belongs to its domain, then

$$\langle \vec{r} \,|\, \hat{A} | \psi \rangle = \hat{A}(\vec{r})\psi(\vec{r}),$$

where $\hat{A}(\vec{r})$ is the expression of the operator $\hat{A}$ in position space.

The orthogonality of the position eigenstates $|\vec{r}\rangle$ and $|\vec{r}'\rangle$ is expressed by the relation

$$\langle \vec{r}' | \vec{r} \rangle = \delta(\vec{r}' - \vec{r}),$$

where, in Cartesian coordinates, the three-dimensional delta function is

$$\delta(\vec{r}' - \vec{r}) = \delta(x' - x)\delta(y' - y)\delta(z' - z)$$

The closure relation for the position eigenstates now reads

$$\int_{-\infty}^{\infty} d^3\vec{r} \, |\vec{r}\rangle\langle\vec{r}| = \hat{1}$$

In three dimensions, the canonical commutation relations for position and momentum read, in state space,

$$[\hat{r}_i, \hat{p}_j] = i\hbar\delta_{ij},$$

where the indices i, j take the values 1,2,3, with 1 standing for the coordinate x, 2 for y, and 3 for z.

Using the same reasoning as in the one-dimensional case, we find that the expressions of the position and momentum operators in position space are, respectively,

$$\hat{\vec{r}}(\vec{r}) = \vec{r} \text{ and } \hat{\vec{p}}(\vec{r}) = -i\hbar\nabla,$$

where ∇ is the del operator. That is, the position operator is the position vector, while the momentum operator is the del operator times $-i\hbar$, which is an obvious generalization from the one-dimensional case.

The momentum eigenfunctions in position space are then derived by solving the momentum eigenvalue equation

$$-i\hbar\nabla p(\vec{r}) = \vec{p}p(\vec{r}),$$

which, in Cartesian coordinates, reads

$$-i\hbar\left(\frac{\partial p(\vec{r})}{\partial x}\hat{e}_x + \frac{\partial p(\vec{r})}{\partial y}\hat{e}_y + \frac{\partial p(\vec{r})}{\partial z}\hat{e}_z\right) = \left(p_x\hat{e}_x + p_y\hat{e}_y + p_z\hat{e}_z\right)p(\vec{r}) \qquad (23)$$

where $\hat{e}_x, \hat{e}_y, \hat{e}_z$ are the unit vectors on the axes x, y, z, respectively. We note that the momentum eigenfunctions are, actually, wave functions, thus they are scalar functions. We can easily solve (23) by separating the variables, i.e. by writing the momentum eigenfunction as

$$p(\vec{r}) = X(x)Y(y)Z(z)$$

Then, substituting into (23), we obtain three differential equations with the same form as (9), thus the momentum eigenfunction $p(\vec{r})$ is the product of three eigenfunctions with the form (15), i.e.

$$p(\vec{r}) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip_x x}{\hbar}\right)\frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip_y y}{\hbar}\right)\frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip_z z}{\hbar}\right)$$

or

$$p(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}}\exp\left(\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right) \qquad (24)$$

We note that with the same reasoning, in n spatial dimensions, the momentum eigenfunctions are, in position space,

$$p(\vec{r}) = \frac{1}{(2\pi\hbar)^{n/2}}\exp\left(\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right),$$

where $\vec{r} = (r_1, ..., r_n)$ and $\vec{p} = (p_1, ..., p_n)$.

Since $p(\vec{r}) = \langle\vec{r}\,|\,\vec{p}\rangle$, (24) is also written as

$$\langle \vec{r} \,|\, \vec{p} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left(\frac{i\vec{p}\cdot\vec{r}}{\hbar} \right)$$

The description in momentum space is completely analogous. The momentum-space wave function is defined as

$$\tilde{\psi}(\vec{p}) \overset{\text{def.}}{=} \langle \vec{p} | \psi \rangle$$

Similarly to (3), if $\hat{A}$ is a state-space operator and the state $|\psi\rangle$ belongs to its domain, then

$$\langle \vec{p} | \hat{A} | \psi \rangle = \hat{A}(\vec{p})\psi(\vec{p}),$$

where $\hat{A}(\vec{p})$ is the expression of the operator $\hat{A}$ in momentum space.

The orthogonality relation of the momentum eigenstates reads

$$\langle \vec{p}' | \vec{p} \rangle = \delta(\vec{p}' - \vec{p}),$$

where, in Cartesian momentum coordinates,

$$\delta(\vec{p}' - \vec{p}) = \delta(p_x' - p_x)\delta(p_y' - p_y)\delta(p_z' - p_z)$$

The momentum eigenstates satisfy the closure relation

$$\int_{-\infty}^{\infty} d^3\vec{p} \,|\, \vec{p} \rangle \langle \vec{p} \,| = \hat{1}$$

The expressions of the position and momentum operators in momentum space are, respectively,

$$\hat{\vec{r}}(\vec{p}) = i\hbar\nabla_p \text{ and } \hat{\vec{p}}(\vec{p}) = \vec{p},$$

where ∇_p is the del operator in momentum, i.e. in Cartesian momentum coordinates,

$$\nabla_p = \frac{\partial}{\partial p_x}\hat{e}_{p_x} + \frac{\partial}{\partial p_y}\hat{e}_{p_y} + \frac{\partial}{\partial p_z}\hat{e}_{p_z}$$

Since

$$r(\vec{p}) = \langle \vec{p} | \vec{r} \rangle = \langle \vec{r} | \vec{p} \rangle^* = p^*(\vec{r}),$$

the position eigenfunctions in momentum space are the complex conjugates of the momentum eigenfunctions in position space, as it happens in one-dimension too. Then, by means of (24),

$$r(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left(-\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right)$$

The relation between the position and momentum space wave functions $\psi(\vec{r})$ and $\tilde{\psi}(\vec{p})$ is derived in the same way as in the one-dimensional case, and we obtain

$$\tilde{\psi}(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} d^3\vec{r} \exp\left(-\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right)\psi(\vec{r}),$$

i.e. the momentum-space wave function is the three-dimensional Fourier transform of the position-space wave function. The three-dimensional inverse Fourier transform then relates the position-space wave function to the momentum-space wave function, i.e.

$$\psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} d^3\vec{p} \exp\left(\frac{i\vec{p}\cdot\vec{r}}{\hbar}\right)\tilde{\psi}(\vec{p})$$

CONCLUSIONS

We have reformulated the physical equivalence of the quantum mechanical state space to the position and momentum spaces in terms of a simple relation and we have provided a presentation of the position and momentum spaces that is intuitively geometric in that it is more constructive and less calculative than the presentations found in standard quantum mechanics textbooks, and thus it is deprived of such mathematical subtleties as the presence of the delta function derivative, which, although well defined, is not well understood by physics students.

By the present work, we hope to offer a perspective complementary to those given in literature, which will help students to acquire functional knowledge of the quantum mechanical position and momentum spaces.

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