Eurasian Journal of Educational Research (2020) 135-168

Eurasian Journal of Educational Research
www.ejer.com.tr

Development of Secondary School Students' Relational Thinking Skills with a Teaching Experiment*

Nilufer YAVUZSOY KOSE[1], Ayhan KIZILTOPRAK[2]

ARTICLE INFO

Article History:
Received: 06 Feb. 2019
Received in revised form: 18 Oct. 2019
Accepted: 10 Nov. 2019
DOI: 10.14689/ejer.2020.85.7

Keywords
mathematics education,
relational thinking,
equality, equal sign

ABSTRACT

Purpose: Since there are a limited number of studies on how to develop relational thinking in secondary school students in mathematics education literature, this study will contribute to the field both in theoretical terms and concerning the implications for in-class applications. In this respect, this study aims to examine how to develop the relational thinking skills of 5th-grade students.
Research Methods: The participants of this study, which was adopted as a research design of the teaching experiment, were six students attending 5th grade in secondary school. The teaching process was eight sessions per week with one session. The main data source of this study was in-class teaching videos. The data were analyzed descriptively.
Findings: The questions, the in-class dialogues directing relational thinking and activities in each session of the teaching experiment conducted with the fifth-grade students were presented under related themes.
Implications for Research and Practice: The most general result was that at the end of the teaching process based on numbers, relationships between numbers, operations and properties, the students made use of equality axioms to evaluate the true/false and open number sentences without any calculation. It was also seen that the students made connections between addition-subtraction, addition-multiplication and multiplication-division and that they made effective use of commutative, associative and distributive properties.

[1] Corresponding Author: Anadolu University, Faculty of Education, TURKEY, e-mail: nyavuzsoy@anadolu.edu.tr, ORCID: https://orcid.org/0000-0001-7407-7498
[2] Ministry of National Education-Mathematics Teacher, PhD Candidate-Anadolu University, TURKEY, e-mail: toprakkizil@hotmail.com, ORCID: https://orcid.org/0000-0002-6820-0565

Introduction

Children are first introduced to arithmetic, and they then start learning algebra with symbols and connections. Algebra allows people to make simple algebraic descriptions or to use letters and symbols while dealing with equations that seem to be complicated for most people (Blanton, 2008). The transition from arithmetic to algebra starts at early age with activities related to numbers and is expected to be generalized towards the end of elementary school education. Generalized arithmetic refers to a generalization of properties of numbers and operations. Boulton et al. (2000) claim that to achieve the transition from arithmetic to algebra in accordance with the consecutive development model of the algebraic knowledge, students should first have the knowledge and skills found in the arithmetic step of this model. This knowledge includes knowledge of fundamental properties of operations, such as commutative property, associative property and distributive property, and the skills include the ability to work backward and to recognize that the values on both sides of an equal sign are the same. The development of basic arithmetic operation skills allows writing number sentences and understanding a number in various forms (7-2=5, 3+2=5, 5+2=7). Students can deal with number sentences given in the form of true/false or open number sentences by focusing on the relationship between equality and numbers. Both the fragmentation of numbers in different ways and the association between equality and numbers are extremely critical for the generalized arithmetic and require relational thinking, which has an important role in the development of algebraic thinking.

According to Koehler (2004), relational thinking provides a different perspective to arithmetic rather than direct calculation and plays a key role in learning arithmetic. Stephens (2006) points out that relational thinking basically depends on students' use and understanding of varieties between numbers in a number sentence. Carpenter, Franke and Levi (2003) claim that relational thinking should be taught to students for two reasons. First, relational thinking, which provides flexibility and allows acting fast in teaching arithmetic, is also a prerequisite to algebraic thinking. During the elementary grades, much instructional time in mathematics is devoted to developing fluency with multiplication (Stephens, Ellis, Blanton & Brizuela, 2017). For example, Carpenter, Levi, Berman and Pligge (2005) found that especially elementary school students use the distributive property in number sentences involving multiplication. Baek (2008) also points out that the primary school students (3.-5.) who understood; especially the associative and distributive properties of multiplication were successful in solving verbal multiplication problems involving multi-digit numbers. This research results show that students intuitively make judgments based on operations properties, such that this process prepares students for algebraic thinking. Secondly, relational thinking provides a substantial basis for the transition to algebra. Students regard four operations as a process of doing operation when they learn using traditional methods. In relational thinking, number sentences are taken into account as a whole rather than as processes that have to be followed step by step, and the purpose is to have students avoid calculations and to help them recognize that both sides of equality represent the same numbers. The focus of this purpose lies in the concept of equality. Therefore, equality and relational thinking should not be separated from one another.

In literature, many researchers focus on the meaning of the equal sign and the concept of equality. In this respect, most of the studies have been carried out with elementary school first, second and third-grade students. For example, Koehler (2004) worked with first and second-grade primary school students, Carpenter et al. (2003) worked with primary school students at all levels, Molina and Ambrose (2006) worked with third-grade students, Molina, Castro and Mason (2008) with eight-year-old students, Molina and Mason (2009) worked with eight and nine-year-olds, Eichhorn, Perry and Brombacher (2018) worked with 2nd and 3rd-grade students with an average age of eight years and four months.

In the literature, there are also studies that focused on the use of the equal sign in mathematics textbooks and on the extent to which these textbooks supported relational thinking (Seo & Ginsburg, 2003; Köse & Tanışlı, 2011). For example, Seo and Ginsburg (2003) stated that the contents of the elementary school mathematics textbooks they studied were limited in supporting relational thinking, that equality was matched with performing an operation, and that the number sentences were predominantly given in a standard format, such as a + b = c or a-b = c.

Some of the studies in the literature examined the relationship between the relational meaning of the equal sign and secondary school students' solving equation problems (Alibali, Knuth, Hattikudur, McNeil & Stephens, 2007) or simple linear equations (Knuth et al., 2006). These researchers agreed that students did not have any disposition towards operational understanding while forming the relational meaning and that this situation was a reflection of the process of teaching the concepts (Stephens, Ellis, Blanton & Brizuela, 2017, p. 391). Thanks to this agreement, the researchers started to focus on how teaching processes should be while forming the relational meaning of the equal sign and to develop relational thinking. In studies carried out with students at early ages, the findings showed that early algebra teaching involving the generalized arithmetic approach allowed students to recognize basic operations and properties of operations, such as commutative, to produce different ways of thinking in their reasoning about numbers, operations and properties of operations and even to make various generalizations (Carpenter et al., 2003; Bastable & Schifter, 2008; Blanton et al., 2015; Steinweg, Akinwunmi & Lenz, 2018). When the related literature is examined, it could be stated that the studies examining the teaching processes and focusing on the relational meaning of the equal sign and the development of relational thinking were mostly carried out with preschool and primary school students (Carpenter et al., 2003; Blanton et al., 2015; Steinweg et al., 2018; Strachota, Knuth & Blanton, 2018) and that relevant studies conducted with secondary school students were limited (Napaphun, 2012). The main reason for this is that equality is a key concept in mathematics since preschool. On the other hand, the secondary school fifth grade, which involves a transition from arithmetic and algebra could be regarded as a key grade for the development of students' thinking (Kızıltoprak & Yavuzsoy Köse, 2017). In this respect, it is important to reveal students' ways of thinking and reasoning with the help of a teaching process, which will develop relational thinking. In the first phase of this study, in which clinical interviews were used to examine relational thinking skills of fifth-grade students before and after the teaching process (Kızıltoprak &Yavuzsoy Köse, 2017), it was seen that the students answered the open and true/false number sentences based on the result-oriented operational process during the pre-interviews; that they answered these open and true/false number sentences based on the relationships between numbers and operations during the post-interviews; and that they all developed their relational thinking skills. This result could be said to reflect the teaching process. The present study, which is the second phase, aimed to examine the teaching process conducted and to present in detail with the help of sample activities and in-class discussions on how the properties of operations were used for the development of the students' relational thinking and which concrete materials were used.

Research Purpose

In this study, the purpose was to examine how to develop relational thinking skills of fifth-grade students. In line with this purpose, the following research question was directed:

- How are the number sentences and properties of operations used in the teaching process to develop students' relational thinking skills?

This study, which focused on the development of relational thinking, is thought to act as a guide for mathematics teachers for teaching the relational meaning of the concept of equality as well as for providing a holistic view regarding operations and operational properties. The designed teaching process will not only allow presenting the given concepts, their order of presentation and the related materials but also help explain how and which number sentences will be used for the development of students' thoughts. Given that the concept of equality and arithmetic operations are taught starting from the primary school first grade, the present study could be said to be beneficial for those interested in curriculum development, for those authoring mathematics textbooks and mathematics teachers.

Equality and Relational Thinking

Equality is one of the first mathematical concepts that students start learning at the primary school level. In addition, studies carried out after 1980s at different levels ranging from preschool to high school revealed that students fail to understand the concept of equality and the equal sign (e.g. Falkner, Levi, & Carpenter, 1999; McNeil & Alibali, 2005; Matthews, Rittle-Johnson, McEldoon & Taylor, 2012; Sáenz-Ludlow & Walgamuth, 1998). Understanding mathematical equality requires that the values on both sides of the equal sign must be the same and that this is not an easy process (Kızıltoprak & Yavuzsoy Köse, 2017). Unfortunately, most primary, elementary and middle school students do not focus on the relational meaning of the equal sign, which is an indicator of the concept of equality. However, they tend to regard its operational meaning as a command to be used for the application of arithmetic operations (Rittle-Johnson, Matthews, Taylor &McEldoon, 2011). For instance, Falkner and colleagues (1999) reported that most primary school students (first and second grades) regarded the equal sign in an open number sentence like 8+4=…+5 as the application of the operation and that they put 12 or 17 in the blank given in the number sentence. In the study, only a few students reported that the equal sign represented a relationship and that the values on both sides of the equality should be the same. This study by Falkner et al. (1999) could be said to constitute a ground for many studies regarding the equal sign. The results of many other studies demonstrated that students tended to focus on the operational meaning of the equal sign and to regard the sign as the answer, as the total or as adding the numbers given before the sign (Byrd, McNeil, Chesney & Matthews, 2015; Matthews & Rittle-Johnson, 2009; McNeil & Alibali, 2005; Sáenz-Ludlow & Walgamuth, 1998).

The structure of the number sentence also has an important role in understanding the equal sign. For example, students not only experience difficulty in dealing with equalities which do not have the standard structure of a+b=c given as operations-equals-answer but also think that the number sentences in the structure of only operations-equals-answer are true while evaluating whether a number sentence is true or false (Falkner et al., 1999; Rittle-Johnson & Alibali, 1999). Seo and Ginsburg (2003) reported that the students did not accept the equalities which included operations on the right side (c=a+b) or on both sides (a+b=c+d) or no operations (c=c). Given that students mostly face the operational meaning of the equal sign both in their textbooks (Seo & Ginsburg, 2003; Köse & Tanışlı, 2011) and in in-class activities (Carpenter et al., 2003) makes it more difficult to understand the concept of equality. However, a developmental process involving continuity requires understanding the concept of equality (Rittle-Johnson et al., 2011), and it is important that students meet different equality structures not only in in-class activities but also in resources they use (e.g., textbooks, workbooks).

In Table 1, it is seen that the equality structures regarding the concept of equality whose development has been presented from Grade 1 to Grade 4 differ and that the students' understanding of these structures changes. This situation shows students' transition (in relation to their knowledge of equality) from the equality structures in the form of operations-equals-answer to equalities involving operations on the right side or no operations and eventually to equalities involving operations on both sides. In addition, studies revealed a relationship between knowledge of the relational meaning of the equal sign and achievement in equalities involving operations on both sides (Alibali et al., 2007; Rittle-Johnson & Alibali, 1999).

Table 1.

Development of Knowledge of Mathematical Equality

Level	*Description*	*Core equation structure*
Level 4: Comparative relational	Successfully solve and evaluate equations by comparing the expressions on the two sides of the equal sign, including using compensatory strategies and recognizing that performing the same operations on both sides maintains equivalence. Recognize the relational definition of the equal sign as the best definition.	Operations on both sides with multidigit numbers or multiple instances of a variable.
Level 3: Basic relational	Successfully solve, evaluate, and encode equation structure with operations on both sides of the equal sign. Recognize and generate a relational definition of the equal sign.	Operations on both sides: $a+b=c+d$ $a+b-c=d+e$
Level 2: Flexible operational	Successfully solve, evaluate, and encode atypical equation structures that remain compatible with an operational view of the equal sign.	Operations on right: $c=a+b$ or No operation: $a=a$
Level 1: Rigid operational	Only successful with equations with an operations-equals-answer structure, including solving, evaluating, and encoding equations with this structure. Define the equal sign operationally.	Operations on left: $a+b=c$ (including when blank is before the equal sign)

(Rittle-Johnson, Matthews, Taylor & McEldoon, 2011, p. 87)

Students examine and solve the equality structure by comparing the sentences on both sides of the equal sign at the comparative relational level, which is determined to be the top level. In this process, students can make deductions regarding the numbers and operations in equality without any calculation and confirmation. For example, while determining the number to put in the box in the open number sentence of 28+ 42= 27+ □, students may avoid subtracting 27 from the sum of 28 and 42 and can compare the equality and recognize that 27 equals to 28 minus 1. Eventually, they can find 43 as the number to be put in the box. This process is the ability defined exactly as "relational thinking". Therefore, students with the awareness of the relational meaning of the equal sign are likely to evaluate and transform the given number sentences by focusing on the structure of the equality, to relate numbers and operations and to apply different strategies while choosing appropriate numbers and this process is defined as the relational thinking skill. For instance, a student can solve the number sentence of 35+ 48+ 65 = □ by doing calculation from the left to the right. However, the student can also find the result in an easier way by equalizing the sum of the numbers to 100 (35+65). To be able to think in this way, students should see the number sentence as a whole before doing a direct calculation and should be aware of such properties of operations as associative and commutative properties (Jacobs et al., 2007). In relational thinking, the purpose is to have students begin examining the relationships without calculating the answer. In this way, students can transform number sentences by using the relationships between numbers and the fundamentals properties of operations. This situation is beyond doing easy operations, and it allows students to acquire a different thinking skill and most importantly constitutes the basis of learning algebra.

In the present study, number sentences on both sides at third and fourth levels (given in Table 1) were adopted so that the students could evaluate and solve the given number sentences, and the focus was on the students' comparing the structures on both sides of equality and on their effective use of equality axioms to reach the fourth level.

Method

Research Design

In the present study the teaching experiment design was used. The teaching experiment design can be defined as a teaching-based research design in which researchers can reveal their students' knowledge of mathematics and examine the changes in their knowledge in learning environments designed (Steffe &Thompson, 2000).

Participants

The participants in the study were six secondary school fifth-grade students from an average state school. While determining the participants, the criterion sampling method, one of the purposeful sampling methods, was used. The first criterion considered in criterion sampling was to select fifth-grade students. The second criterion was that the selected fifth-grade students were expected to have high levels of oral expression skills. The third criterion was to select fifth-grade students with different levels of achievement. In addition, volunteerism was also considered, and the necessary consents of the related individuals and institutions were taken. Lastly, while presenting the findings, the students' names were kept confidential, and nicknames were used for anonymity.

Procedures

The teaching process was conducted by one of the authors of this paper who was an experienced teacher of mathematics with an M.A. degree in the field of mathematics teaching. The teacher frequently played the role of directing the in-class discussions during the sessions. The discussions were conducted with an inquiry-based approach. The inquiry-based environment defines as student-centered, rich in communication and cooperation, and based on research and asking questions (Chapman, 2011).

The basic purpose of a teaching experiment is to discover students' thinking processes in the learning process, and with the help of appropriate learning environments prepared in line with this discovery, the teacher can experience how mathematical knowledge regarding the target subject or concept is structured (Steffe & Ulrich, 2014). The in-class observations before the teaching process revealed that the students had difficulty telling different meanings of the equal sign and that they used equality to find the result of an operation. It was also seen that the students demonstrated calculation-based thinking without establishing a relationship between the numbers and operations in the number sentences involving arithmetic operations and that they failed to recognize certain properties, such as commutative, associative and distributive properties and had problems, especially with the division. In this respect, a teaching process that aimed to develop the students' relational thinking skills was planned. The teaching process was conducted in eight sessions in eight weeks (once a week). Considering the ages of the students, the sessions were planned in a way to last 30-40 minutes, and breaks were given when necessary. Figure 1 presents the subjects the teacher focused on each week in the research process.

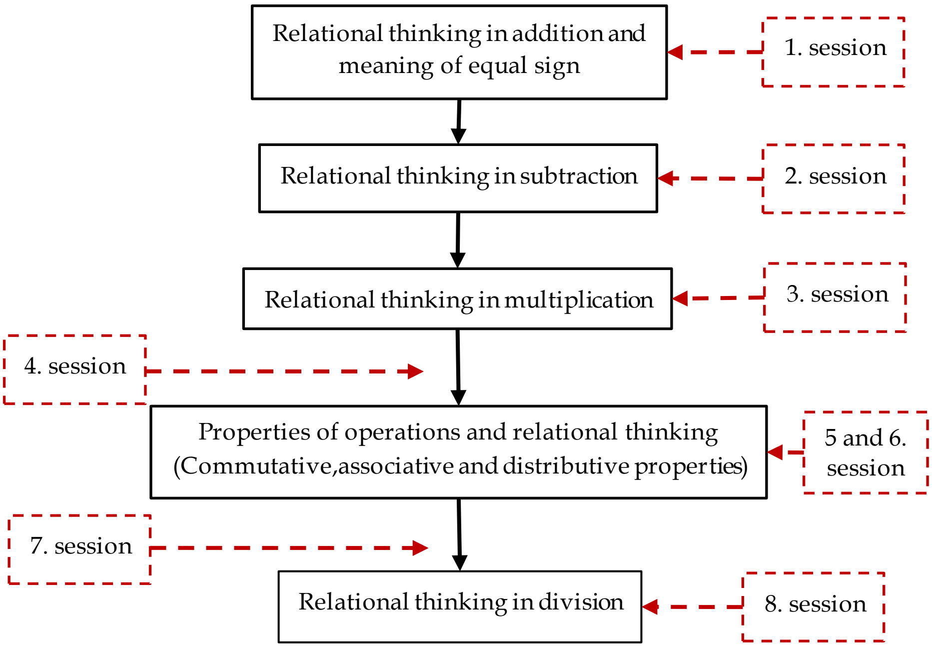


Figure 1. *Teaching Process*.

During the sessions, both individual and group works of the students were supported. In the groups formed, it was paid attention that the students were at different levels. For addition and subtraction, various equality structures which the students were familiar and unfamiliar with like $a \mp b=c, c=a \mp b, a \mp b=c \mp d, a \mp b=c \mp d \mp e, a \mp b \mp c=d \mp e \mp f$ were given in true/false and open number sentences. In the true/false and open number sentences involving multiplication, number sentences involving both addition-subtraction and multiplication like $axb=(axc) \mp d, (axb)+c=dxb$ were given. In the number sentences in the sessions, first, two-digit numbers were used, and in other examples, the number of the digits was gradually increased. During the teaching process, it was important for students to express their thinking easily, the reasons for their thinking were questioned and a discussion environment was created in the classroom. In this process, the teacher provided proper directions.

Data Analysis

While collecting the research data regarding a teaching experiment, the main data collection tool included video records and observations regarding the teaching sessions.

Table 2.

Themes and Components Explaining the Development of the Students' Relational Thinking Skills

Themes		Scope of the Theme
Relational thinking in addition and meaning of the equal sign		• Modeling of equality-addition with concrete material and its expression in a number sentence • Relational thinking in open number sentences involving addition • Meaning of equal sign
Relational thinking in subtraction		• Modeling of subtraction with concrete material and its expression in a number sentence • Discovering the relationship between the minuend, subtrahend and difference • Relational thinking in open and true/false number sentences on both sides involving subtraction
Relational thinking in multiplication		• Addition-multiplication relationship Modeling of multiplication with concrete material and its expression in a number sentence • Discovering the relationship between the factors and multiplication • Multiplication-division relationship • Relational thinking in open number sentences on both sides involving multiplication and addition
Relational thinking based on properties of operations	*Commutative and associative properties*	• The commutative property, use of commutative property in true/false number sentences • The associative property, use of associative property in true/false number sentences • Use of commutative and associative properties in open number sentences on both sides
	Distributive property	• Use of commutative and associative properties in modeling activities • Discovering the distributive property, one of the modelling activities with concrete material, and its expression in a number sentence • Use of distributive property in open and true/false number sentences on both sides
Relational thinking in division		• Expressing division using a number sentence on both sides with the help of problem and concrete material • Discovering the relationship between the dividends and divisors • Relational thinking in open and true/false number sentences on both sides involving division

The data collected were analyzed not only in the research process but also at the end of the process, and the findings obtained using the analyses act as a source both for explaining the students' thoughts and for forming the hypotheses regarding their ways of probable subsequent learning (Steffe and Thompson, 2000). As the present study examined the development of the students' relational thinking skills, retrospective analyses were conducted both in the process and at the end of the process to see the changes in the students' thoughts. The researchers evaluated the students with the help of the analyses regarding the learning process after each teaching session, and they designed learning environments that would help create the grounds for the development of the students' relational thinking. After the sessions ended, the video records of all the teaching sessions were examined again by two mathematics teachers independently, and five main themes explaining the development of the students' relational thinking were obtained. Table 2 presents these five main themes and the components for the development of relational thinking in these themes.

Validity and Reliability

In the research process, validity and reliability principles were considered, and experts were asked for their views about the validity of the contents and plans used in the teaching process. The components thought to be important for the development of relational thinking in textbooks in literature are considered to be in-class discussions and number sentences used in the process. In this respect, two experts with a PhD degree in the field of mathematics education and two experienced mathematics teachers were asked for their views about whether the concepts, models and related number sentences to be used in each session were appropriate to the research purpose. In line with their views, the teaching process was revised by doing the necessary corrections in the number sentences and in the modeling, and it was piloted with fifth-grade students who did not participate in the research application. At the end of the pilot, it was observed that the number of sentences was developed to improve relational thinking and the activities prepared were in accordance with the age level of the students.

For the analysis of the teaching process, two researchers (who are also the authors of the article) monitored the sessions independently each week and evaluated whether the teachings were appropriate or not. This evaluation not only included an examination of how the students established relationships between the operations and numbers in the open and true/false number sentences but also focused on how they used the properties of operations in the process of dealing with the equalities. To support the relationships recognized with the help of different number sentences following the third and sixth sessions, the sessions were repeated. Thus, the number of total sessions was increased to eight.

Findings

In this part, the questions, the in-class dialogues directing relational thinking and the in-class activities in each session of the teaching experiment conducted with the fifth grade students have been presented under related themes.

Relational thinking in addition and the meaning of the equal sign

Figure 2 shows the flow of the first session held with the students.

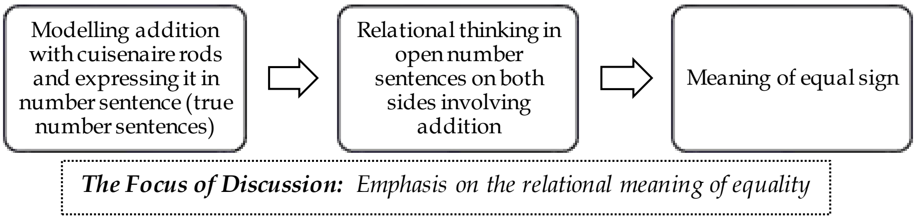
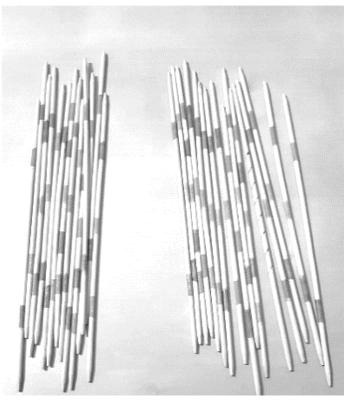


Figure 2. *Flow of the Teaching Design and Main Idea of Classroom Discussion*.

The activities designed using Cuisenaire rods were carried out with the students first within the context of addition. During the activities, each student was asked to divide 30 Cuisenaire rods into two groups by putting any number of rods they wanted in either group and to write them down. Following this, they were asked to take any number of the rods from one of the groups, to put them in the other group and to show this in table t. The students recognized the random changes of the rods as a relational change in table t (an increase and a decrease in the number of the rods). The number sentences formed were shown by the students in table t.



Addend	Addend	Sum
14	16	14+16=30
+1 15	-1 15	15+15=30
+2 16	14 -2	16+14=30
17	13	17+13=30
18	12	18+12=30
…	…	…

Figure 3. *Modelling of the Equalities with Cuisenaire Rods whose Total Numbers were 30.*

Following the activity, the open number sentences were given to the students (for example, …+7=20+8; 13+…=25+15; 26+28=…+29; 129+…=65+132; 971+108=112+…; 571+102+…= 574+105+261), and they were asked to explain their thinking processes without any calculation. At the end of this lesson, the question of "What does equality mean to you?" was directed to the students. The students stated that the equal sign referred to a balance, and they recognized the equality axioms. Examples of the students' explanations regarding equality included "balance", "equivalence", "equality on both sides", "an equal increase, an equal decrease".

Relational Thinking in Subtraction

In the second session, as can be seen in Figure 4, first, unit cubes (30 in number) were distributed to the students so that they could do relational thinking in subtraction and establish relationships between the minuend, subtrahend, and difference, and they were asked to remove any number of cubes they wanted and to write the number of the remaining cubes.

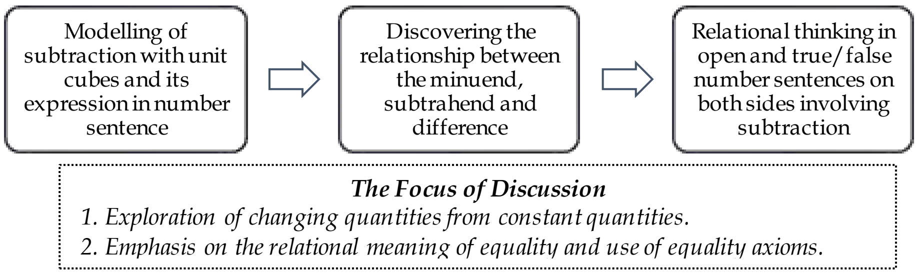


Figure 4. *The flow of the Teaching Design and Main Idea of Classroom Discussion.*

For each subtraction, discussions were held with the students who managed to determine the minuend, subtrahend and difference. Sample in-class dialogues were as follows:

Students : 30-8=22, 30-5=25, 30-6=24, 30-2=28, 30-22=8, 30-3=27.
Teacher : Which one is invariant?
Students : 30, or the minuend.
Teacher : Minuend, very good; which ones changed then?
Tülay : The subtrahend and the difference.
Teacher : Well, how did the subtrahend and the difference change? Let's see!
Ozan : We obtain different results because different numbers are subtracted.
Teacher : What else? Is there any relationship when we look at the subtrahend and the difference?
Tülay : The sum of both makes the minuend.
Teacher : The sum of both makes 30, very good, who else?
Gaye : As the subtrahend increased, the difference decreased. For example, if we subtract 3 from 30, it makes 27, and if we subtract 22 from 30, it makes 8. I mean the difference decreased more.
Teacher : How so?
Gaye : The difference decreases and increases. For example, we subtracted 3 from 30, and when we subtracted less it made 27, and when we subtracted more (30-22), it made 8.

Teacher : When we subtract a number smaller than 30, we obtain a big number, and when we subtract a big number, we get a small number. Well done! For example, here (writes 30-8=22; 30-5=25 on the board). The 30s are the same. What kind of relationship is there between the subtrahends? Let's make comparisons between the subtrahends and between the differences.

Hakkı : I subtracted 5 from 8, and it made 3. If I add 3 to 22, it makes 25.

Teacher : Very good. You mean there is a difference of 3 between 8 and 5, and again there is a difference of 3 between 25 and 22. When 3 was added to 22, it made 25.

Hakkı : I equalized them.

As can be seen in the students' in-class discussions, when the minuend remained the same in the number sentence of "30-a=b", the relationships of "a+b=30" and "30-b=a" between the subtrahend and difference were emphasized. In other words, the students concluded that b decreased when an increased and that a decreased when b increased. Therefore, the students tended to discuss the changing quantities for subtraction by giving up discussing the constant quantities.

In the second phase of this activity, the students were asked to write four number sentences, including subtraction with a constant minuend, and they were also expected to model the operation by using unit cubes. In this process, the teacher asked them whether they observed the relationship between the subtrahend and the difference and requested them to examine the number of sentences in pairs. Below are the number of sentences and expressions provided by three of the students:

Gaye	Hakkı	Ozan
30-1=29 30-4=26 -3 +3 30-7=23 30-8=22 -1 +1 "A decrease of 1 from 8 to 7; an increase of 1 from 22 to 23; a decrease of 3 from 4 to 1; an increase of 3 from 26 to 29."	30-2=28 +2 -2 30-4=26 +2 -2 30-6=24 +2 -2 30-8=22 "I subtracted 2 from each. As there was a decrease of 2 in the difference, an increase of 2 occurred in the subtrahend."	22-1=21 +1 -1 22-2=20 +1 -1 22-3=19 +1 -1 22-4=18 "There was an increase of 1 in the subtrahend and a decrease of 1 in the difference."

When the students examined the activities related to the number of sentences using the tables t, they revealed the relationship of 30-(a+x)=b-x from the relationship of 30-a=b. In this activity, which basically included equality axioms, the students stated that to maintain equality, the difference in subtraction should decrease in line with the increase in the subtrahend. To have the students better understand the relationships between the subtrahend and the difference, they were given number sentences on both sides, and they tried to interpret the equality first in true/false number sentences (e.g. "9-5=12-8", "33-27=34-26", "471-382=474-385", "674-389=664-379") and then in open number sentences without doing any calculation. Following this, they began to work on open number sentences considering the relationships they discovered in true/false sentences. For example, one of the students stated in relation to the number sentence of "33-27=34-26" that "When we subtract 33 from 34, we obtain a difference of 1, so it should not be 26 but 28", while regarding the number sentence of "471-382=474-385", another student said "True. 471 increased by 3, and it made 474. 382 increased by 3, and it made 385. I mean the difference is the same". Based on these comments of the students, it could be stated that they were able to define equality as a balance. Following the true/false number sentences, the students were given open number sentences on both sides like "92-57=…-56", "56-…=58-25", "…-37=75-38", "92-57=94-56-…", "56-23=59-25-…", "573-368=571-370+…". The students who initially established false relationships were then directed to the correct relationships as follows:

For 67-49= □-46:

Tülay	: When we subtract 46 from 49, it makes 3. If we add 3 to 67, it makes 70.
Teacher	: Do all of you think in the same way?
Ozan	: It should be 64.
Teacher	: What about you? (turning to another student)
Gaye	: I found the same number, 64, too.
Teacher	: You said 70 for the box, didn't you?
Hakkı	: No, there was a decrease of 3 from 49 to 46, so there should be a decrease of 3 from 67, and it makes 64.
Teacher	: We should focus on the difference.
İrem	: Teacher, here (67-49), the subtrahend is a bigger number than the one in the other (□– 46). For the difference to remain the same, we should decrease it. Thus, it is 64.

In a given open number sentence like a-b=…-d, all the students correctly found the difference between the numbers of b and d, but some of the students initially thought that they should add this difference to a (…=a+(b-d)). It was revealed using in-class discussions that the difference between b and d should be subtracted from a based on the relational meaning of equality, and the students were told to focus on the difference.

For 92-57=94-56-□:

Hakkı : When we subtract 92 from 94, we obtain a difference of 2, and when we subtract 56 from 57, there is a difference of 1. If we add 2 to 1, it makes 3.

Teacher : Actually, I didn't understand what you mean. Well, Ozan can you explain it, please?

Ozan : Teacher, there is a difference of 2 between 92 and 94 and a difference of 1 between 57 and 56. We added the two numbers.

Teacher : Why did you add them?

Ozan : Because both sides of the equality were equal. Thus, I wrote 3.

In an open sentence like a-b=c-d-□, about which a sample in-class discussion has been presented above, all the students correctly found the difference of 2 between the minuends (a and c) and the difference of 1 between the subtrahends (b and d). Given that the students added these differences and found 3 for the number to be put in the box indicated that they made use of the relational meaning of equality.

Relational Thinking in Multiplication

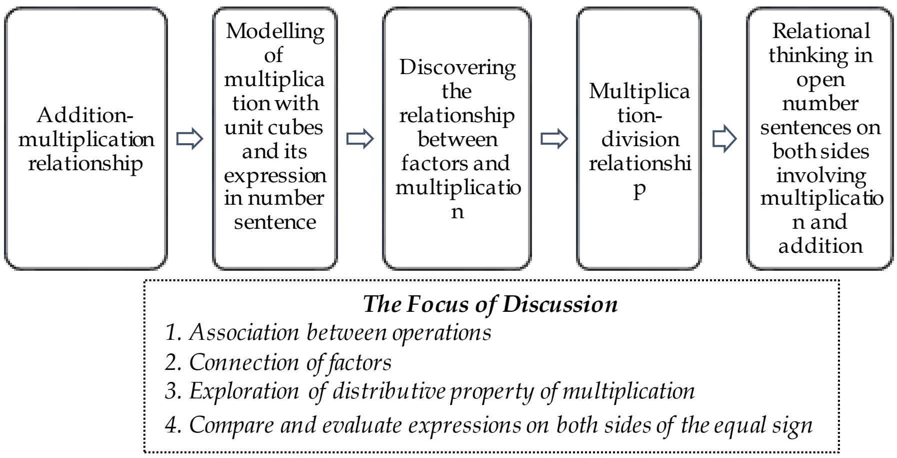


Figure 5. *The flow of the Teaching Design and Main Idea of Classroom Discussion.*

The introductory activity in the third session, which aimed to develop relational thinking in multiplication and the general flow of which is presented in Figure 5, asked the students to separate 24 unit cubes into groups with three-unit cubes in each and to discuss how to write this operation in a number sentence. In this activity, the students first wrote the number of sentences of "3+3+3+3+3+3+3+3=24" and "8x3=24". They then stated that multiplication was actually an operation of addition, and they wrote the equality of 3+3+3+3+3+3+3+3=8x3. In the activity, the students were then asked to re-group the 24 unit cubes, and they wrote a related number sentence. They modeled the operations of 24x1, 1x24, 4x3, 3x4, 6x4, 4x6, 8x3, 3x8, 12x2 and 2x12 using the unit cubes. In this way, the students identified the factors of the number 24. Following this, the teacher wrote these operations in order and interrogated the relationships between the numbers together with the students.

24x1	Tülay : It goes on like 1,2,3,4 on the right side while the other side equals to 24.
12x2	Hakkı : When 24 is divided by 12, it makes 2; when 24 is divided by 3, it makes 8; and when 24 is divided by 4, it makes 6.
8x3	Ozan : Teacher, as the factor increases, this side decreases.
6x4	Teacher: These are all the answers I have expected. All of them are correct.

When the students' views were examined, it was seen that for the operations of 24x1, 12x2, 8x3 and 6x4, Tülay focused on the invariant result of the multiplication despite the changes in the factors; that Hakkı focused on the relationship between the dividend, divisor and quotient; and that Ozan focused on the relationship between the factors. Here, the teacher asked the students to give different examples and expected them to discover the relationships between the factors. In this way, the students recognized the relationships between '1 and 2' and '120 and 60' in equality like 120x1=60x2=40x3. The teacher explained this situation stating that equality was maintained. Following this, the students wrote the equality of 1000x1=500x2=250x4 and mentioned the multiplication and division relationships between the numbers saying that 500 was half of 1000 and that 1000 was 2 times 500.

In the second lesson of the third session, imitation monetary coins and banknotes were distributed to the students. The students were given four groups of money, and each group of money made 20 TLs in sum. The students were given 20 coins of 1 TL, 4 banknotes of 5 TLs, 2 banknotes of 10 TLs and 1 banknote of 20 TLs, and they were asked to state the relationships between these coins or banknotes of 20 TLs. The students stated that all were equal to 20 TLs, and they wrote the equalities using the number sentences of 20x1=4x5=2x10=1x20. At that time, the teacher asked them to show these equalities using the table t of coin-banknote/money and to state the relationships between the factors. Accordingly, the students formed two different tables t. Thanks to this, the teacher had the opportunity not only to emphasize the commutative property but also to let the students see the multiplication-division relationship more clearly in table t.

20:2 10x1

Number	TL
20	1
4	5
2	10
1	20

:10 x10

Number	TL
20	1
4	5
2	10
1	20

x4 :4

Number	TL
1	20
2	10
4	5
20	1

Teacher : What is the relationship between the numbers?
İrem : The result does not change when we change their places.
Teacher : Yes, then, we can say 1x20=20x1.
We call this property the commutative property.
What other relationships are there regarding the numbers?
Gaye : Teacher, 20 is divided by 2, and it makes 10, and when 1 is multiplied by 10, it makes 10. One side is division, and the other is multiplication. (20:2=10x1)
Hakkı : The factor and the divisor are the same,
Teacher : How so?
Hakkı : For example, when 20 is divided by 10, it makes 2, and when 1 is multiplied by 10, it makes 10. (20:10=2, 2 refers to 10 in table t); 1x10=10)

Teacher : Yes, for example, 4 is 4 times 1, and what is the relationship between 20 and 5?
Students : It will be divided by 4.
Teacher : Why?
Hakkı : To equalize them.

In the activity, the commutative property was emphasized with 1x20=20x1, and the multiplication-division relationship was emphasized with 20:2=1x10, 1x4=20:5. At the end of this activity, the open number sentences on both sides involving both multiplication and addition like 3x6=(3x5)+…, (3x4)+…=10x4, 10+10+10+10-…=4x8, (5x9)+ …=10+10+10+10+10 were given, and they tried to interpret the equality without any calculation. Lastly, an open number sentence on both sides involving two unknowns were given, and a related discussion was held as follows:

For 13x10= (10x □)+∆:

Ozan : Teacher, a lot of numbers are possible.
Teacher : Are they? Let's start with Tülay!
Tülay : We can write 13 in the box and 0 in the triangle.
Teacher : Yes.
Tülay : Also, we can write 1 in the box and 3 in the triangle.
Other students: That's not right.
Teacher : Let's have a look (writes 13x10=(10x1)+3 on the board). There should be 13 times 10, right? Is this true for here (points to the other side of the equality)?
Tülay : No.
Gaye : Teacher, I wrote 10 in the box and 30 in the triangle.
Teacher : Why did you think so?
Gaye : There is only one 10 in 13, and the remaining is 30.
Teacher : What else?

Hakkı : Teacher, we write 8 in the box and 50 in the other (triangle), and we write 7 in the box and 60 in the other, also 6 there (box) and 70 in the other just like table t...

In an open number sentence of axb=(bx□)+Δ involving two unknowns as in the activity, there are a wide variety of numbers that can be written in the box and in the triangle. It was seen that most of the students transformed 13 into the number sentences of (10x13)+0, (10x10)+30, (10x8)+50, (10x7)+60. Though the students did not explicitly state it, they actually made use of the commutative and distributive properties. They regarded 13x10 as (13+0)x10, (10+3)x10, (8+5)x10, (7+6)x10. In this example, it was also seen that the students referred to table t they had used in previous examples.

Relational Thinking based on Properties of Operations

Figure 6 presents the flow of the teaching design in this session. The students were first directed the question of "What does changing the places of the numbers mean to you" to let them interrogate whether addition and subtraction involve the commutative property.

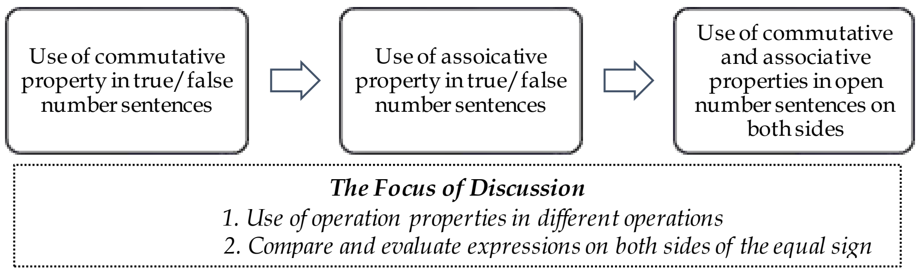


Figure 6. *Flow of the Teaching Design and Main Idea of Classroom Discussion*.

In the interrogation process, first, the number sentence of 3+8=8+3 was discussed. Following this, the students were asked whether the same relationship existed in the number sentence of 6-5=5-6. Examples of related in-class discussions were as follows:

For 6-5=5-6:

Ozan : This question is the same as the previous one (3+8=8+3). Only their places have been changed, and the numbers are the same. Also, the results of the operations are the same. Thus, it is true.

Teacher : Well, is there anyone with different views?

Semih : No, it is false.

Teacher : Why?

Semih : Teacher, in one of them, the minuend is smaller than the subtrahend, and in the other, it is bigger.

Ozan : But, the numbers are the same.

Semih : But, the result is different because it is subtraction. Thus, the equality is wrong.

Ozan : Teacher, changing the places of the numbers in addition and multiplication does not change the result, but it changes in division and subtraction.
Teacher : Good.

As can be seen in the dialogue, there were students who thought that the commutative property was true in subtraction as it is in addition. To change this view, an in-class discussion was held regarding the minuend and subtrahend, and the students stated that the properties valid in addition and multiplication may not always be true in subtraction and division. After emphasizing the commutative property, the students' attention was drawn to the associative property. The students were given true/false number sentences like 5+(3+8)=(5+3)+8, 3x(8x7)=(3x8)x7, 10-(7-2)=(10-7)-2, 20:(10:2)=(20:10):2 involving the associative property, and they were given time to examine these number sentences. The students were directed the question of "Do you need to do an operation?", and they were expected to interrogate which sentences were false and why. In this way, the students were provided with the opportunity to make generalizations regarding the properties of operations. As a result, it was emphasized that the properties valid for addition and multiplication were not true for subtraction and division. Following this, in the session, the students were given open number sentences like 9+16=□+18, 313+…=52+316, 198+980=980+…, 125+…+74=76+127+888, 113+315+801=…+316+799, 9-5=…-6, 85-…=88-36, …-21=52-23, and possible numbers to be put in the box were discussed.

For 85- □ =88-36:
İrem : 33.
Teacher : Why?
İrem : Because 88 increased 3, and I subtracted 3 from 36 to maintain the balance.
Ozan : I found 39.
Teacher : Why 39?
Hakkı : Teacher, 88 is bigger than 85 by 3. Thus, to maintain the balance with 36, we should increase 36 by 3.
Teacher : Let's think about it. That is 85, and it is 3 minus 88. Here (the number to be put in the box), if you increase it by 3, you will subtract more, won't you?

85- □=88-36

Hakkı : We have decreased the minuend.
Teacher : Then, what will the subtrahend be? It will decrease. Why?
Hakkı : Because the differences should be equal.

The first example of the in-class discussions presented above was the open number sentence given as a+□+b=(b+2)+(a+2)+888. In this example, which involved the use of the commutative and associative properties, it was seen that some of the students found the answer to be 884 at first glance. They then thought it should be 892 because through the discussions among themselves, they had agreed that the 4 on the right side of the equation should be added to the left side. In the second example (a-□=(a+3)-36), which was similar to the first one and which should be basically solved depending on the relational meaning of the equal sign, the students initially considered subtracting the difference between the minuends from the subtrahend (36). Therefore, the students were reminded that equality should be maintained and that the difference should be bigger than 36 by 3.

Relational Thinking Based on the Properties of Operations-2

Figure 7 shows the flow of the teaching design for the sixth session, which started with a modeling activity regarding the use of the commutative and associative properties in a problem. The students were directed towards the activities that would help discover the distributive property. In this respect, the students were asked how to place biscuits in a parcel. Depending on the students' responses, related experiments were carried out using a parcel brought into the class. There were boxes in the parcel and biscuits in the boxes. The students were asked how to calculate the number of biscuits.

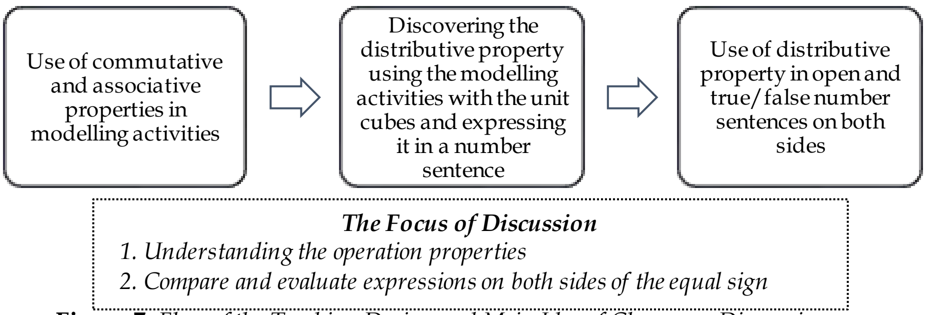


Figure 7. *Flow of the Teaching Design and Main Idea of Classroom Discussion*.

Next, the students were directed the question of "There are 10 biscuits in one package of biscuits. 20 packages of biscuits are placed in a box. You can put a total of 25 boxes in one parcel. Accordingly, could you write the sentence that will show the number of the biscuits in one parcel?", and a related in-class discussion was held:

Teacher : Well, I don't want you to find the result. I just want to see the number sentence.
Semih : We multiply 10 by 20, and it makes 200. So, it makes one package.
Hakkı : No, it is a box.
Semih : Next, we multiply 200 by 25.
Teacher : Why?
Semih : As one parcel can include 25 boxes, I have multiplied it by 25. The result is 5000.
Teacher : Is there anyone to do a different operation?
Hakkı : We can first multiply 20 by 10 and then 25 by 200.
Ozan : Teacher, he means we can change their places.

The teacher wrote the operations of (20x10)x25 and (10x20)x25 side by side and asked the students whether these sentences reminded them of anything from previous lessons. The students stated that the commutative and associative properties were used.

In the follow-up activity, all the students were given a square prism with the top open, and each of them was distributed 32 unit cubes. First, the students were asked how to place the unit cubes. In this phase, the unit cubes in different colors were chosen to let the students discover the distributive property. As can be seen in Figure 8, the teacher first showed the unit cubes placed differently by two of the students. Following this, the teacher asked the students how they could find the number of the total unit cubes. All the students stated that the cubes were placed differently and that the total numbers of the cubes were equal. The students wrote the related number sentences. An example for the in-class discussions regarding the number sentence of (4x4)x2=(2x4)x2x2 was as follows:

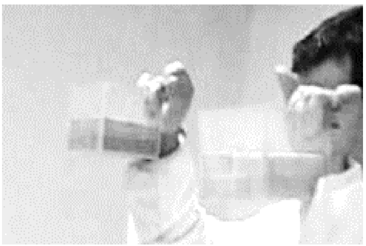


Figure 8. *Photos from the Fifth Session.*

Hakkı : Here, we can put four unit cubes on one side of the bottom of the box and another four on the other side of the bottom. As the surface area, 4 multiplied by 4 makes 16. Then, we multiply it by 2, and the result is 32.
Teacher : Well, can we do the same operation, or a different one?
Semih : We can multiply 2 by 4, and it makes 8.
Ozan-Semih: Then we multiply it again by 2.
Semih : 16. I mean it is because there are different colors. We multiply 16 by 2, and it makes 32. I mean because I used two-unit cubes twice and because I have divided into two halves, I multiply 2 by 4 and get 8. Then, I multiply 8 by 2, and it makes 16. As the others (referring to the second floor) are the same, I multiply it by 2.

Next, for the purpose of allowing the students to discover the distributive property, the teacher asked them how many unit cubes there were in total. Here, the students recognized that the total numbers of the unit cubes in two boxes were equal and managed to write the equalities of (4x4)x2=(2x8)+(2x8)=2x(8+8):

İrem : Teacher, I will say the same thing again. There are 4 times 8 unit cubes (the total number of yellow and orange cubes)
Salih : Adding 8 to 8 means multiplying 8 by 2. Thus, when multiplied by 2, it makes two 8s. It is the same to multiply it by 2 and add 2.
Teacher : Well, what would be the name of this property?

Semih : It is the property of cancelation.
Teacher : You are almost right.
Gaye : Decomposition
Teacher : We call this property the distributive property.

The teacher-directed different questions by changing the number of factors with the unit cubes and changing the numbers of the colored cubes to examine the students' ability to use the distributive property. Following this, in-class discussions were held regarding the true/false number sentences on both sides like $5x(6+7)=(5x6)+(5x7)$, $(3x4)+(1x4)=4x4$, $8x4=(4x4)+(4x4)$, $(8x3)+(8x1)=8x4$ and open number sentences on both sides like $6x(8+7)=(6x\ldots)+(6x7)$, $\ldots x(2+3)=(\ldots x2)+(\ldots x3)$, $(3x8)+(4x\ldots)=7x8$, $4x(18+2)=(4x\square)+(\Delta x2)$ involving the use of the distributive property.

Relational Thinking in Division

Figure 9 presents the teaching flow of the last session, in which the students were given the problem of "The numbers of the triangle pyramids and of the square prisms are equal. The total face number of the square prisms is 36. Now please find the number of the faces of the pyramids."

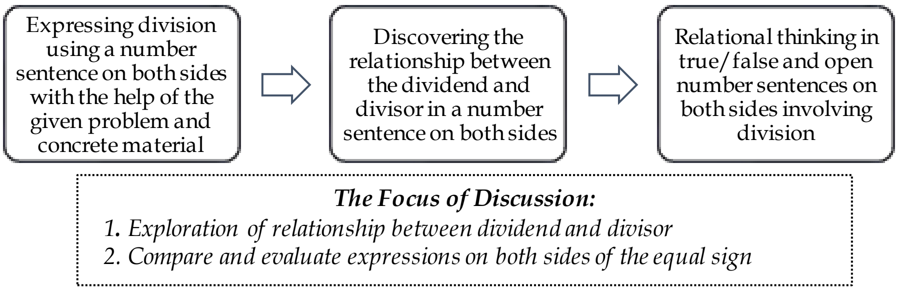


Figure 9. *Flow of the Teaching Design and Main Idea of Classroom Discussion*.

The students were asked to examine the face numbers of the triangle pyramids and square prisms brought into class. They were expected first to find the number of the prisms and then to find the face number in the pyramids with the help of the equality of the numbers of the prisms and pyramids. Depending on the result they found, in-class discussion was held regarding whether it was possible to write the equality of "24:4=36:6", and the data were transferred to table t.

Teacher : How many faces does a square prism and pyramid have?
Students : 6 and 4.
Teacher : The total face number of the square prism is given as 36. Then, how many prisms are there?
Semih : 6.
Teacher : How did you find that 6?
Semih : It has 6 faces, and I divided 36 by 6.
Teacher : You divided 36 by 6 (writes 36:6 on the board). What do think about the other?

Semih : I multiplied 6 by 4, and I found the face number of the pyramids.
Teacher : Well, in 36:6, 6 is the number of faces, and the result is the number of prisms. The numbers of the triangle pyramids and of the square prisms are equal, so we can write the face number of the triangle pyramid on the right side of the equality, right?
Students : Yes.
Teacher : Then, if write 36:6=□:4, what is the number we can put in the box?
Semih : To make it equal, it is 24.
Teacher : How can you find it without any calculation? Just consider it like table t.

Total face number	*Number of objects*
36	*6*
?	*4*

Students : 24
Teacher : Now, it is much easier, right? Why?
Ozan : Here (the first line), it is 6 times more. In the other one (the second line), it should be 6 times more, too. Thus, it is 24.

The session continued with open number sentences which aimed to develop relational thinking in division (for example; 10:2=5:…, 66:…=22:2, 350: 14=50:…).

10:2=5: …
İrem : 1
Teacher : Why?
Students : Because, teacher, here, 2 times 5 is 10. It was divided by 2. Now, 10 was divided by 2, and it made 5. I divided 2 by 2, and it made 1.
Teacher : Very good.
Hakkı : 5 is half of 10. To maintain the equality, it should be half of 2. I mean it should be 1.

In an open number sentence like ka:b=a:□ exemplified above, the students stated that the relationship (k) between the dividends could also be established between the divisors and that the number to be put in the box should be b/k. The session ended with the open number sentences like 60:□=20:Δ, which had more than one answer and with the true/false number sentences like 10:(5:5)=(10:5):5, which aimed to evaluate whether the commutative-associative properties were valid in the division.

Discussion, Conclusion and Recommendations

Relational thinking, which could be associated with the relational meaning the equal sign, is a skill that can be developed using teaching based on the generalized arithmetic approach at the level of secondary school fifth grade. This situation was also determined previously using the clinical interviews held before and after the teaching process carried out in the first phase of the present study (Kızıltoprak & Yavuzsoy Köse, 2017). Relational thinking not only covers numbers, operations and relationships between operations but also requires understanding and effectively using the fundamental properties of operations. Therefore, relational thinking involves basic mathematical ideas regarding the development of students' algebraic thinking. This capacity of relational thinking also made it necessary to examine the teaching process, which constituted the second phase of the present study. Accordingly, the most general result was that at the end of the teaching process based on numbers, relationships between numbers, operations and properties of the operations, the students made use of equality axioms to evaluate the true/false and open number sentences without any calculation. Parallel to the results of other studies carried out to develop primary school third-grade students' relational thinking using teaching processes (Carpenter et al., 2003; Koehler, 2004; Molina, Castro & Ambrose, 2005), the present study revealed that the students managed to make use of relational thinking while evaluating and solving the given true/false and open number sentences at the end of the teaching process. It was also seen that the students made connections between addition-subtraction, addition-multiplication and multiplication-division and that they made effective use of commutative, associative and distributive properties.

In the relevant literature, there are many studies that showed that students perceive the equal sign as a command for the application of arithmetic operations and they thus consider the equal sign to have an operational meaning (Sáenz-Ludlow & Walgamuth, 1998; Yaman, Toluk & Olkun, 2003; McNeil & Alibali, 2005; Matthews & Rittle-Johnson, 2009; Byrd et al., 2015; Rittle-Johnson et al., 2011). Given that students encounter mostly with the operational meaning of the equal sign not only in their textbooks (Seo & Ginsburg, 2003; Köse & Tanışlı, 2011) but also in their in-class learning process (Carpenter et. al., 2003) makes it more difficult to understand the concept of equality. To overcome this difficulty and to let students understand the concept of equality, teaching processes in the phase of introduction to the concept and then in the phase of transition to addition could be beneficial. In relation to this, Seo and Ginsburg (2003) reported that teachers could use rods and coins to contribute to students' views about equality. Researchers point out that in an activity involving the use of rods, the relational meaning could be attributed to numbers, operations and equality. With the help of this approach, in the first session of the teaching process, the students were able to recognize the related changes in the number sentences by modeling addition with Cuisenaire rods and by showing the number sentences in table t. To clarify this better, it could be stated that using table t, the students were able to represent the number sentences whose sum was 30 and which they modeled with Cuisenaire rods. Also, the students were thus able to recognize that the increases and decreases between the two addends were equal. Thinking over correct number sentences not only helps students see number sentences as a whole but also supports the relational meaning of the concept of equality (Molina & Ambrose, 2008). In the first session, in the activity involving use of Cuisenaire rods, which was used as the introductory activity, transition from true number sentences involving addition to the maintenance of equality was a planned transition to support the relational meaning of the concept of equality. Thus, it was an important starting point that at the end of the first session, the students used expressions like "scale, balance, equality on both sides" regarding the concept of equality.

In the second session of the teaching process, subtractions were modeled using unit cubes, and all the students correctly expressed the minuend, subtrahend and difference. In a number sentence like 30-a=b, the students recognized the change between the subtrahend and the difference by keeping the minuend constant. In the follow-up activity, the students were asked to model different subtractions which they themselves formed with unit cubes and to show the number of sentences related to these operations in table t. The students recognized that the difference decreased/increased when they increased/decreased the subtrahend in the number sentences (they showed in table t) in a certain pattern. The students thought in that way because they examined their true number sentences two by two (for example, 30-1=29 and 30-4=26; 22-1=21 and 22-2=20). In other words, the students related the change in the difference to the change in the subtrahends in the true number sentences two by two. This process also contributed to the students' relational thinking while evaluating the true/false number sentences on both sides involving subtraction. In this way, the students agreed that equality was maintained by keeping the difference constant in a number sentence on both sides involving subtraction. Therefore, during the in-class discussions, the students reported that the difference (2) between the minuends (10 and 8) for the number sentence of 10-5=8-3 should be equal to the difference (2) between the subtrahends (5 and 3). The in-class discussions regarding especially the true/false number sentences supported the students' analysis of the given number sentences from a holistic perspective without doing any calculation. This result is consistent with the results of another study carried out with primary school third-grade students by Molina and Ambrose (2008), who pointed out that the relational meaning of the equal sign is supported by the teaching process. The researchers reported that the true/false number sentences they used in their teaching process helped develop the students' understanding of the relational meaning of the equal sign and allowed their transition from the computational approach to the structural/analytical approach. In the present study, following the true/false number sentences involving subtraction in the second session, the students were given open number sentences. The students managed to generalize not only the maintenance of the difference between the minuends and subtrahends in the true/false number sentences but also the number sentence on both sides, and they even achieved relational thinking in more complex number sentences. Carpenter et al. (2003) claim that discussions regarding true/false and open number sentences are fairly beneficial for students' understanding of the equal sign. This claim put forward by the researchers is also supported by the related findings obtained in the present study. The sessions starting with true/false number sentences and the relational thinking strategies applied for the evaluation of these number sentences made it easy for the students to solve the open number sentences without doing any calculation.

Understanding the relationship between addition and multiplication allows students to multiply with the help of their knowledge about addition (Carpenter et. al., 2003). Thus, in the third session of the teaching process, unit cubes and imitation coins and banknotes were used to increase the degree of relating addition to multiplication and to understand multiplication. In this process, the number sentences of "24x1=12x2=6x4=8x3" and "20x1=4x5=2x10=1x20" were given to the students together with the table t used in the previous sessions. The students developed different thinking strategies using the table t, focused on the relationship between the factors and mentioned the commutative property. In addition, some of the students began to establish relationships between the dividend, divisor and quotient in the number sentences in table t and thus established relationships between multiplication and division. Given that the students were able to recognize and establish all these relationships thanks to table t was an important finding obtained in the present study. In the third session, another striking result was obtained through an open number sentence on both sides involving two unknowns like 13x10=(10x □)+∆. Number sentences involving two unknowns like 18+(Box A)= 20+(Box B) could direct students towards relational thinking (Stephens & Ribeiro, 2012). In one study, Napaphun (2012) found that open number sentences involving one unknown and two unknowns developed students' relational thinking skills. Molina and Ambrose (2006) point out that asking students to form true number sentences in the form of …+…=…+…; …−… =…−…; …+…=…−… could be fairly beneficial for clarifying and consolidating their relational understanding. Parallel to these results, in the present study, the students formed different number sentences using relational thinking in open number sentences on both sides involving two unknowns. While forming these number sentences, the students, though they did not state it explicitly, made effective use of the commutative and distributive properties.

In the fifth session, the students were encouraged to recognize the commutative and associative properties with the help of true/false number sentences. When the students stated which operations were suitable for using these properties, they started to deal with related problems and modeling. Especially the distributive property used together with the associative and commutative properties plays a key role in the development of arithmetic (like mental calculations, algorithms, rules) and algebraic thinking (like the transformation of sentences, recognition of the equality relationship) (Malara&Navarra, 2006). In addition, unit cubes with different colours used in the teaching process to help the students discover the distributive property was fairly useful for the calculation of the square prism. This activity, which is basically used for the associative and commutative properties, contributes to the recognition of the distributive property. In the present study, the students made a direct transition from the equality of (4x4)x2=(2x4)x2x2 to the equality of 2x(8+8)=(2x8)+(2x8) without mentioning the distributive property. The students did relational thinking while writing these equalities in multiplication. This result is consistent with the results obtained by Carpenter, Levi, Berman and Pligge (2005), who reported in their study that especially primary school students intuitively use the distributive property in number sentences involving multiplication. Baek (2008) points out that the third and fifth-grade students who understood especially the associative and distributive properties of multiplication were successful in solving verbal multiplication problems involving multi-digit numbers. As another important finding obtained in the present study, the students stated that the equalities of (a+b)xc=(axc)+(bxc) they formed based on the square prisms were correct, and they effectively used the distributive property as well as the associative and commutative properties in open number sentences.

In the light of the results obtained in the present study, with the help of appropriate teaching environments and thanks to the in-class discussions guided by the teacher, the students managed to give a relational meaning to the concept of equality. Depending on the results of the present study, which presented a teaching design, learning trajectories that aim to develop relational thinking could be developed and tested on different participants. Moreover, studies could be designed on elementary school teachers' teaching processes regarding the concept of equality. In the present study, the findings suggest that tables t were considerably influential on evaluating the number of sentences based on relational thinking. In this respect, mathematics teachers and especially elementary school teachers could use table t to show the arithmetic operations and the related number sentences which they provide in problem contexts and which they have modeled with various concrete materials. In this way, the relationships between the numbers discovered in table t could be related with operations. In this study, the focus was especially on the distributive property of multiplication over addition and multiplication. On the other hand, examples like 184:8=(80+80+24): 8= (80:8)+(80:8)+(24:8) or 180: 6=(120:6)+(60:6) involving different usage of the distributive property were out of the scope of the present study. In this respect, further research could examine which strategies students use especially in division within the context of relational thinking.

References

Alibali, M. W., Knuth, E. J., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning*, *9*(3), 221-247.

Baek, J.-M. (2008). Developing algebraic thinking through explorations in multiplication. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and Algebraic Thinking in School Mathematics: Seventieth yearbook of the National Council of Teachers of Mathematics* (NCTM; pp. 141–154). Reston, VA: NCTM.

Bastable, V., & Schifter, D. (2008). Classroom stories: Examples of elementary students engaged in early algebra. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the Early Grades* (pp. 165–184). Mahwah, NJ: Lawrence Erlbaum/Taylor & Francis Group; Reston, VA: NCTM.

Blanton, M.L. (2008). *Algebra and the elementary classroom: Transforming thinking, transforming practice*. Portsmouth: Heinemann.

Boulton G. M., Lewis G., Cooper T. J., Atweh B., Pillay H. & Wills L. (2000). Readiness of algebra. In T. Nakahara & M. Koyama (Eds.), *Proceeding of the 24th Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 89-105). Hiroshima, Japan: International Group for the Psychology of Mathematics Education.

Blanton, M., Stephens, A., Knuth, E., Gardiner, A., Isler, I., & Kim, J. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education*, *46*(1), 39–87.

Byrd, C. E., McNeil, N. M., Chesney, D. L., & Matthews, P. G. (2015). A specific misconception of the equal sign acts as a barrier to children's learning of early algebra. *Learning and Individual Differences, 38*, 61-67.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in the elementary school.* Portsmouth, NH: Heinemann.

Carpenter, T. P., Levi, L., Berman, P., & Pligge, M. (2005). Developing algebraic reasoning in the elementary school. In T. A. Romberg, T. P. Carpenter, & F. Dremock (Eds.), *Understanding Mathematics and Science Matters* (pp. 81–98). Mahwah, NJ: Lawrence Erlbaum.

Chapman, O. (2011). Elementary school teachers' growth in inquiry-based teaching of mathematics. *ZDM Mathematics Education, 43*(6), 951-963.

Eichhorn, M.S., Perry, L.E., & Brombacher, A. (2018). Students' early grade understanding of the equal sign and non-standard equations in Jordan and India. *International Journal of Research in Education and Science*, *4(*2), 655-669. doi:10.21890/ijres.432520

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics, 6,* 232-236.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education, 38*(3), 258-288.

Kızıltoprak, A. & Yavuzsoy Köse, N. (2017). Relational thinking: The bridge between arithmetic and algebra. *International Electronic Journal of Elementary Education, 10* (1), 131-145.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297-312.

Koehler, J.L. (2004). *Learning to think relationally: Thinking relationally to learn* (University of Wisconsin-Madison) Retrieved from ProQuest Dissertations and Theses. (AAT 3143187).

Köse, N., & Tanışlı, D. (2011). Equal sign and relational thinking in elementary mathematics textbooks. *Necatibey Faculty of Education Electronic Journal of Science & Mathematics Education, 2*(5), 251-277.

Malara, N., & Navarra, G. (2006). Approaching the distributive law with young pupils. In M. Bosch (Ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education* (pp. 363-373). Sant Feliu de Guíxols: FUNDEMI IQS –Universitat Ramon Llull.

Matthews, P., & Rittle-Johnson, B. (2009). In pursuit of knowledge: Comparing self-explanations, concepts, and procedures as pedagogical tools. *Journal of Experimental Child Psychology, 104*, 1–21. doi:10.1016/j.jecp.2008.08.004

Matthews, P., Rittle-Johnson, B., McEldoon, K., & Taylor, R. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education, 43*(3), 316-350

McNeil, N., & Alibali, M. (2005). Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of Cognition and Development, 6*(2), 285-306. doi:10.1207/s15327647jcd0602_6

Molina, M., & Ambrose, R. C. (2006). Fostering relational thinking while negotiating the meaning of the equals sign. *Teaching Children Mathematics, 13*(2), 111-117.

Molina, M., & Ambrose, R. (2008). From an operational to a relational conception of the equal sign. Thirds graders' developing algebraic thinking. *Focus on Learning Problems in Mathematics, 30*(1), 61-80.

Molina, M., Castro, E., & Mason, J. (2008). Elementary school students approaches to solving true/false number sentences. PNA. *Revista de Investigación en Didáctica de la Matemática, 2*(2), 75-86.

Molina, M., & Mason, J. (2009). Justifications-on-demand as a device to promote shifts of attention associated with relational thinking in elementary arithmetic. *Canadian Journal of Science, Mathematics and Technology Education, 9*(4), 224-242.

Napaphun, V. (2012). Relational thinking: Learning arithmetic in order to promote algebraic thinking. *Journal of Science and Mathematics Education in Southeast Asia, 35* (2), 84-101.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*, 175–189. doi:10.1037/0022-0663.91.1.175

Rittle-Johnson, B. , Matthews, P. G., Taylor, R. S. & McEldoon, K. L. (2011). Assessing knowledge of mathematical equivalence: A construct modeling approach. *Journal of Educational Psychology, 103* (1), 85-104. doi: 10.1037/a0021334

Sáenz-Ludlow, A. & Walgamuth, C. (1998). Third graders' interpretations of equality and the equal symbol. *Educational Studies in Mathematics, 35*(2), 153-87. doi:10.1023/A:1003086304201

Seo, K.-H., & Ginsburg, H. P. (2003). ''You've got to carefully read the math sentence...'': Classroom context and children's interpretations of the equals sign. In A. J. Baroody & A. Dowker (Eds.), *The Development of Arithmetic Concepts and Skills.* Mahwah, NJ: Lawrence Erlbaum.

Steinweg, A. S., Akinwunmi, K. & Lenz, D. (2018). Making implicit algebraic thinking explicit: Exploiting national characteristics of German approaches. In C. Kieran (Ed.), *Teaching and Learning Algebraic Thinking with 5-to 12-Year-Olds* (pp.283-307). ICME-13 Monographs. Cham: Springer.

Steffe, L. P. & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 267-307). Hillsdale: Erlbaum.

Steffe, L. P. & Ulrich, C. (2014). The constructivist teaching experiment. In S. Lerman (Ed.) *Encyclopedia of Mathematics Education* (pp. 102-109). Springer, Berlin

Stephens, M. (2006). Describing and exploring the power of relational thinking. Retrieved from http://www.merga.net.au/documents/RP552006.pdf

Stephens, M. & Ribeiro, A. (2012). Working towards algebra: The importance of relational thinking, *Revista Latinoamericana de Investigación en Mathemática Educativa, 15* (3), 373-402.

Stephens, A.C., Ellis, A. B., Blanton, M. & Brizuel, B. M. (2017). Algebraic thinking in the elementary and middle grades. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (386-420). Reston, VA: NCTM.

Strachota, S., Knuth, E. & Blanton, M. (2018). Cycles of generalizing activities in the classroom. In C. Kieran (Ed.), *Teaching and Learning Algebraic Thinking with 5-to 12-Year-Olds* (pp.351-378). ICME-13 Monographs. Cham: Springer.

Yaman H., Toluk Z., & Olkun S. (2003). How the elementary school students would perceive equal sign? *Hacettepe University Journal of Education, 24*, 142-151.

Ortaokul Öğrencilerinin İlişkisel Düşünme Becerilerinin Bir Öğretim Deneyi Aracılığıyla Geliştirilmesi

Atıf:

Kose. N.Y. & Kiziltoprak, A. (2020). Development of secondary school students' relational thinking skills with a teaching experiment. *Eurasian Journal of Educational Research*, 85, 135-168, DOI: 10.14689/ejer.2020.85.7

Özet

Problem Durumu: Literatürde pek çok araştırmacının eşit işaretinin anlamı ve eşitlik kavramı üzerine yoğunlaştığı görülmektedir. Bu araştırmacılar arasında eşit işaretinin ilişkisel anlamını oluşturmada öğrencilerin eğilimlerinin işlemsel anlama yönünde olmadığı aksine bu durumun kavramlara ilişkin öğretim süreçlerinin bir yansıması olarak oluştuğu konusunda ortak bir uzlaşma söz konusudur (Stephens, Ellis, Blanton &Brizuela, 2017, s. 391). Bu ortak uzlaşı araştırmacıları, eşit işaretinin ilişkisel anlamının oluşturulmasında ve ilişkisel düşünmenin geliştirilmesinde öğretim süreçlerinin nasıl olması gerektiğine doğru yöneltmiştir. Küçük yaşlardaki öğrenciler ile gerçekleştirilen çalışmalarda, genellenmiş aritmetik yaklaşımını içeren erken cebir öğretimi aracılığıyla öğrencilerin temel işlemlerin ve değişme özelliği gibi işlem özelliklerinin farkına varmada önemli kazanımlar sağladıkları, genel olarak sayılara, işlemlere ve işlem özelliklerine ilişkin muhakemelerinde farklı düşünme yolları ürettikleri, hatta çeşitli genellemelere ulaşabildikleri belirlenmiştir (Carpenter vd., 2003; Bastable&Schifter, 2008; Blanton vd., 2015; Steinweg vd., 2018). Literatür incelendiğinde eşit işaretinin ilişkisel anlamına ve ilişkisel düşünmenin geliştirilmesine yönelik öğretim süreçlerini inceleyen araştırmaların ağırlıklı olarak okulöncesi ve ilkokul düzeyinde olduğu, ortaokul düzeyindeki öğrencilere ilişkin araştırmaların sınırlı (Napaphun, 2012) olduğu söylenebilir. Oysaki aritmetikten cebire geçişin sağlandığı ortaokul 5. sınıf düzeyi öğrencilerin düşüncelerinin geliştirilmesinde kilit bir düzey olarak ele alınabilir. Öğrencilerin ilişkisel düşünmeyi geliştirici bir öğretim süreci aracılığıyla muhakemelerinin ve düşünme yollarının ortaya çıkarılması önemlidir. Bu bağlamda bu çalışmada gerçekleştirilen öğretim sürecini incelemek amaçlanmış, öğrencilerde ilişkisel düşünmenin geliştirilmesinde işlem özelliklerinin nasıl kullanıldığı, hangi somut materyallerin ele alındığı etkinlik örnekleri ve sınıf içi tartışmalar ile detaylı olarak sunulmuştur.

Araştırmanın Amacı: Bu araştırma ile ortaokul 5. sınıf öğrencilerindeki ilişkisel düşünme becerisinin nasıl geliştirilebileceğinin incelenmesi amaçlanmaktadır. Bu amaç doğrultusunda "Öğrencilerdeki ilişkisel düşünme becerisinin geliştirilmesinde sayı cümleleri ve işlem özellikleri nasıl kullanılmaktadır?" sorusuna yanıt aranmıştır.

Araştırmanın Yöntemi: Öğrencilerdeki ilişkisel düşünme becerisinin geliştirilmesinin incelendiği bu araştırmanın deseni öğretim deneyidir. Araştırmanın katılımcılarını; Eskişehir ilindeki orta düzey olan bir devlet okulunun 5. sınıfında öğrenim gören 6

öğrenci oluşturmaktadır. Öğretim öncesi sınıf içi gerçekleştirilen gözlemlerde öğrencilerin eşit işaretinin farklı anlamlarını söylemekte zorlandıkları ve eşitliği bir işlemin sonucunu bulma olarak kullandıkları belirlenmiştir. Öğrencilerin aritmetik işlemleri içeren sayı cümlelerinde sayılar ve işlemler arasında bir ilişki kurmaksızın hesaplamaya dayalı düşündükleri, değişme, birleşme ve dağılma özelliklerini fark etmedikleri, özellikle bölme işleminde zorlandıkları görülmüştür. Bu bağlamda öğrencilerin ilişkisel düşünme becerilerinin gelişimini amaçlayan bir öğretim süreci planlanmıştır. Gerçekleştirilen öğretim süreci her hafta 1 oturum olacak şekilde 8 oturum/8 hafta olarak gerçekleştirilmiştir. Öğrencilerin yaş düzeyleri göz önüne alınarak oturumlar 30-40 dakika olacak şekilde planlanmış, gerek duyulduğunda oturumlara ara verilmiştir.

Araştırmanın Sonuçları ve Öneriler: Bu araştırmadan elde edilen en genel sonuç sayılar, sayılar arası ilişkiler, işlemler ve işlem özelliklerine dayalı bir öğretim süreci sonunda öğrencilerin doğru/yanlış ve açık sayı cümlelerini hesaplama yapmadan eşitlik aksiyomlarından yararlanarak değerlendirebilmeleridir. Araştırmada öğrenciler öğretim süreci sonunda verilen doğru yanlış ve açık sayı cümlelerini değerlendirmede ve çözmede ilişkisel düşünmüşlerdir. Öğrencilerin toplama-çıkarma, toplama-çarpma, çarpma-bölme işlemleri arasında ilişkilendirme yapabildikleri ve değişme, birleşme ve dağılma özelliklerini etkili bir biçimde kullanabildikleri görülmüştür.

Öğrencilerin eşit işaretini aritmetik işlemlerin uygulanması için bir komut gibi algılamaları ve dolayısıyla gerek ders kitaplarında gerekse ön öğretimlerinde eşit işaretinin işlemsel anlamı ile karşılaşmaları eşitlik kavramının anlaşılmasını zorlaştırmaktadır. Bu zorluğun üstesinden gelebilmede ve eşitlik kavramının kazandırılmasında kavrama ilk girişteki ve ardından toplama işlemine geçişteki öğretim süreçleri çare olabilir. Öğretimin ilk oturumunda mikado çubukları ile toplama işleminin modellenmesi ve sayı cümlelerinin t tablosunda gösterilmesi aracılığıyla öğrenciler sayı cümlelerindeki ilişkili değişimi fark etmişlerdir. Oturumda, giriş etkinliği olarak seçilen mikado çubukları etkinliğinde toplama işlemindeki doğru sayı cümlelerinden eşitliğin korunumuna geçilmesi eşitlik kavramının ilişkisel anlamının desteklenmesi için gerçekleştirilmiş planlı bir geçiştir.

Öğretimin ikinci oturumunda çıkarma işlemleri birim küplerle modellenmiş, öğrencilerin tamamı eksilen, çıkan ve farkı doğru ifade etmişlerdir. Öğrencilerden etkinliğin devamında, birim küplerle kendi oluşturdukları farklı çıkarma işlemlerini modellemeleri ve bu işlemlere ait sayı cümlelerini t tablosunda göstermeleri istenmiştir. Öğrenciler t tablosunda gösterdikleri sayı cümlelerindeki çıkanı örüntüsel bir şekilde arttırdıklarında/azalttıklarında farkın da azalacağını/artacağını fark etmişlerdir. Bu düşüncenin gelişmesinde öğrencilerin oluşturdukları doğru sayı cümlelerini ikişerli olarak incelemeleri (örn. 30-1=29 ve 30-4=26) etkili olmuştur. Diğer bir ifade ile öğrenciler ikişerli ele aldıkları doğru sayı cümlelerinde çıkanların değişimi ile farkın değişimini ilişkilendirmişlerdir. Sınıf içinde özellikle doğru/yanlış sayı cümleleri ile ilgili tartışmalar öğrencilerin verilen sayı cümlelerini hesaplama yapmadan bütüncül bir bakış açısıyla analiz etmelerini desteklemiştir. Öğrenciler doğru/yanlış sayı cümlelerinde ulaştıkları eksilenler ve çıkanlar arasındaki farkın korunumuna ilişkin genellemeyi çift taraflı açık sayı cümlesine genişletmişler, hatta daha karmaşık sayı cümlelerinde ilişkisel düşünebilmişlerdir. Oturumlara öncelikle doğru/yanlış sayı cümleleri ile başlanması, bu sayı cümlelerinin değerlendirilmesinde kullanılan ilişkisel düşünme stratejileri öğrencilerin açık sayı cümlelerini hesaplama yapmadan çözmelerini kolaylaştırmıştır.

Öğretimin üçüncü oturumunda toplama ve çarpma işlemleri arasındaki ilişkilendirmenin arttırılması ve çarpma işleminin anlamlandırılması için birim küpler ve oyun paraları kullanılmıştır. Öğrenciler t tablosu aracılığıyla farklı düşünme stratejileri geliştirmişler, çarpanlar arasında kat ilişkisine odaklanmışlar ve değişme özelliğini kendileri ifade etmişlerdir. Ayrıca bazı öğrenciler t tablosundaki bu sayı cümlelerinde bölünen, bölen ve bölüm arasında ilişkilendirme yapmış ve dolayısıyla çarpma ve bölme işlemleri arasında da ilişki kurmuşlardır. Tüm bu ilişkilerin fark edilmesinde ve kurulmasında t tablosunun oldukça etkisi olması araştırmada ulaşılan önemli sonuçlardandır. Oturumdaki en dikkat çekici sonuçlardan bir diğeri ise 13x10=(10x □)+∆ gibi iki bilinmeyen içeren çift taraflı bir açık sayı cümlesinde ortaya çıkmıştır. Öğrenciler, iki bilinmeyen içeren çift taraflı açık sayı cümlesinde ilişkisel düşünerek farklı sayı cümleleri oluşturmuşlardır. Bu sayı cümlelerini oluştururken öğrenciler ifade etmeseler de değişme ve dağılma özelliklerini etkili biçimde kullanmışlardır.

Beşinci oturumda ele alınan değişme ve birleşme özellikleri doğru/yanlış sayı cümleleri ile fark ettirilmeye çalışılmış, öğrencilerin bu özelliklerin hangi işlemler için geçerli olduğunu belirtmeleri ile birlikte problemlere ve modellemelere geçilmiştir. Öğretim sürecinde dağılma özelliğinin keşfi için farklı renklerde kullanılan birim küpler ile kare prizmanın hacminin hesaplaması son derece etkili olmuştur. Özellikle öğrencilerin (a+b)xc=(axc)+(bxc) eşitliklerinin doğruluğunu savunmaları ve ardından açık sayı cümlelerinde birleşme ve değişme özellikleri ile birlikte dağılma özelliğini de etkili bir biçimde kullanmaları ulaşılan diğer önemli sonuçlardandır.

Araştırmadan elde edilen sonuçlar ışında, uygun öğretim ortamları ve öğretmenin sınıf içi tartışmaları aracılığıyla öğrencilerin eşitlik kavramına ilişkisel bir anlam yükleyebildikleri görülmüştür. Bir öğretim tasarımı sunan bu araştırmanın sonuçlarına dayalı olarak ilişkisel düşünmeyi geliştirici öğrenme yörüngeleri geliştirilebilir ve farklı katılımcılar üzerinde test edilebilir. Hatta sınıf öğretmenlerinin eşitlik kavramına ilişkin öğretim süreçleri üzerine araştırmalar desenlenebilir.

Anahtar Sözcükler: Matematik eğitimi, ilişkisel düşünme, eşitlik, eşit işareti.