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Comparing Extracted and Stipulated Definitions in Algebra 1 Textbooks and *Khan Academy*

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ABSTRACT

Previous research has established the importance of definitions in mathematics and have distinguished the difference between extracted and stipulated definitions. Much remains unknown, however, about the role definitions serve in developing students' disciplinary literacy. In this study, we analyze how definitions from four US Algebra 1 textbooks and *Khan Academy* define vocabulary in the context of quadratics. Results show that *Khan Academy* tends to use extracted definitions, where textbooks tend to use stipulated definitions. Implications from this study are that there is a need to teach students both stipulated and extracted definitions.

Keywords: mathematical definitions, stipulated, extracted, disciplinary literacy, Khan Academy

INTRODUCTION

Research has shown that students are often intimidated and do not learn as well from a traditional textbook (Buehl, 2017). As a result, students will use alternative resources, such as *Khan Academy* to give them the individualized help that they need. *Khan Academy* has over 12,000,000 users a month, and features 7-14 minute long instructional videos, which provide explanations and definitions of concepts (Khan Academy, 2017a; Thompson, 2011). Furthermore, since students develop mathematical literacy by understanding texts produced by others, there is a need for students to learn directly from a textbook or *Khan Academy* (Hillman, 2014).

Mathematics has a distinct language, as everyday words often have a different and precise meaning in a mathematical context (Edwards & Ward, 2004; Hillman, 2014). This is because mathematics is a deductive theory, starting with primary notions and axioms. The role of definitions in mathematics is critical to students learning outcomes as understanding of concepts are mainly acquired through their definitions in mathematics (Vinner, 1991). Therefore, analyzing how different textbooks and *Khan Academy* define vocabulary offers insight into how each respective text defines key vocabulary. Thus, the objective of this investigation is to analyze how different US Algebra 1 textbooks and *Khan Academy* define key vocabulary in the context of quadratics. This will be done by using the concept of disciplinary literacy to justify the analysis of definitions.

Specifically, definitions will be classified as either stipulated, or extracted. Extracted definitions will be defined as "definitions based on examples of actual usage" (Landau, 2001, p.165). Thus, extracted definitions have a truth value, meaning they are not always true. Stipulated definitions will be defined as "explicit and self-conscious setting up of the meaning-relation between some word and some object" (Robinson, 1962, p.135). Note that stipulated definitions have no truth value, and thus they will always be free from connotation (Edwards & Ward, 2001, p.223). Classifying definitions as either extracted or stipulated is important because

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"the role and use of mathematical definitions is deeply imbedded in the culture of working mathematicians" and thus mathematical definitions should be differentiated from everyday language (Edwards & Ward, 2001, p.224).

Students must use stipulated definitions in order to become discourse insiders into the subject of mathematics, as using and understanding stipulated definitions is a characteristic of mathematicians (Buehl, 2017, Edwards, B. S., & Ward, 2004). Stipulated definitions allow students to communicate ideas, defend their reasoning, construct arguments, and draw logical conclusions. Furthermore, using stipulated definitions correctly is one way for students to demonstrate mathematical literacy (Hillman, 2014).

In contrast, mathematical terminology can often confuse students, and a more conceptual way of understanding content is needed (Vinner, 1991). Students need to connect mathematics to the real world in order to develop an understanding of a concept, as students have an easier time making meaning of words through personal connections, such as concept images (Bruun, Diaz, & Dykes, 2015). Thus, extracted definitions are best for students development of a concept image. It follows that there is a need for extracted mathematical definitions to help students understand the content. Thus, a dialectic is formed between extracted and stipulated definitions, as there is a simultaneous need for both types of definitions in the classroom.

While textbooks do not always dictate how vocabulary is defined in the classroom, research has shown that textbooks are the most influential resource that impacts what happens in the classroom (Son & Diletti, 2017). Thus, this research ask's the question of how do textbooks and *Khan Academy* define vocabulary in the context of quadratics? The purpose of this study is to see the relationship between stipulated and extracted definitions within Algebra 1 textbooks and *Khan Academy*.

THEORY

Disciplinary Literacy

Disciplinary literacy is the ability to read, write, listen, speak, and think critically in the context of a given discipline (Hillman, 2014; Shanahan & Shanahan, 2008). This is important as research about disciplinary literacy has shown that each discipline understands and uses language differently (Fang & Schleppegrell, 2010; Schleppegrell, 2007). In mathematics, disciplinary literacy includes the ability to defend one's reasoning, construct arguments, communicate mathematically, and draw logical conclusions (Hillman, 2014).

Furthermore, mathematical literacy lends structure to students' reasoning. This is because the language of mathematics is precise, as every word is included deliberately, and the slightest change will alter the meaning of an entire proposition (Edwards, B. S., & Ward, 2004; Shanahan & Shanahan, 2008). Understanding mathematical terminology allows students to communicate mathematical ideas to others, as it provides students with a common language that represents ideas. Teaching students how to think like mathematicians is imperative, as the National Council of Teaching Mathematics claimed that teaching disciplinary literacy is the goal of teaching mathematics (NCTM, 1989).

Common Core State Standards mathematical practice standards also include constructing viable arguments and critiquing the reasoning of others. Common core state standards states, "mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments" (Standards for Mathematical Practice, 2010). Furthermore, CCSS includes attention to precision when communicating to others, as mathematically proficient students, "use clear definitions in discussion with others and in their own reasoning" (Standards for Mathematical Practice, 2010). Thus, this research is valuable to both researchers and practitioners, as the focus of this project is the role definitions play in mathematical literacy.

The Importance of Multiple Representations

Hillman states that the ability to express mathematical ideas with multiple representations is the fourth CCSS practice (2014). Research has stated that mathematical representations are fundamental for learning mathematics, as representing objects in multiple ways plays an important role for mathematical understanding (Dreher & Kuntze, 2015). There is also great importance in making connections between different representations of objects, as this helps students to develop mathematical literacy (Duval, 2006). Furthermore, previous research has found that there is a lack of multiple representations used in 8th grade Algebra 1 textbooks (Alkhateeb, 2019).

Mathematical Definitions

Mathematical definitions play a different role than everyday language in mathematics. Edwards and Ward state that "mathematical definitions are of fundamental importance in the axiomatic structure that characterizes mathematics" (2001, p.223). This is because mathematical definitions give structure to mathematics, giving mathematicians the ability to communicate ideas through definitions. Furthermore, mathematical definitions also play a role in how students understand a given concept, as Edwards and Ward state, "definitions are often used as a vehicle toward a more robust understanding of a given concept" (2001, p.223). So, students come to understand mathematical concepts through their definitions. Thus, the differences between stipulated and extracted definitions will be examined.

The role of stipulated definitions

"Precise mathematical" definitions will be referred to as stipulated definitions in the context of this research. Lexicographer Sidney Landau states that stipulated definitions "are imposed on the basis of expert advice" with the goal of "ease of accuracy of communication between those versed in the language" (2001, p.165). This implies that students use stipulated definitions to defend their reasoning, construct arguments, communicate mathematically, and draw logical conclusions, which helps students develop mathematical literacy (Hillman, 2014). Furthermore, stipulated definitions align with CCSS mathematical practices by giving students the language necessary to construct viable arguments and communicate with others (Standards for Mathematical Practice, 2010). By understanding and using stipulated definitions, students are able to develop disciplinary literacy. Therefore, students should be taught stipulated definitions, and the importance of them to mathematics.

The role of extracted definitions

"Student friendly" definitions will be referred to as extracted definitions in the context of this research. Landau stated that extracted definitions are "definitions that are based on examples of actual usage, definitions *extracted* from a body of evidence" (2001, p.165). Vinner discusses how definitions should be taught by giving various examples and counterexamples of a definition. This helps students to create a concept image, which is something non-verbal associated in students mind with the concept name (1991). This relates to extracted definitions as the relationship between math and the real world play a central role in developing a deeper understanding of a mathematical concept (Kaiser & Willander, 2005). Therefore, students should be taught extracted definitions, to help them create concept images that relates definitions to the real world.

METHODS

Textbook Selection

The textbooks selected for this project come from four different US public school districts and were all used for eighth grade students during the 2018-2019 school year. The textbooks analyzed are the following; *Algebra 1 Common Core*, *Algebra 1 A Common Core Curriculum*, *Core Connections Algebra*, and *Springboard Algebra One*. The *Khan Academy* videos selected were *Visual introduction to parabolas* and *Solving equations using the zero-product property* (Khan Academy, 2017a, 2017b). These videos were selected as they covered the same concepts about quadratics as in the textbooks. Furthermore, both videos have over 239,000 YouTube views respectively, so these *Khan Academy* videos are viewed by a large audience.

Topic Selection

The specific topic selected for this project was quadratics. Quadratics was selected as CCSS includes the following standard:

CCSS.MATH.CONTENT.HSF.IF.C.7.A

Graph linear and quadratic functions and show intercepts, maxima, and minima (High School: Functions » Interpreting Functions., n.d.).

Furthermore, the topic of quadratics was selected because many studies have revealed that students have difficulties in understanding the concept of quadratics (Eraslan, 2008; Kotsopoulos, 2007; Zaslavsky, 1997). The topic of quadratics was also selected as if students develop a good understanding of quadratic functions, then it becomes easier for them to build a good understanding of more complex and different types of functions and concepts (Hoon, Singh, & Halim, 2018).

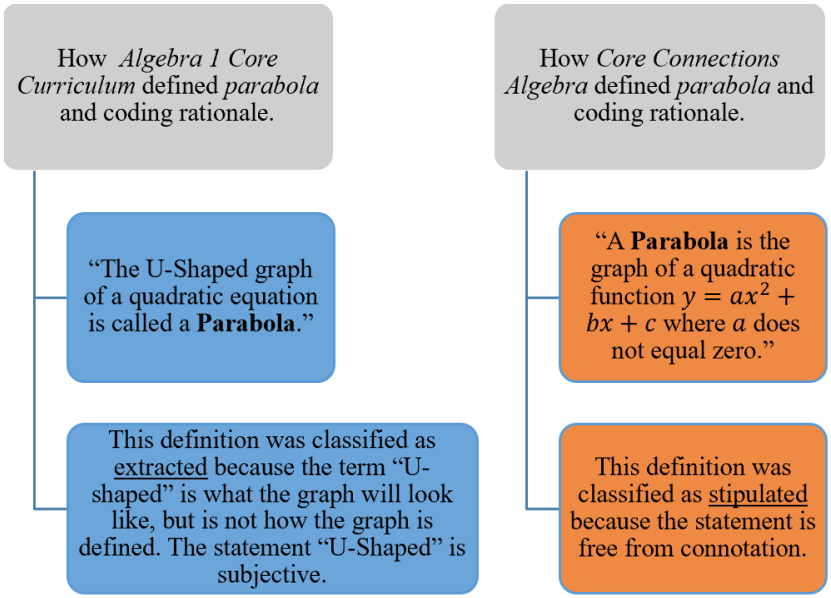


Figure 1. Example of coding *parabola*

Definitions Selection

The focal definitions analyzed are the following: *parabola, quadratic function, quadratic equation, axis of symmetry, vertex, minimum, maximum, roots of the equation, zero product property,* and *intercepts*. These definitions were analyzed as each text has at least seven of the eleven focal definitions introduced in their respective lessons introducing quadratics. There were 42 definitions coded overall. *Algebra 1 A common Core Curriculum* contributed ten focal definitions, *Algebra 1 Common Core* and *Core Connections Algebra* contributed nine focal definitions respectively, and *Springboard Algebra One* and *Khan Academy* contributed seven focal definitions respectively.

Method of Analysis

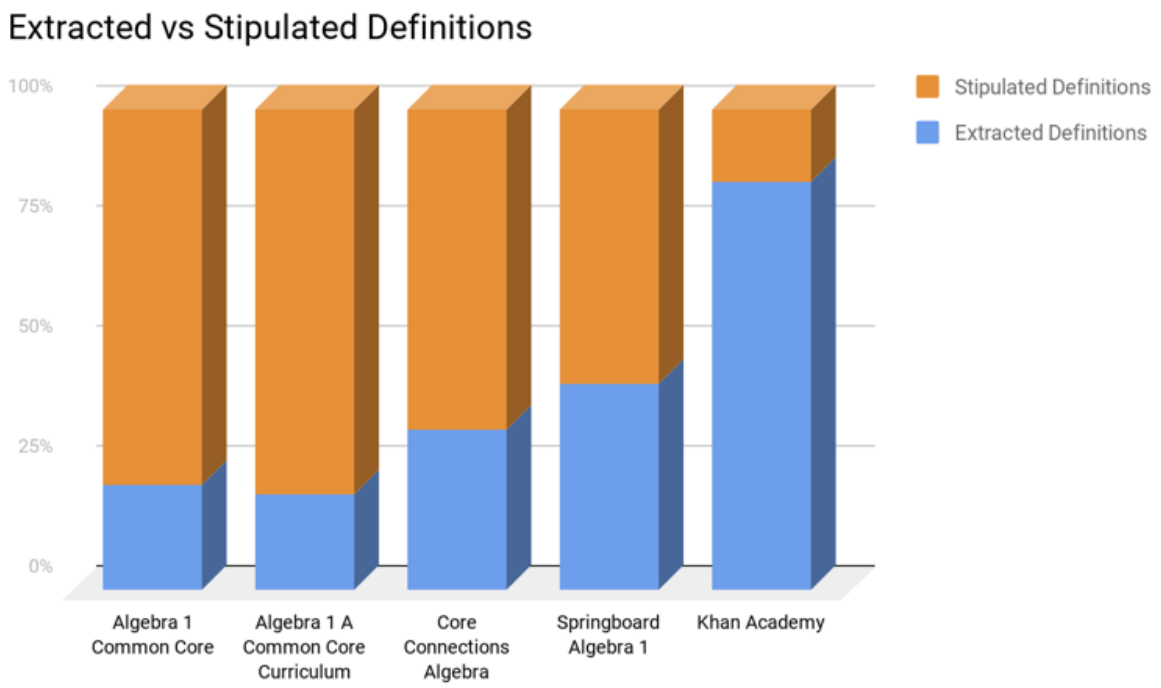
Each of the 42 focal definitions were coded according to whether they met the criteria established by Edwards, B. S., & Ward of being a stipulated or extracted definition (2004). Definitions that have a truth value were classified as extracted definitions. A definition has a truth value if it is defined by an example, uses non-mathematical terminology, or if a counterexample could be found where that definition would not be true. Furthermore, definitions that strictly used mathematical terminology and were found to always be free of connotation were classified as stipulated definitions, as they have no truth value. To verify the reliability of the coding, the second author double-coded 100% of the definitions, reaching 95% agreement. Differences were resolved by discussion.

Algebra 1 Core Curriculum's definition of *parabola* was coded as extracted as it is a definition based on actual usage, since the term "U-Shaped graph" is used in defining *parabola* (Landau, 2001). Furthermore, *Core Connections Algebra's* definition of *parabola* has no truth value and was thus coded as stipulated Edwards, B. S., & Ward, 2004).

RESULTS

Graph 1 shows the gathered data from the coding of the 42 focal definitions. This data includes percentage of definitions defined as stipulated and extracted.

The graph reveals that each of the textbooks defined the majority of their focal definitions in a stipulated fashion. For example, *Algebra 1 Common Core* featured 22% extracted definitions, and 78% stipulated definition. In contrast, *Khan Academy* defined a 85% of focal definitions in an extracted manner. Note that the definitions for *Quadratic Function, Quadratic Equation,* and *Roots of the Equation* were all coded as stipulated definitions for all texts except *Khan Academy*. Furthermore, the definition *Vertex* was classified as stipulated for all texts and *Khan Academy*.



Graph 1. Extracted vs Stipulated Definitions

FINDINGS

Data Results

Results show that overall, a greater percentage of stipulated definitions were used in textbooks, where *Khan Academy* had a greater percentage of extracted definitions. Note that extracted and stipulated definitions were found in all of the textbooks. Furthermore, no definitions were coded as both stipulated and extracted, as each text only defined each term once.

Since all of the textbooks and *Khan Academy* contain both stipulated and extracted definitions, it is implied that they all recognize students' need for both types of definitions. However, none of the textbooks or *Khan Academy* define the same term in a stipulated and extracted manner. Thus, students who are only reading one of the textbooks, or watching *Khan Academy* are exposed to one type of definition for each term.

Addressing Textbooks

The extensive use of stipulated definitions in textbooks is valuable to students, as it helps them develop a "more robust understanding of a given topic" (Edwards, B. S., & Ward, 2004, p.223). Furthermore, Landau states that stipulated definitions help students to accurately communicate ideas to others who understand mathematical language (2001). Thus, the use of stipulated definitions directly aligns with the CCSS of constructing viable arguments and communicating with others (Standards for Mathematical Practice, 2010).

Textbooks' use of stipulated definitions also helps students develop disciplinary literacy. This is because stipulated definitions help students defend reasoning, construct arguments, communicate mathematically, and draw logical connections (Hillman, 2014). Therefore, students should learn stipulated definitions to meet CCSS and to develop mathematical literacy.

Addressing Khan Academy

Khan Academy had a greater overall percentage of extracted definition usage when compared to the textbooks. Results showed that *Khan Academy* would often define terms by connecting them to the real world, which helps students create a concept image for definitions (Vinner, 1991). One example is how *Khan Academy* defines *parabola* in the video *Visual introduction to parabolas,* as they state, "well my brain immediately imagined this is the trajectory, that this is the path that is a pretty good approximation for the path of things that are actually thrown" (2017a). By relating the term *parabola* to the real-life application of measuring the

trajectory of a thrown object, *Khan Academy* helps students create a concept image (Kaiser & Willander, 2005; Vinner, 1991). Even though this definition does not rigorously define what a *parabola* is, it still gives students an understanding of what *parabolas* are used for.

Khan Academy also helps students create concept images by giving various examples and nonexamples when defining terms (Vinner, 1991). One instance of this is how *Khan Academy* defines *zero product property* in the video *Solving equations with zero product property*. *Khan Academy* presents viewers with an equation of a factored polynomial set equal to zero, and then instructs viewers to, "pause this video and see if you can figure out the x values that would satisfy this equation" (2017b). By using an example to explain *zero product property*, *Khan Academy* helps students understand what the concept is by means of its applications. Furthermore, *Khan Academy* is teaching students mathematical literacy by having them draw logical conclusions (Hillman, 2014). Both of these examples help students to make meaning of concepts through personal connections (Bruun, Diaz, & Dykes, 2015). Thus, students should be taught extracted definitions to help them develop concept images.

Addressing Different Representations

Khan Academy is different from textbooks, as they deliver content through videos, allowing students to both listen and see the lecture as it progresses. Furthermore, students can pause the videos and rewatch them as needed. This is different from traditional textbooks, which promote mathematical literacy by having students read mathematical texts produced by others (Hillman, 2014). While textbooks and *Khan Academy* are fundamentally different representations, the data shows that the two mediums also defined definitions in different ways.

The implications from the data are that utilizing both *Khan Academy* and textbooks would be one way to give 8th grade Algebra 1 students different representations of the same concept (Alkhateeb, 2019). By giving students different representations of concepts, this would help them to develop a better understanding of the definitions (Dreher & Kuntze, 2015). Furthermore, it is implied that *Khan Academy* should be integrated with learning from a textbook, as making connections between different types of definitions is crucial to developing mathematical literacy (Duval, 2006).

DISCUSSION/FURTHER RESEARCH

There is a need for teaching students extracted definitions, as the relationship between mathematics and the real world must play a central role for students to develop a deeper understanding of mathematical concepts (Kaiser & Willander, 2005). This helps students develop concept images to enhance their understanding of definitions (Vinner, 1991). Simultaneously, there is a need for teaching stipulated definitions, as this is one of the defining characteristics of mathematics, thus understanding stipulated definitions in order to develop mathematical literacy in students (Edwards, B. S., & Ward, 2004, Hillman, 2014). Research shows that there is great importance in using multiple representations of concepts, so teaching students both stipulated and extracted definitions would be effective teaching pedagogy (Alkhateeb, 2018). Furthermore, engaging students in the process of stipulating and extracting definitions, as having students make connections between different representations of objects helps them to develop mathematical literacy (Duval, 2006).

Implications for Mathematics Teachers

The implications for mathematics teachers are that they should integrate both stipulated and extracted definitions into their lessons. To help students develop mathematical literacy, mathematics teachers should take time to explain to students the differences between stipulated and extracted definitions. Furthermore, it would be worthwhile to explain to students why definitions are so important to mathematics, and how mathematical definitions are different from other subjects.

Further implications are that mathematics teachers should encourage students to create a glossary of definitions (Hillman, 2014). In these glossaries' students should be encouraged to define terms both in a stipulated and extracted manner. This would help students recognize the need for stipulated definitions, as well as giving them the extracted definitions to help them understand a term more conceptually. An effective exercise would be to engage students in the process of stipulating definitions, as this would help students understand the role of stipulated definitions in mathematics.

Implications for Further Research

Further research should investigate how these focal definitions are actually defined in the classroom. Are these focal definitions defined to the class as they appear in the book, or do teachers give students alternative definitions? Research should be done on if textbooks vary their use of stipulated and extracted definitions based on grade level. For example, would Algebra 1 textbooks designed for ninth graders yield different results?

Investigating how teachers integrate stipulated and extracted definitions in their classroom would beneficial as well, as reporting effective techniques would provide mathematics teachers valuable strategies to teach definitions. Finally, a study similar to Mosvold & Fauskanger's study that focused on US teachers' beliefs about teaching definitions would give insight into their understanding of stipulated and extracted definitions (2013).

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