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REPEATING DECIMALS AND IRRATIONAL NUMBERS ON THE NUMBER LINE: THROUGH THE LENS OF PRE-SERVICE AND IN-SERVICE MATHEMATICS TEACHERS

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Abstract: The purpose of the study is to examine how pre-service and in-service mathematics teachers locate repeating decimals and irrational numbers on the number line. The participants of the study included 274 pre-service and 106 in-service mathematics teachers. Data were collected through a written questionnaire including four open-ended questions. In the questionnaire, pre-service and in-service teachers were asked to determine whether the given numbers 0.444…,$\sqrt{2}$, π and e can be located on the number line, they cannot be located on the number line or they do not have exact places on the number line. Additionally, they were asked to justify their responses to each of the questions in the questionnaire. Open coding was performed while analysing data. Findings indicated that while majority of the participants stated that given repeating decimal, i.e. 0.444…, can be located on the number line, for the given irrational numbers, i.e. $\sqrt{2}$, π and e, only small number of participants considered that they can be located on the number line. The main sources of the consideration that the given numbers do not have an exact place or they cannot be located on the number line were identified as approximation, infinity, irrationality and uncertainty in the justifications.

Key words: Repeating Decimals, Irrational Numbers, Number Line, Pre-service Teachers, In-service Teachers

1. Introduction

The set of real numbers include both the sets of rational numbers and irrational numbers. Rational numbers can be defined as a ratio of two integers whereas irrational numbers are numbers that cannot be expressed as a ratio of two integers m and n (n ≠ 0) (Adams & Essex, 2009). Moreover, irrational numbers can be also described as "numbers that cannot be represented as a terminating or repeating decimal" (OCG, 2005, p. 127) or numbers in decimal form that neither terminate nor repeat a sequence of digits (McKeague, 2014). Students learn the natural numbers in primary school, extend them to the set of integers, and rational numbers including repeating and terminating decimals and then irrational numbers, and finally they reach the set of the real numbers. Thus, it can be important to understand repeating decimals and irrational numbers in order to extend and reconstruct the concept of number from the set of rational numbers to real numbers (Sirotic & Zazkis, 2007a). Moreover, without fully understanding of repeating decimals, it is unlikely for the teachers to support students' understanding of important concepts in the elementary school level. In addition, understanding the concept of irrational numbers can enable the learners to realize the completeness of the set of real numbers and influence their understanding of the continuity and limit of a function (Hayfa & Saikaly, 2016).

In Turkey, according to the Turkish Mathematics Curriculum (Ministry of National Education, 2018), students are introduced to repeating decimals at 6th grade level and they learn how to represent repeating decimals as rational numbers and rational numbers as repeating decimals at 7th grade level. In order to represent repeating decimals on the number line, firstly, it is required to convert the given repeating decimal to the rational number and then to find location of the rational number on the number line. For instance, when we convert 0.444… to the rational number, we get 4/9. When we divide the space between 0 and 1 into 9 equal parts on the number line, each part obtained represents 1/9 and moving 4 parts from 0 towards 1 gives us 4/9.

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Students encounter with irrational numbers for the first time at 8th grade level which is the last year of elementary education and they continue to learn the concept of irrationality in high school years. In elementary level, it is expected from 8th grade students to identify π as an irrational number and to understand that square roots of non-square numbers cannot be specified as rational numbers. π can be represented on the number line by geometric construction. When we take a circle with a diameter of 1unit, its circumference would be π. As can be observed in Figure 1, in order to represent π on the number line, we mark a point on the circle, put that point at 0 point on the number line and rotate the circle. The point at which the marked point touches the number line for the first time will be the location of π on the number line.

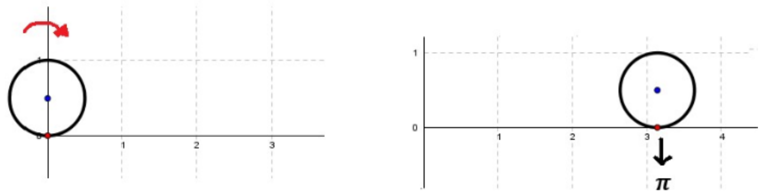
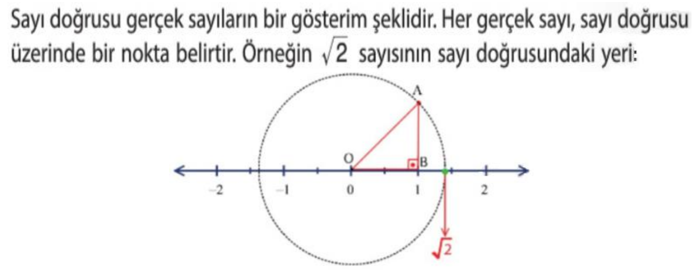


Figure 1. *Task regarding finding the location of π on the number line*

In high school level, students encounter with irrational numbers at the first year of secondary education and 9th grade students learn the relationship between sets of natural numbers, integers, rational numbers, irrational numbers and real numbers. In addition, 9th grade students are required to locate irrational numbers such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ on the number line. Figure 2 shows one of the tasks that exists in 9th grade school textbook regarding finding the location of $\sqrt{2}$ on the number line. In order to find the location of $\sqrt{2}$ on the number line, an isosceles right triangle with the length of congruent sides is equal to 1 unit is drawn on the number line as can be observed in the figure. With the help of Pythagorean Theorem, the length of hypotenuse is found as $\sqrt{2}$. Then, by aligning the compass needle on the point O and pecil lead on the point A, when we draw a circle, the point where the circle intersects the number line on the positive side is the location of $\sqrt{2}$ on the number line.



Number line is a representation of real numbers. Each real number corresponds to a point on the number line. For instance, the place of $\sqrt{2}$ on the number line:

Figure 2. *Task regarding finding the location of $\sqrt{2}$ on the number line*

At the last year of secondary education, it is expected from teachers to emphasize that the number *e* is an irrational number while 12th grade students are introduced to logarithmic functions and its usage in mathematics and other disciplines. Furthermore, in both elementary and secondary school textbooks, there are some information and tasks related to the objectives about irrational numbers. In order to represent the number e on the number line, function-graph approach can be used. When we draw the graph of a function $f\ (x)\ =\ e^{x}$, when $x = 1$, the value of $e^{x} = e$. That is, height of the curve at the point 1 will give the number *e*.

When we think about both the importance of repeating decimals and irrational numbers and the place of these concepts in the mathematics curriculum, it is expected from both elementary and secondary pre-service and in-service mathematics teachers that they should know these concepts and expand their knowledge in these concepts in order to teach them well. Also, pre-service and in-service teachers' knowledge about the concept of repeating decimals and irrational numbers and how they associate them with other types of numbers are essential to promote students' understanding regarding these concepts.

In the literature, there are various research studies conducted about irrational numbers with pre-service and in-service teachers. These studies addressed the definition of irrational numbers (Fischbein, Jehiam & Cohen, 1995; Guven, Cekmez & Karatas, 2011), identifying given numbers as rational or irrational numbers (Arcavi, Bruckheimer & Ben-Zvi, 1987; Fischbein, Jehiam & Cohen, 1995; Zazkis & Sirotic, 2010), locating irrational numbers on the number line (Guven, Cekmez & Karatas, 2011; Peled & Hershkovitz, 1999; Sirotic & Zazkis, 2007a, 2007b), and the relationship between the sets of numbers (Fischbein, Jehiam & Cohen, 1995; Sirotic & Zazkis, 2007a). In addition, there are some studies conducted with high school students (Hayfa & Saikaly, 2016; Kidron, 2018) and undergraduate university students (Mamolo, 2009) about irrational numbers and repeating decimals and their locations on the number line.

Fischbein, Jehiam and Cohen (1995) found that pre-service teachers had a difficulty in describing the concepts of rational, irrational and real numbers and in specifying these numbers in given examples related to numbers. Similar to the findings of the study conducted with the pre-service teachers, Arcavi, Bruckheimer and Ben-Zvi (1987) found that in-service teachers also had a difficulty in identifying numbers as being rational or irrational. In addition, most of these teachers believed that irrationality relies upon decimal representations. This was an indicator of the teachers' lack of knowledge regarding historical emergence of decimal numbers and believing that the origin of irrational numbers depends on decimals and not related to geometry as happened in history (Arcavi, Bruckheimer & Ben-Zvi, 1987).

In the study conducted by Peled and Hershkovitz (1999), it was found that most of the pre-service teachers were not good at the tasks in which representations of the given numbers such as $\sqrt{5}, \pi$, and 0.333. . . on the real number line are required. They generally claimed that because of having infinite number of digits in their decimal representations, these numbers cannot be placed on the real number line. In addition, some participants asserted that since these numbers are irrational numbers, they have no place on the real number line. In a similar vein, Mamolo (2009) examined the connections that undergraduate university students made between points on the number line and real numbers. The students were able to associate rational numbers with points on the number line because of identifying rational numbers as finite quantities whereas they could not associate repeating decimals and irrational numbers with points since they regarded these numbers as infinite quantities. In a study conducted to investigate how high school students approach the existence of irrational numbers, Kidron (2018) found that some students recognized irrational numbers as non-repeating infinite decimals, and they stated that only rational numbers are located on the number line. Furthermore, some students asserted that since a point on the number line is well-defined, it could not be explained as an infinite decimal. Hayfa and Saikaly (2016) examined high school students' knowledge and the ways of thinking while defining and identifying irrational numbers and locating them on the number line. Students thought that irrational numbers do not have an exact location on the number line because of nonending process after the decimal.

Sirotic and Zazkis (2007a) found that pre-service teachers' intuitive and formal dimensions of knowledge about the relations between rational and irrational sets were not consistent. In another study, the same researchers investigated how pre-service teachers represent an irrational number $\sqrt{5}$ on a number line and they realized that conception of pre-service teachers regarding real number line is limited to decimal rational number line. Also, findings indicated that while some of the pre-service teachers used geometric approach, numerical approach, or function-graph approach, the others stated that $\sqrt{5}$ cannot be placed exactly on the number line because of its infinite digits (Sirotic & Zazkis, 2007b). In addition, Guven, Cekmez and Karatas (2011) in their study reported that some of the pre-service teachers considered that irrational numbers' exact locations on the number line cannot be determined. On the other hand, although majority of the pre-service teachers responded to the questions about locating irrational numbers on the number line correctly, they could not provide formal justifications for their responses. As the researchers stated that conducting additional research which examine the reasons for pre-service teachers' misunderstandings about the concept of irrational numbers is worthwhile.

Previous research studies have shown that students, pre-service and in-service teachers had a difficulty in determining how to locate repeating decimals and irrational numbers on the number line. Bearing those studies in our minds, as different than the mentioned studies, this research study focuses on how both pre-service and in-service teachers locate repeating decimals and irrational numbers on the number line and what are the possible sources of their difficulty in locating these numbers on the number line. Examining this situation and determining the possible sources of both pre-service and in-service teachers' difficulty in this situation is important for future research studies and mathematics learning and teaching process. Furthermore, this study also involves the number of *e* which is one of the most important and well-known irrational numbers in addition to other irrational numbers $\sqrt{2}$ and π. Moreover, as there is a limited number of research studies that investigated how pre-service and in-service teachers locate repeating decimals and irrational numbers on the number line in the related literature, there is limited understanding of how they understand these concepts and their locations on the number line. Thus, it is believed that examining the mentioned situation can elaborate on previous research and contribute to the literature. In addition, the findings of the study can enlighten teacher educators about mathematics teacher education programs and the content of the mathematics courses. Thus, the purpose of the study is to examine how pre-service and in-service mathematics teachers locate repeating decimals and irrational numbers on the number line. For this purpose, following research questions were formulated:

- What are the pre-service and in-service mathematics teachers' responses to the questions related to locating repeating decimals and irrational numbers on the number line?
- How do pre-service and in-service mathematics teachers justify their responses regarding locating repeating decimals and irrational numbers on the number line?

2. Method

In the present study, qualitative methodology was used to investigate how pre-service and in-service mathematics teachers locate repeating decimals and irrational numbers on the number line. For this purpose, participants' justifications for their responses to the questions were examined in detail.

2. 1. Participants

In line with the purpose of the study, we reached a large number of participants consisting of pre-service and in-service mathematics teachers at both elementary and secondary level. Pre-service elementary and secondary mathematics teachers were enrolled in elementary and secondary mathematics education programs in two state universities in Turkey and in-service elementary and secondary mathematics teachers were teaching at different middle and high schools in Turkey at the time of the study. The number of participants is presented in Table 1.

Table 1. *The number of participants*

	Pre-service mathematics teachers					In-service mathematics teachers
	1st year	2nd year	3rd year	4th year	Total	
Elementary	59	41	43	29	172	60
Secondary	29	25	28	20	102	46

2. 2. Data Collection

In the study, data were collected through a written questionnaire which was developed by the researchers according to the literature. After the questionnaire was developed, it was shared by the mathematics education researchers to ensure the content validity and put into final form. In the questionnaire, pre-service and in-service teachers were given four open-ended questions consisting the numbers $x = 0.444\ldots$, $y = \sqrt{2}$, $\pi = 3.14159$ ..., and $e = 2.71828$ .... Irrational numbers that are most familiar to and known by individuals who engage with mathematics were chosen while developing the questionnaire. For each of the given numbers, pre-service and in-service teachers were asked to determine whether it can be located on the number line, it cannot be located on the number line or it does not have an exact place on the number line. In addition, they were asked to justify their responses.

2. 3. Data Analysis

Open coding was performed while analysing data. Firstly, frequencies and percentages of the participants' responses were calculated. Then, justifications of the participants' responses to each question were analysed by the researchers individually. After that, researchers compared and discussed results of their analyses and codes on which there was disagreement were revised and discussed until an agreement was built. Intercoder reliability between the two coders was calculated as 91.4%. Additionally, another researcher from mathematics education field assessed the codes.

3. Findings

Findings will be presented in three sections. In the first section, participants' responses to the questions in terms of frequencies and percentages will be presented. In the second section, participants' justifications for their responses to the questions and in the third section, examples for these justifications will be presented.

In the tables, abbreviations will be used for both participants and their responses. PE stands for pre-service elementary teachers, PS stands for pre-service secondary teachers, IE stands for in-service elementary teachers and IS stands for in-service secondary teachers. In addition, L stands for the response that given number can be located on the number line, NE stands for the response that given number does not have an exact place on the number line, and NL stands for the response that given number cannot be located on the number line.

3. 1. Participants' Responses to the Questions

Responses of the participants to the questions in terms of frequencies and percentages are given in Table 2. As can be observed in Table 2, for the first question, majority of the total participants indicated that 0.444… can be located on the number line and the percentages of the pre-service elementary, pre-service secondary, in-service elementary and in-service secondary teachers who claimed that 0.444… has a place on the number line are close to each other. For the second question, most of the total participants considered that $\sqrt{2}$ has no exact place on the number line. When the responses of the participants were examined in detail, it was seen that for the pre-service elementary, pre-service secondary and in-service elementary teachers, the percentages of the response that the location of $\sqrt{2}$ is not exact are higher than the percentages of the other responses. However, for in-service secondary teachers, the percentage of the response that $\sqrt{2}$ cannot be located on the number line is the highest one among the other responses.

Table 2. *Responses of the participants to the questions*

	0.444 …			**$\sqrt{2}$**			**π**			**e**		
	L	**NE**	**NL**	**L**	**NE**	**NL**	**L**	**NE**	**NL**	**L**	**NE**	**NL**
PE	113 (65.7)	42 (24.4)	17 (9.9)	29 (16.9)	111 (64.5)	32 (18.6)	4 (2.3)	105 (61)	63 (36.6)	4 (23)	101 (58.7)	67 (39)
PS	69 (67.6)	21 (20.6)	12 (11.8)	14 (13.7)	56 (54.9)	32 (31.4)	3 (2.9)	45 (44.1)	54 (52.9)	-	46 (45.1)	56 (54.9)
IE	43 (63)	11 (19.6)	6 (17.4)	18 (30)	27 (45)	15 (25)	5 (8.3)	23 (38.3)	32 (53.3)	2 (3.3)	25 (41.7)	33 (55)
IS	29 (69.5)	9 (18.6)	8 (11.9)	12 (26.1)	15 (32.6)	19 (41.3)	5 (10.9)	10 (21.7)	31 (67.4)	5 (10.9)	10 (21.7)	31 (67.4)
Total	**254 (66.8)**	**83 (21.8)**	**43 (11.3)**	**73 (19.2)**	**209 (55)**	**98 (25.8)**	**17 (4.5)**	**183 (48.2)**	**180 (47.4)**	**11 (2.9)**	**182 (47.9)**	**187 (49.2)**

For the third question, it can be said that small percentage of the total participants indicated that π can be located on the number line. On the other hand, the percentage of the total number of participants who asserted that the location of π is not exact and π cannot be located on the number line is close to each other. When the responses of the participants were examined in detail, it was seen that for the pre-service elementary teachers, the percentage of the response that the location of π is not exact is higher than the percentage of the other responses. Yet, for the pre-service secondary, in-service elementary and in-service secondary teachers, it can be said that most of them stated that π has no place on the number line. Similar to the findings for the third question, for the fourth question, extremely small percentage of the total participants indicated that e can be located on the number line. However, the percentage of the total participants who considered that e has no place on the number line and the percentage of participants who stated that the location of e on the number line is not exact are higher and close to each other. In addition, it was seen that none of the pre-service secondary mathematics teachers stated that e can be located on the number line. Similar to the findings for π, most of the pre-service secondary teachers and in-service elementary and in-service secondary teachers stated that e has no place on the number line.

When the participants' responses to the four questions were compared, it was seen that there is a decrease in the percentage of participants who considered that the given number can be located on the number line from first question to the last question. That is, most of the participants were able to determine that 0.444…has a place on the number line. However, for the questions related to the irrational numbers the percentage of participants who considered that they can be located on the number line decreased from the second question to the fourth question. Moreover, among $\sqrt{2}, \pi$ and e the participants did not consider that particularly π and e has a place on the number line in general. In addition, for $\sqrt{2}$, the percentage of the total number of participants who stated that the location of it is not exact is higher than the percentage of the total number of participants who stated that it cannot be located on the number line. Conversely, for π and e, the percentages of participants' responses that the given number has no exact place and has even no place on the number line are close to each other.

Sequences of the responses of the participants to the questions were presented in Table 3 in terms of frequencies and in Figure 3 in terms of percentages. As can be observed in Table 3 and Figure 3, L-NE-NE-NE, L-NE-NL-NL, and L-NL-NL-NL were the most frequently given sequences of the responses by the participants, respectively. On the other hand, it was seen that L-L-L-L sequence was found to be very low. In addition, NE-NE-NE-NE and NL-NL-NL-NL sequences have also emerged. While the sequence L-NE-NE-NE was seen in the responses of PE, PS and IE mostly, the sequence L-NL-NL-NL was seen in the responses of PS mostly. In conclusion, it can be said that there is a consistency in the responses to the questions (L-L-L-L, NE-NE-NE-NE, NL-NL-NL-NL) given by some of the participants. These participants either considered that all given numbers can be located on the number, or they have no exact place on the number line or they cannot be located on the number line. On the other hand, some of the participants' responses changed according to the questions and there was not consistency in their responses to the questions.

Table 3. *Sequences of the responses of the participants to the questions in terms of frequencies*

	PE	PS	IE	IS	Total
L-L-L-L	2	0	2	5	9
L-L-NE-NE	14	5	3	1	23
L-L-NL-NL	4	5	7	4	20
L-NE-NE-NE	46	23	15	3	87
L-NE-NL-NL	25	15	4	6	50
L-NL-NL-NL	10	10	9	10	39
NE-NE-NE-NE	20	8	4	5	37
NE-NE-NL-NL	6	4	1	1	12
NE-NL-NL-NL	7	4	2	2	15
NL-NL-NL-NL	3	5	4	6	18
Others	35	23	9	3	70
Total	172	102	60	46	380

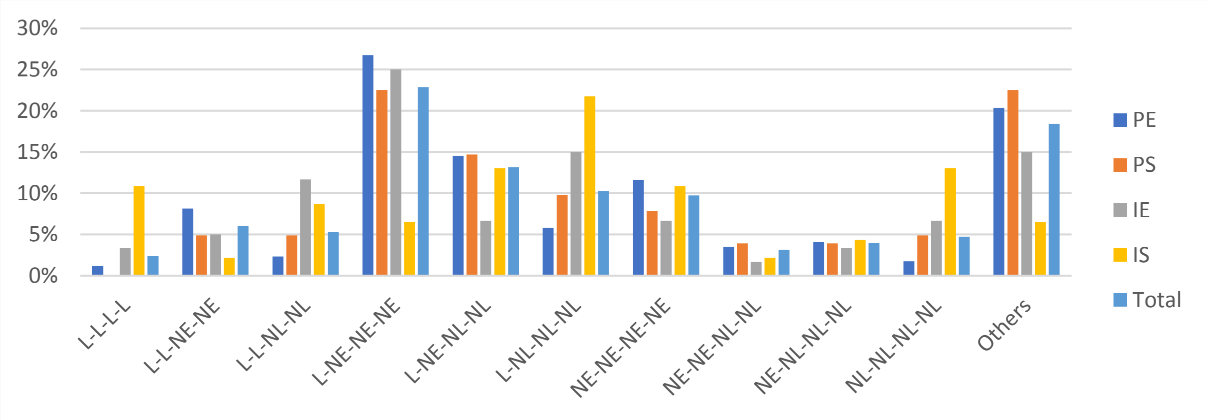


Figure 3. *Sequences of the responses of the participants to the questions in terms of percentages*

3. 2. Justifications of Participants for Their Responses

3.2.1. Justifications of participants for their responses to locating 0.444… on the number line. Categories and codes emerged from participants' justifications for locating 0.444… on the number line are presented in Table 4.

Table 4. *Categories and codes for 0.444…*

	Abbr	Category	Code
1	**VR**	**Visual Representation**	Drawing a number line
2	**A**	**Approximation**	Between 0 and 1 Between 0.4 and 0.5
3	**C**	**Conversion**	$0.\overline{4}$ 4/9 By converting the repeating decimal to the fraction
4	**F/R**	**Fraction/ Rational number**	Proper fraction Fraction Rational number
5	**I**	**Infinity**	Digits after the decimal point continue infinitely Decimal expansion goes on by repeating 4 repeats continuously after the decimal point 4 goes on forever
6	**O**	**Other**	Convergent/converging Non exact value

As can be observed in Table 4, six categories emerged from the participants' responses to the first question. Participants expressed their responses about the location of 0.444… on the number line as visual representation, approximation, conversion, a fraction or a rational number, infinity, and convergence and value. Justifications regarding convergence and value were grouped and named as other.

Table 5 shows participants' justifications for their responses to the first question in terms of both frequency and percentage. Also, responses of the participants in the categories in terms of percentage regarding the first question are represented in Figure 4.

Table 5. *Frequencies and percentages of participants' justifications for locating 0.444…*

0.444…		**PE**			**PS**			**IE**			**IS**		
		L	**NE**	**NL**	**L**	**NE**	**NL**	**L**	**NE**	**NL**	**L**	**NE**	**NL**
VR	**f**	34	1	-	24	-	-	8	-	1	8	-	-
	%	19.8	0.6	-	23.5	-	-	13.3	-	1.7	17.4	-	-
A	**f**	1	6	1	1	4	1	1	3	-	1	7	-

	%	0.6	3.5	0.6	1	3.9	1	1.7	5	-	2.2	15.2	-
C	f	57	10	3	32	3	3	23	3	-	19	1	-
	%	33.1	5.8	1.7	31.4	2.9	2.9	38.3	5	-	41.3	2.2	-
FR	f	18	3	1	10	1	2	10	5	-	1	-	-
	%	10.5	1.7	0.6	9.8	1	2	16.7	8.3	-	2.2	-	-
I	f	1	16	10	1	7	5	1	-	5	-	1	8
	%	0.6	9.3	5.8	1	6.9	4.9	1.7	-	8.3	-	2.2	17.4
O	f	2	6	2	1	6	1	-	-	-	-	-	-
	%	1.2	3.5	1.2	1	5.9	1	-	-	-	-	-	-
Total	f	**113**	**42**	**17**	**69**	**21**	**12**	**43**	**11**	**6**	**29**	**9**	**8**
	%	**65.7**	**24.4**	**9.9**	**67.6**	**20.6**	**11.8**	**71.7**	**18.3**	**10**	**63**	**19.6**	**17.4**

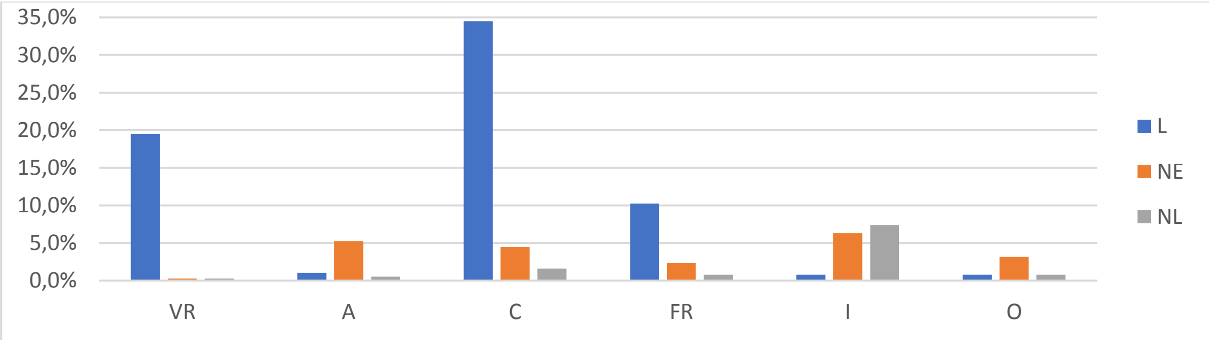


Figure 4. *Responses of the participants in the categories in terms of percentages regarding the first question*

As can be observed in Table 5 and Figure 4, most of the participants who stated that 0.444… can be located on the number line justified their responses by converting given repeating decimal to the fraction 4/9. In addition, justifications of some participants who considered that 0.444… has a place on the number line included visual representation and fraction/rational number. Furthermore, use of infinity in justifications was generally preferred by the participants who indicated that location of 0.444… on the number line is not exact or it has no place on the number line. Indeed, all in-service secondary teachers who indicated that 0.444… cannot be located on the number line expressed their responses by utilizing infinity.

3.2.2. Justifications of participants for their responses to locating $\sqrt{2}$, π and *e* on the number line. Categories and codes emerged from participants' justifications for locating irrational numbers ($\sqrt{2}$, π and *e*) on the number line are presented in Table 6.

Table 6. *Categories and codes for* $\sqrt{2}$, π *and e*

	Abbr	Category	Code
1	**GR**	**Geometric Representation**	Pythagorean Theorem, circumference of a circle, graph
2	**A**	**Approximation**	(Approximate value) between 1-2/3-4/2-3
3	**IR**	**Irrationality**	Irrational number Not a rational number/ not a fraction Not represented as a/b Rational numbers/integers are represented on the number line Irrational numbers are not represented on the number line
4	**U**	**Uncertainty**	All digits are not known The end of the number is non-exact The end after decimal point is unknown Digits after decimal point have not been found yet

5	**I**	**Infinity**	Continues infinitely No end, number that continues, long number Number with an infinite number of digits The numbers after the decimal point continue infinitely Numbers are added continuously after the decimal point
6	**O**	**Other**	The number line is not divided into equal parts of $\sqrt{1}$, $\sqrt{2}$, … Famous/special/different irrational number Number that contains all number combinations The numbers after the decimal point do not form a pattern

As can be observed in Table 6, six categories emerged from the participants' responses to the questions related to irrational numbers. Participants justified their responses about locating irrational numbers on the number line as geometric representation, approximation, irrationality, uncertainty, infinity, number line, speciality and pattern. Justifications regarding number line, speciality and pattern were grouped and named as other. Some of the participants used Pythagorean Theorem to show the place of $\sqrt{2}$ on the number line. Moreover, some participants drew a circle and utilized circumference of the circle to represent π and some of the participants drew graph of y = e^x to represent *e* on the number line. Codes related to Pythagorean Theorem, circumference of a circle, and graph were categorized as visual representation. Additionally, some of the participants about the number line noted that the number line cannot be divided into equal parts of $\sqrt{1}$, $\sqrt{2}$… regarding the location of $\sqrt{2}$ on the number line. While some of the participants emphasized speciality of the numbers π and *e* by stating that these numbers are famous, special or different irrational numbers, some participants indicated that there is no pattern in decimal expansion of irrational numbers. Furthermore, in the justifications it was mentioned that the number π contains all number combinations; hence, its speciality was emphasized.

Participants' justifications for their responses to the second question in terms of both frequency and percentage is represented in Table 7. Also, responses of the participants in the categories in terms of percentage regarding the second question are represented in Figure 5.

Table 7. *Frequencies and percentages of participants' justifications for locating* $\sqrt{2}$

$\sqrt{2}$		PE			PS			IE			IS		
		L	**NE**	**NL**	**L**	**NE**	**NL**	**L**	**NE**	**NL**	**L**	**NE**	**NL**
GR	**f**	20	3	-	13	-	-	18	-	-	12	-	-
	%	11.6	1.7	-	12.7	-	-	30	-	-	26.1	-	-
A	**f**	-	74	1	-	29	1	-	18	-	-	11	1
	%	-	43	0.6	-	28.4	1	-	30	-	-	23.9	2.2
IR	**f**	-	18	15	-	7	10	-	4	9	-	4	6
	%	-	10.5	8.7	-	6.9	9.8	-	6.7	15	-	8.7	13
U	**f**	1	7	2	-	15	2	-	3	4	-	-	6
	%	0.6	4.1	1.2	-	14.7	2	-	5	6.7	-	-	13
I	**f**	1	5	2	-	3	15	-	2	2	-	-	6
	%	0.6	2.9	1.2	-	2.9	14.7	-	3.3	3.3	-	-	13
O	**f**	7	4	12	1	2	4	-	-	-	-	-	-
	%	4.1	2.3	7	1	2	3.9	-	-	-	-	-	-
Total	**f**	**29**	**111**	**32**	**14**	**56**	**32**	**18**	**27**	**15**	**12**	**15**	**19**
	%	**16.9**	**64.5**	**18.6**	**13.7**	**54.9**	**31.4**	**30**	**45**	**25**	**26.1**	**32.6**	**41.3**

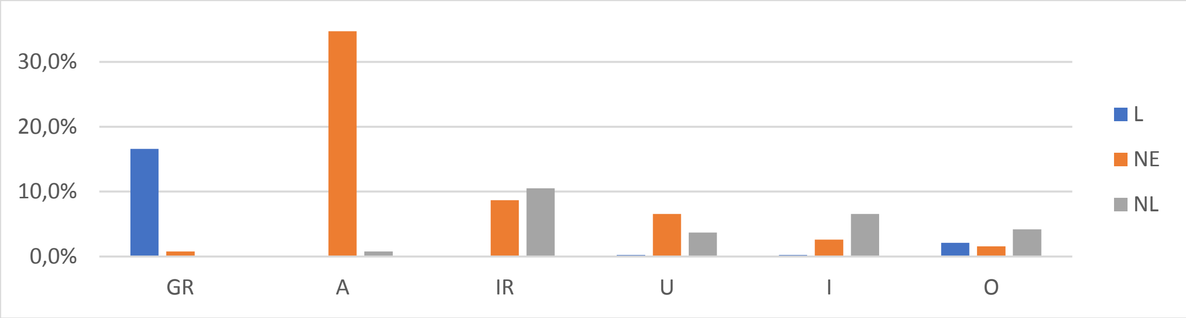


Figure 5. *Responses of the participants in the categories in terms of percentages regarding the second question*

As can be observed in Table 7 and Figure 5, participants who indicated that $\sqrt{2}$ can be located on the number line justified their responses generally by using geometric representation. Participants who stated that the location of $\sqrt{2}$ on the number line is not exact justified their responses mostly by using approximation. In other words, these participants specified that $\sqrt{2}$ is somewhere between 1 and 2 on the number line, but its' place cannot be determined exactly. Additionally, participants (particularly pre-service secondary and in-service secondary teachers) who indicated that $\sqrt{2}$ cannot be located on the number line justified their claims generally with infinity. Conversely, the pre-service elementary teachers mostly justified their responses that $\sqrt{2}$ cannot be located on the number line as irrationality. In addition, for the response that $\sqrt{2}$ cannot be located on the number line, some of the participants emphasized uncertainty in their justifications which was mostly utilized by in-service secondary teachers.

Table 8 shows participants' justifications for their responses to the third question in terms of both frequency and percentage. Also, responses of the participants in the categories in terms of percentage regarding the third question are represented in Figure 6.

Table 8. *Frequencies and percentages of participants' justifications for locating π*

π		PE			PS			IE			IS		
		L	**NE**	**NL**	**L**	**NE**	**NL**	**L**	**NE**	**NL**	**L**	**NE**	**NL**
GR	**f**	3	-	-	3	-	-	5	-	-	4	-	-
	%	1.7	-	-	2.9	-	-	8.3	-	-	8.7	-	-
A	**f**	-	11	-	-	5	2	-	9	1	-	6	1
	%	-	6.4	-	-	4.9	2	-	15	1.7	-	13	2.2
IR	**f**	1	8	10	-	4	5	-	6	12	-	3	7
	%	0.6	4.7	5.8	-	3.9	4.9	-	10	20	-	6.5	15.2
U	**f**	-	29	19	-	15	13	-	2	7	-	1	6
	%	-	16.9	11	-	14.7	12.7	-	3.3	11.7	-	2.2	13
I	**f**	-	41	17	-	18	31	-	5	11	-	-	15
	%	-	23.8	9.9	-	17.6	30.4	-	8.3	18.3	-	-	32.6
O	**f**	-	16	17	-	3	3	-	1	1	1	-	2
	%	-	9.3	9.9	-	2.9	2.9	-	1.7	1.7	2.2	-	4.3
Total	**f**	**4**	**105**	**63**	**3**	**45**	**54**	**5**	**23**	**32**	**5**	**10**	**31**
	%	**2.3**	**61**	**36.6**	**2.9**	**44.1**	**52.9**	**8.3**	**38.3**	**53.3**	**10.9**	**21.7**	**67.4**

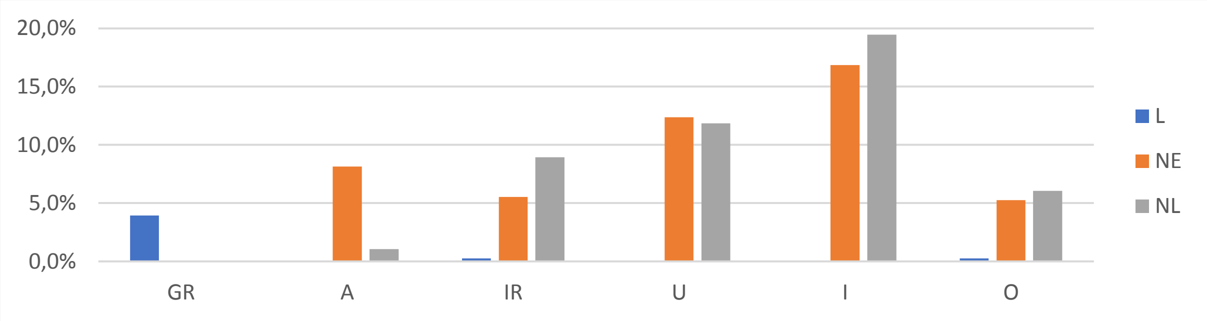


Figure 6. *Responses of the participants in the categories in terms of percentages regarding the third question*

Similar to the findings for $\sqrt{2}$, participants generally justified their responses that π can be located on the number line by using geometric representation as can be observed in Table 8 and Figure 6. These participants showed the place of π on the number line by utilizing circumference of a circle. Moreover, while pre-service teachers who considered that the location of π on the number line is not exact expressed their consideration mostly with infinity and uncertainty, in-service teachers explained the same consideration generally with approximation and irrationality. In addition, it was seen that the percentage of the total participants who stated that the location of π is not exact and the percentage of the total number of participants who stated that π cannot be located on the number line are in the infinity category at most.

Participants' justifications for their responses to the fourth question in terms of both frequency and percentage is presented in Table 9. Also, responses of the participants in the categories in terms of percentage regarding the fourth question are represented in Figure 7.

Table 9. *Frequencies and percentages of participants' justifications for locating e*

e		PE			PS			IE			IS		
		L	**NE**	**NL**	**L**	**NE**	**NL**	**L**	**NE**	**NL**	**L**	**NE**	**NL**
GR	**f**	1	-	-	-	-	-	2	-	-	3	-	-
	%	0.6	-	-	-	-	-	3.3	-	-	6.5	-	-
A	**f**	-	12	-	-	7	-	-	10	1	-	6	1
	%	-	7	-	-	6.9	-	-	16.7	1.7	-	13	2.2
IR	**f**	1	10	10	-	7	7	-	5	12	-	3	8
	%	0.6	5.8	5.8	-	6.9	6.9	-	8.3	20	-	6.5	17.4
U	**f**	-	27	15	-	13	13	-	4	8	-	1	7
	%	-	15.7	8.7	-	12.7	12.7	-	6.7	13.3	-	2.2	15.2
I	**f**	1	36	19	-	17	33	-	5	11	-	-	14
	%	0.6	20.9	11	-	16.7	32.4	-	8.3	18.3	-	-	30.4
O	**f**	1	16	23	-	2	3	-	1	1	2	-	1
	%	0.6	9.3	13.4	-	2	2.9	-	1.7	1.7	4.3	-	2.2
Total	**f**	**4**	**101**	**67**	-	**46**	**56**	**2**	**25**	**33**	**5**	**10**	**31**
	%	**2.4**	**58.7**	**39**	-	**45.1**	**54.9**	**3.3**	**41.7**	**55**	**10.9**	**21.7**	**67.4**

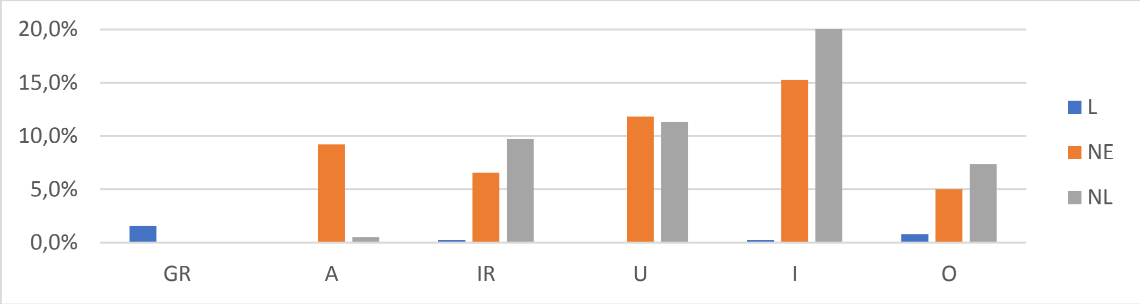


Figure 7. *Responses of the participants in the categories in terms of percentages regarding the fourth question*

As can be observed in Table 9 and Figure 7, extremely small number of participants indicated that e has a place on the number line and they justified their assertations generally by drawing the graph of y=e^x as geometric representation. Furthermore, among the participants who stated that the place of e on the number line is not exact, while pre-service teachers generally justified their responses with infinity and uncertainty, in-service teachers generally justified their responses with approximation and irrationality similar to the finding for π. Participants who indicated that e cannot be located on the number line generally justified their responses with infinity, uncertainty, and irrationality. Similar to the findings for π, the percentage of the total number of participants who stated that the location of e is not exact and the percentage of the total number of participants who stated that e cannot be located on the number line are in the infinity category at most.

Figure 8 shows responses of the participants in the categories in terms of percentage regarding the total of the second, third and fourth questions.

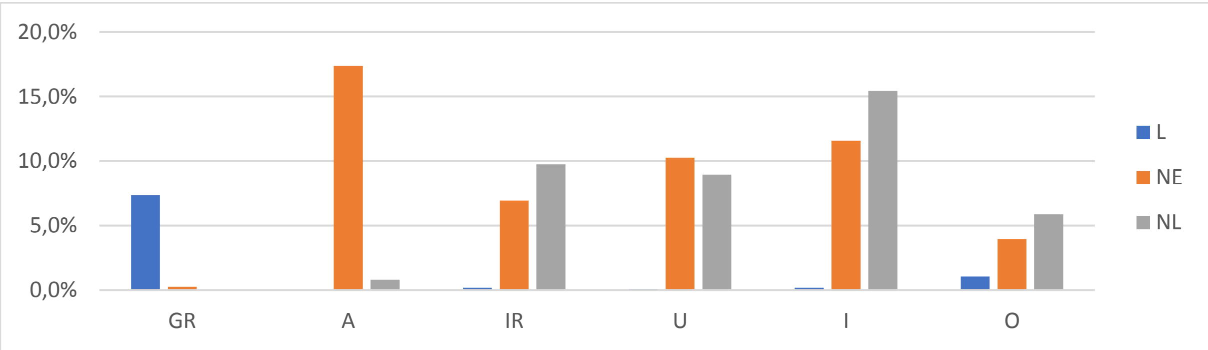


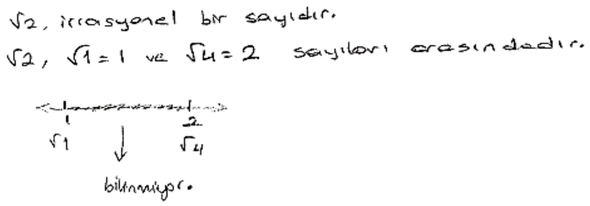
Figure 8. *Responses of the participants in the categories in terms of percentage regarding the total of the second, third and fourth questions*

When the participants' responses to the questions related to irrational numbers were examined in detail, it was seen that they generally preferred to use geometric representation to justify their responses that irrational numbers ($\sqrt{2}$, π and e) can be located on the number line (see Figure 8). In addition, they mostly utilized approximation to explain that locations of given irrational numbers on the number line are not exact. In addition to approximation, participants also used infinity, uncertainty and irrationality while expressing that the given irrational numbers have not exact places on the number line. Furthermore, participants who stated that given irrational numbers cannot be located on the number line mostly justified their responses with infinity. In addition, uncertainty and irrationality were also used by the participants in their justifications for the response that irrational numbers cannot be located on the number line.

3. 3. Examples for Justifications of Participants for Their Responses to Locating $0.444\ldots$, $\sqrt{2}$, π and e on the Number Line

Common categories were recognized in participants' justifications for their responses to the questions related to the repeating decimal and irrational numbers. These categories are approximation, infinity and rationality/irrationality.

Some of the participants mentioned that the location of 0.444… is somewhere between 0.4 and 0.5 or between 0 and 1 on the number line as approximation. Similarly, some participants estimated that the place of $\sqrt{2}$ is somewhere between 1 and 2 (see Figure 9), the place of π is somewhere between 3 and 4 and the place of *e* is somewhere between 2 and 3 on the number line by using approximation These participants who justified their responses by using approximation generally stated that given numbers do not have an exact place on the number line.

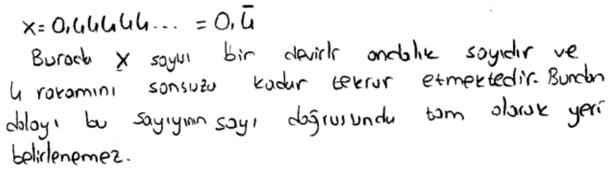


$\sqrt{2}$ is an irrational number.
$\sqrt{2}$ is between the numbers $\sqrt{1} = 1$ and $\sqrt{4} = 2$.

Not known

Figure 9. *Justification for the location of* $\sqrt{2}$ *on the number line as approximation*

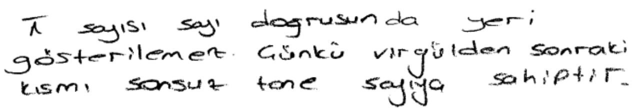
In addition to approximation, infinity was common in the participants' justifications for their responses to the questions about the repeating decimal and irrational numbers. Some participants emphasized infinity in their justifications that 4s in decimal expansion of 0.444… continue infinitely (see Figure 10).



Here x is a repeating decimal and 4 goes on forever. Hence, the location of this number on the number line cannot be determined exactly.

Figure 10. *Justification for the location of 0.444… on the number line as infinity*

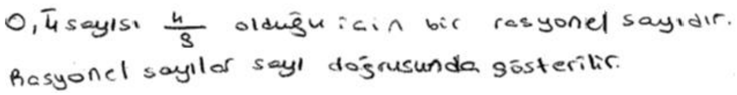
In the same way, some of the participants' justifications included that there are infinite number of digits in decimal expansion of irrational numbers (see Figure 11).



The number π cannot be located on the number line because the part after the decimal point has an infinite number of digits.

Figure 11. *Justification for the location of* π *on the number line as infinity*

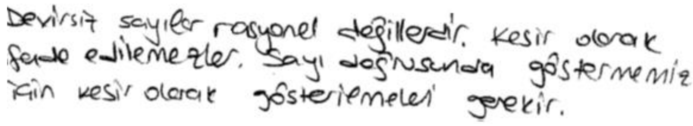
Furthermore, some participants referred to rationality/irrationality while explaining both the location of 0.444… and given irrational numbers. Rationality was used by the participants while explaining that 0.444… can be written as a rational number; hence, it can be located on the number line (see Figure 12).



The number $0.\overline{4}$ is a rational number since it is equal to $\frac{4}{9}$. Rational numbers can be located on the number line.

Figure 12. *Justification for the location of 0.444… on the number line as rationality*

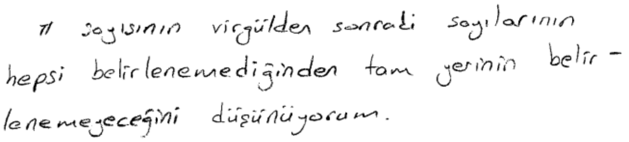
On the other hand, some of the participants used irrationality in the way that $\sqrt{2}$, π and e are not rational numbers, or these numbers cannot be represented as a/b. Therefore, they generally considered that either these numbers do not have an exact place, or they do not have even a place on the number line (see Figure 13).



Non-repeating decimals are not rational. They cannot be represented as fractions. They must be represented as fractions to represent on the number line.

Figure 13. *Justification for the location of e on the number line as irrationality*

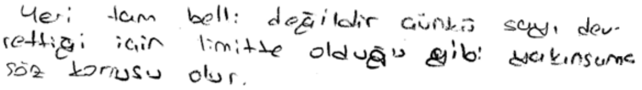
Some of the participants referred to uncertainty in the justifications for their responses to the questions related to irrational numbers. According to these participants, since all digits after the decimal point are not known or have not been found yet, they emphasized uncertainty of irrational numbers (see Figure 14). Among these participants, whereas pre-service teachers used uncertainty generally to justify that irrational numbers do not have an exact place, in-service teachers generally utilized uncertainty to justify that irrational numbers cannot be located on the number line.



Since all digits of the number π after the decimal point are not determined, I think that its exact location cannot be determined.

Figure 14. *Justification for the location of π on the number line as uncertainty*

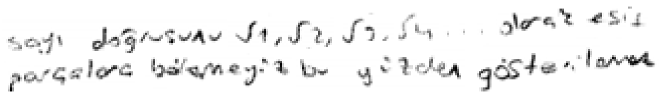
Other category emerged from the participants' responses to the questions including both the repeating decimal and irrational numbers. For the repeating decimal, other category includes justifications regarding convergence of 0.444… (see Figure 15) and its non-exact value. Some of the pre-service teachers used them mostly to justify that the place of 0.444… on the number line is not exact.



Its location is not exact because the number repeats, there is convergence as in the limit.

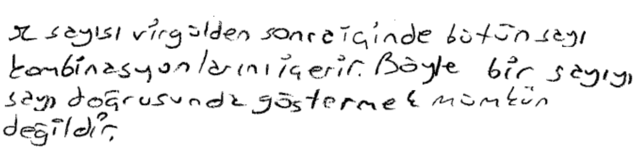
Figure 15. *Justification for the location of 0.444… on the number line as convergence*

For the irrational numbers, justifications regarding the characteristic of the number line, speciality of the given numbers and pattern in decimal expansion are included in the other category (see Figure 16, Figure 17 and Figure 18) and they were used generally by the pre-service elementary teachers to mostly emphasize that irrational numbers cannot be located on the number line.



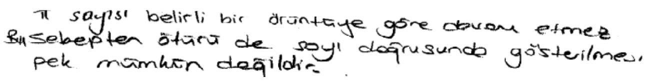
We cannot divide the number line into equal parts of $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$... Hence, it cannot be located on the number line.

Figure 16. *Justification for the location of $\sqrt{2}$ on the number line regarding characteristic of the number line*



The number π contains all number combinations after the decimal point. It is not possible to locate such a number on the number line.

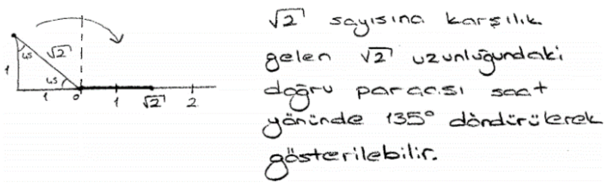
Figure 17. *Justification for the location of π on the number line regarding speciality*



The number π does not continue according to a certain pattern. For this reason, it is unlikely to locate it on the number line.

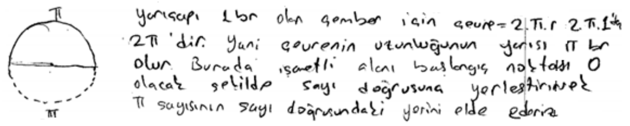
Figure 18. *Justification for the location of π on the number line regarding pattern in decimal expansion*

The participants who stated that given irrational numbers can be located on the number line justified their responses by showing how to locate given numbers on the number line by geometric construction (see Figure 19 and Figure 20).



It can be located by rotating the line segment whose length correspond to $\sqrt{2}$ clockwise by an angle 135°.

Figure 19. *Justification for the location of $\sqrt{2}$ on the number line as geometric representation*



For the circle whose radius is 1-unit: circumference is $2.\pi.r = 2.\pi.1 = 2.\pi$. That is, half of the circumference of the circle is π unit. Here, if we place the dotted part on the number line in a way that the starting point is 0, we get its place on the number line.

Figure 20. *Justification for the location of π on the number line as geometric representation*

4. Discussion, Conclusion and Recommendations

Teaching the concepts of repeating decimals and irrational numbers requires an appropriate knowledge base of teachers to develop their students' understanding. However, previous research studies have shown that pre-service and in-service teachers and students have a difficulty in determining how to locate repeating decimals and irrational numbers on the number line (Guven, Cekmez & Karatas, 2011; Hayfa & Saikaly, 2016; Kidron, 2018; Mamolo, 2009; Peled & Hershkovitz, 1999; Sirotic & Zazkis, 2007b). Investigating the sources of their difficulty is important as indicated in the related literature; hence, in the present study, we examined how pre-service and in-service mathematics teachers locate repeating decimals and irrational numbers on the number line and the sources of their difficulty in locating these numbers.

The findings of the study indicated that while majority of the participants stated that given repeating decimal, i.e. 0.444…, can be located on the number line, for the given irrational numbers, i.e. $\sqrt{2}$, π and *e*, only small number of the participants considered that these numbers can be located on the number line. This finding is consistent with the related literature since several research studies reported that pre-service and in-service teachers had a difficulty in locating irrational numbers on the number line (Guven, Cekmez & Karatas, 2011; Peled & Hershkovitz, 1999; Sirotic & Zazkis, 2007b).

Majority of the participants asserted that 0.444… has a place on the number line. This situation reveals that participants have a knowledge of representing repeating decimals on the number line and this is a common knowledge among the participants. On the other hand, although the given numbers $\sqrt{2}$, π and *e* are generic examples for irrational numbers (Sirotic & Zazkis, 2007a), it was found that few participants know that they can be located on the number line. Furthermore, the number of participants who indicated that π and *e* has a place on the number line is small when it is compared with the number of participants who stated that $\sqrt{2}$ has a place on the number line. This can be due to the fact that participants might have been more familiar to $\sqrt{2}$ than π and *e*; hence, they could show the place of this number on the number line. In addition, the percentage of the pre-service teachers who indicated that given irrational numbers can be located on the number line is smaller than the percentage of the in-service teachers who specified that they can be located on the number line. This situation might have resulted from the fact that in-service teachers might have been learning irrational numbers while teaching students them. On the other hand, although there are tasks related to locating irrational numbers on the number line in textbooks, it was seen that the percentage of in-service teachers who stated that the given irrational numbers have no place on the number line is higher than the percentage of in-service teachers who specified that they can be located on the number line. This situation can be the result of that in-service teachers might have forgotten this concept because time has passed since they have learned, or they could not conceptualize it while learning during their undergraduate studies.

The findings of the study also indicated that the total number of the participants who stated that π and e have no exact place on the number line, and the total number of those who stated that they cannot be located on the number line are close to each other and higher than the number of participants who stated that they can be located on the number line. In addition, in the present study, although the expected sequence is L-L-L-L, different sequences were emerged in the responses of the participants to the questions. Sequences of the responses L-L-L-L, NE-NE-NE-NE, and NL-NL-NL-NL showed consistency but the NE-NE-NE-NE, and NL-NL-NL-NL are actually incorrect responses. That is, participants either considered that all given numbers can be located on the number line or they have no exact place on the number line or they cannot be located on the number line. On the other hand, there were inconsistencies in some of the participants' responses. It can be said that these participants' responses changed according to the questions and the given numbers in the questions. This situation might be attributed to the participants' lack of knowledge about the given numbers in the four questions. These findings indicate the importance of determining what are the justifications for responses that the given numbers cannot be located or their locations are not exact on the number line. In this study, since these justifications were revealed in categories, teaching and learning environments can be designed in a way that will not lead to them.

Based on the second research question, findings showed that participants generally justified their responses that 0.444…can be located on the number line by converting it to 4/9, drawing a number line and representing it on the number line or emphasizing that fractions has a place on the number line. For irrational numbers, we identified that there are two basic types of knowledge that participants should have to be able to locate irrational numbers on the number line. First one is the knowledge of geometric representations of given irrational numbers. This knowledge is specific because it requires also knowing other things such as Pythagorean theorem for $\sqrt{2}$, circumference of a circle for π and graph of y= e^x for e. Second type of knowledge is knowing that all real numbers are included on the number line and this knowledge is more general than the first one. These types of knowledge can be gained from textbooks and in mathematics lessons at middle and high school, and in mathematics courses in undergraduate studies.

On the other hand, participants who stated that the given numbers do not have an exact place, or they cannot be located on the number line justified their responses generally by using approximation, infinity, irrationality, and uncertainty. Approximation was commonly used by the participants who stated that the locations of given repeating decimal and irrational numbers are not exact. These participants made inferences by determining intervals in which given numbers lie. This finding is consistent with the findings of the study conducted by Sirotic and Zazkis (2007b) because in which pre-service teachers used rough approximation, i.e. between 2 and 3, for the location of $\sqrt{5}$ on the number line. Actually, it can be said that these participants responded neither completely wrong nor correct, but their knowledge is incomplete. In addition, in the present study it was seen that infinity, irrationality and uncertainty are the general sources of the consideration of the participants that given numbers cannot be located on the number line or their locations on the number line is not exact. Participants who asserted that 0.444… has no place on the number line generally justified their responses by using infinity and this finding is consistent with the related literature (Mamolo, 2009; Peled & Hershkovitz, 1999; Sirotic & Zazkis, 2007b). Indeed, all in-service secondary teachers utilized infinity in order to justify that 0.444… cannot be located on the number line. Similarly, for irrational numbers π and e, almost one-third of the pre-service secondary and in-service secondary teachers utilized infinity in order to justify that these numbers cannot be located on the number line. This can be due to the fact that these secondary school teachers might use repeating decimals and irrational numbers while teaching limit concept to high school students and they might have associated infinity in limit concept with these numbers. In the present study, participants emphasized infinity because of infinitely many digits in the decimal expansion of irrational numbers and this led them to consider that either the exact locations of them on the number line cannot be found or they have no place on the number line confirming the findings of the previous studies (Guven, Cekmez & Karatas, 2011; Hayfa & Saikaly, 2016; Mamolo, 2009; Peled & Hershkovitz, 1999; Sirotic & Zazkis, 2007b). In terms of irrationality, in the present study participants asserted that rational numbers can be represented on the number line, but irrational numbers cannot be located on the number line similar to conclusions drawn by Peled and Hershkovitz (1999), Guven, Cekmez and Karatas (2011) and Kidron (2018). This could be attributed to the fact that participants might have never seen an irrational number located on the number line or they do not know that each real number corresponds to a point on the number line (Sirotic & Zazkis, 2007b). In the present study, participants also mentioned uncertainty of the irrational numbers. The idea that irrational numbers cannot be real since they cannot be written down fully (Stewart, 1995) might have led them to consider that irrational numbers cannot be located on the number line or their locations on the number line are not exact.

In addition to infinity, irrationality and uncertainty, some of the participants indicated that given irrational numbers cannot be located on the number line because there is no pattern in the decimal expansion of them. These participants might have considered that pattern in the decimal expansion of a number is a requirement for having a place on the number line. This finding is consistent with the findings of the study conducted by Zazkis and Sirotic (2010) in which pre-service secondary teachers stated that the number is rational if there is a pattern and the number is irrational if there is no pattern in decimal expansion of the number. This conflict might have stemmed from "poor understanding of the relationship between fractions and their decimal representations" (Zazkis & Sirotic, 2010, p. 15) by the pre-service and in-service teachers in the present study.

Some of the participants asserted in their justifications that irrational numbers, particularly π, are special numbers. The consideration of the pre-service and in-service teachers that irrational numbers are special numbers may be due to the emphasis on the speciality of the number π. In other words, proclaim of UNESCO regarding celebrating Pi Day as International Day of Mathematics and existence of the contents about inclusion of all number combinations in π in social media and films may have had an effect on the pre-service and in-service teachers' responses. Moreover, some of the participants justified their responses regarding that $\sqrt{2}$ cannot be located on the number line by stating that the number line is not divided into equal parts of $\sqrt{1}, \sqrt{2}, \sqrt{3}, \ \sqrt{4}\ldots$ and so on. This can be due to the fact that the number line is generally drawn on the board only representing the integers in order of such as…, -3, -2, -1, 0, 1, 2, 3, … since elementary school.

In conclusion, although 20 years have passed since the first research studies have conducted about locating repeating decimals and irrational numbers on the number line, it was seen that pre-service and in-service teachers still make an error in locating these numbers on the number line in the year of 2020. We realized that main sources of their errors identified in their justifications are approximation, infinity, irrationality and uncertainty. Therefore, it is necessary to eliminate these sources of errors which lead to thought that repeating decimals and irrational numbers cannot be located on the number line among the pre-service and in-service teachers. For this purpose, we should help pre-service and in-service teachers gain two types of knowledge mentioned before. Firstly, we should teach them how to represent irrational numbers on the number line geometrically. Secondly, we should provide opportunities for them to realize that for each real number there is a point on the number line. In addition to these two types of knowledge, most importantly, we should emphasize the number line in teaching and learning process of approximation, infinity, irrationality and uncertainty that emerged in this study and we should teach them how to represent irrational numbers especially $\sqrt{2}$, π and e and repeating decimals on the number line. Thus, we recommend that there could be more content in teacher education courses regarding repeating decimals and irrational numbers and locating them on the number line. In-service teachers could be unaware of the presence of irrational numbers beyond π, e, and common square roots (Sirotic & Zazkis, 2007a). Hence, training and seminars should be provided to in-service teachers in order to enhance their knowledge and experience in irrational numbers. For future research studies, it is recommended that teaching and learning environments for conceptual and deep understanding of repeating decimals and irrational numbers can be designed. In addition, studies related to approximation, infinity, irrationality, and uncertainty as specific to location of numbers on the number line can be conducted.

References

Adams, R. A., & Essex, C. (2009). *Calculus: A complete course* (7th ed.). Canada: Pearson Education.

Arcavi, A., Bruckheimer, M., & Ben-Zvi, R. (1987). History of mathematics for teachers: The case of irrational numbers. *For the Learning of Mathematics*, *7*(2), 18–23.

Fischbein, E. R., Jehiam, R., & Cohen, C. (1995). The concept of irrational number in high school student and prospective teachers. *Educational Studies in Mathematics, 29*(1), 29–44.

Guven, B., Cekmez, E., & Karatas, I. (2011). Examining preservice elementary mathematics teachers' understanding about irrational numbers. *Problems, Resources and Issues in Mathematics Undergraduate Studies*, *21*(5), 401–416. https://doi.org/10.1080/10511970903256928

Hayfa, N., & Saikaly, L. (2016). Dimensions of knowledge and ways of thinking of irrational numbers. *Athens Journal of Education, 3*(2), 137-154. https://doi.org/10.30958/aje.3-2-3

Kidron, I. (2018). Students' conceptions of irrational numbers. *International Journal of Research in Undergraduate Mathematics Education*, 4, 94–118. https://doi.org/10.1007/s40753-018-0071-z

Mamolo, A. (2009). Intuitions of 'infinite numbers': Infinite magnitude vs. infinite representation. *The Montana Mathematics Enthusiast, 6*(3), 305-330.

McKeague, C. P. (2014). *Elementary Algebra* (2nd ed.). New York: Academic Press.

Ministry of National Education [MoNE] (2018). *Matematik dersi öğretim programı (ilkokul ve ortaokul 1, 2, 3, 4, 5, 6, 7 ve 8. sınıflar)* [Elementary and Middle School Mathematics Education Program (Grades 1, 2, 3, 4, 5, 6, 7, 8)]. Ankara: T.C. Milli Eğitim Bakanlığı.

OCG. (2005). The Ontario Curriculum Grades 1-8: Mathematics, 2005. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/math18curr.pdf.

Peled, I., & Hershkovitz, S. (1999). Difficulties in knowledge integration: Revisiting Zeno's paradox with irrational numbers. *International Journal of Mathematical Education in Science and Technology, 30*(1), 39–46.

Sirotic, N., & Zazkis, R. (2007a). Irrational numbers: The gap between formal and intuitive knowledge. *Educational Studies in Mathematics, 65*(1), 49–76. https://doi.org/10.1007/s10649-006-9041-5

Sirotic, N., & Zazkis, R. (2007b). Irrational numbers on the number line – where are they? *International Journal of Mathematical Education in Science and Technology, 38*(4), 477–488. https://doi.org/10.1080/00207390601151828

Stewart, I. (1995). *Nature's numbers*. New York: Basic Books.

Zazkis, R., & Sirotic, N. (2010). Representing and defining irrational numbers: Exposing the missing link. *CBMS Issues in Mathematics Education, 16*, 1-27. doi: 10.1090/cbmath/016/01

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