Special case of the three-dimensional Pythagorean gear

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Introduction

In mathematics, three integer numbers or triples have been shown to govern a specific geometrical balance between triangles and squares. The first to study triples were probably the Babylonians, followed by Pythagoras some 1500 years later (Friberg, 1981). This geometrical balance relates parent triples to child triples via the central square method (Teia, 2015). The great family of triples forms a tree—the Pythagorean tree—that grows its branches from a fundamental seed (3, 4, 5) making use of one single very specific motion. The diversity of its branches rises from different starting points (i.e., triples) along the tree (Teia, 2016). All this organic geometric growth sprouts from the dynamics of the Pythagorean geometric gear (Figure 1). This gear interrelates geometrically two fundamental alterations in the Pythagorean theorem back to its origins—the fundamental process of summation $x + y = z$ (Teia, 2018). But reality has a three dimensional nature to it, and hence the next important question to ask is: what does the Pythagorean gear look like in three dimensions?

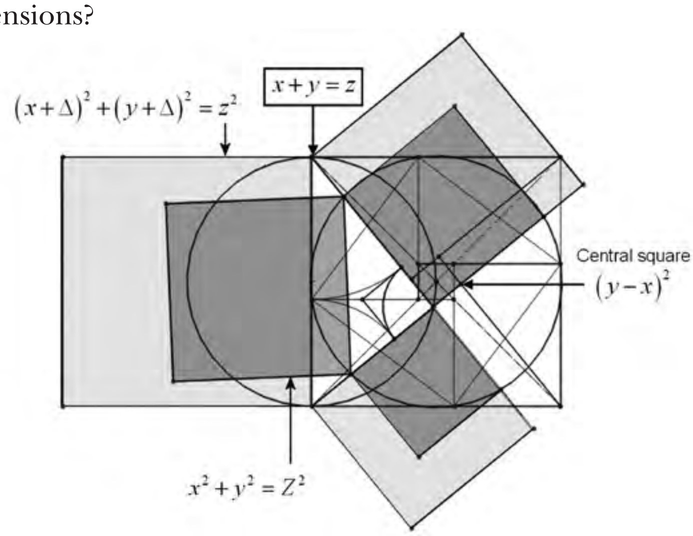

Figure 1. Overall geometric pattern or 'geometric gear' connecting the three equations (Teia, 2018).

A good starting point for studying any theory in any dimension is the examination of the exceptional cases. This approach was used in eminent discoveries such as Einstein's theory of special relativity where speed was considered constant (the exceptional case) that then evolved to the theory of general relativity where acceleration was accounted for (the general case) (Einstein, 1961). Following on his approach, and in particular for the case of the Pythagorean theorem this exceptional condition occurs when the right triangle is isosceles, that is, when both legs have the same length $x = y = 1$ and hold the same angles (Figure 2a). This special case is seen to play particular importance in ancient times (as indicated by the clay tablet of the Babylon's named YBC 7289, estimated to date back to circa 1800–1600 BCE; Beery & Swetz, 2018). It was found that this table contained in itself the geometrical and mathematical expression of this special case, where the later holds an impressive approximation to the triangle's hypotenuse length $\sqrt{2}$ (Flannery, 2006), given as:

$$\sqrt{2} \approx 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = 1.41421296296296\ldots$$

Indeed, ancient Greek philosophers such as Aristotle and Euclid studied this number $\sqrt{2}$, contemporarily termed in mathematics as the Pythagoras' constant (Finsch, 2003), and presented proofs of its irrational nature (Euclid, 300 BC).

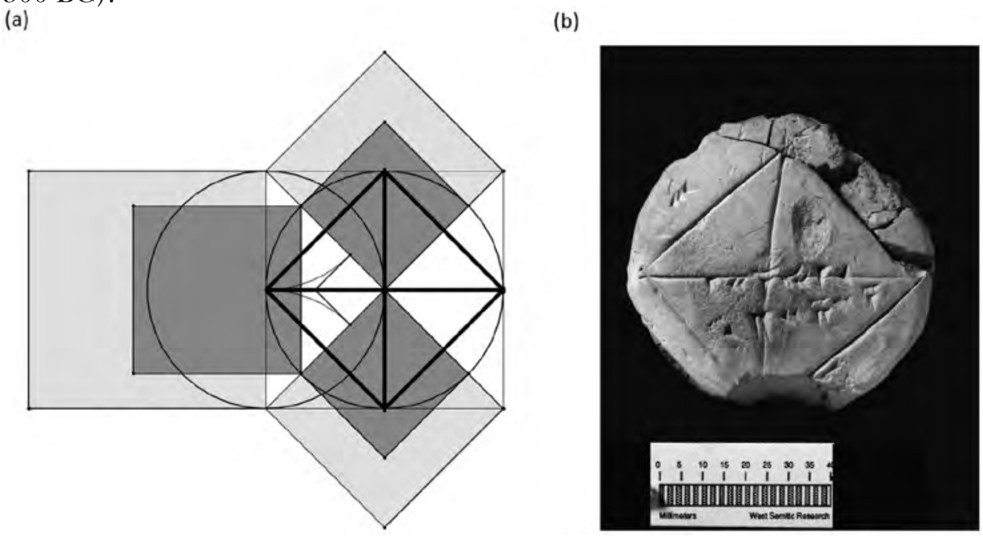

Figure 2. Special case x = y: (a) the Pythagorean gear and (b) Babylonian clay tablet YBC 7289 (courtesy of the Yale Babylonian Collection).

Hypothesis

The hypothesis of this paper is that the special case $x = y$ of the three dimensional version of the Pythagorean geometric gear is constructed from several of the two dimensional versions aligned along the three orthogonal planes.

Theory

Three-dimensional Pythagorean gear

Consider the isosceles right triangle (two legs $x = y = 1$ and corresponding angles equal) and point 0 at the centre of the hypotenuse aligned along the XZ plane (Figure 3a). Rotating the triangle into the XY plane, while maintaining the point 0 at the centre of the hypotenuse, and perpendicular to the previous triangle, gives Figure 3b. Rotating now into the YZ plane, again while maintaining the point 0 at the centre of the hypotenuse and perpendicular to the previous triangle gives Figure 3c. The final outcome gives the construct in Figure 3d holding a hexagonal face at the centre. This hexagonal face is formed by the three hypotenuse of each triangle aligned in each orthogonal plane having as a common point the centre 0.

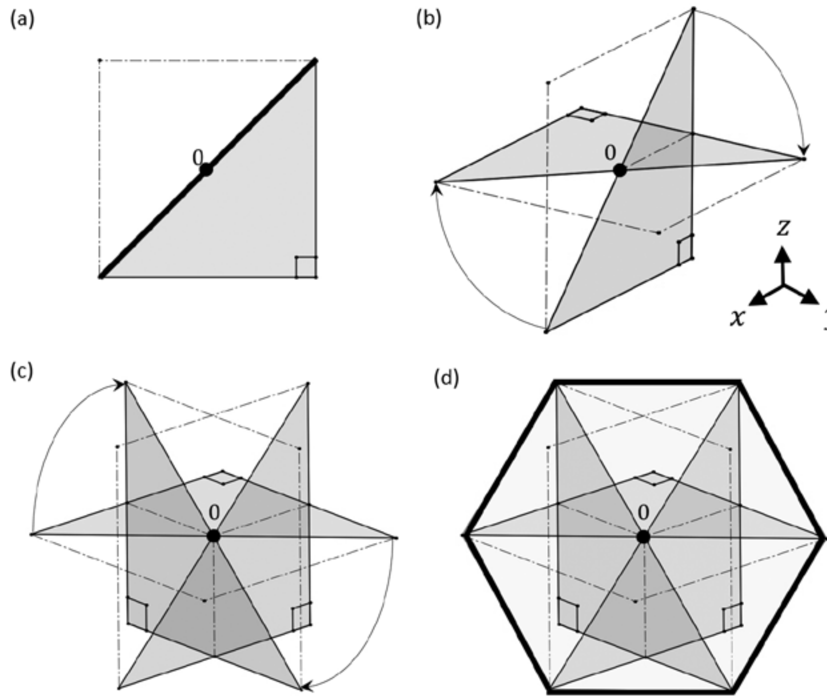

Figure 3. Formation of the three dimensional diagonal: (a) triangle in XZ plane, (b) rotation into XY plane, (c) rotation into YZ plane and (d) joining the edges gives a hexagon.

This approach suggests that just like two perpendicular lines form in between them a triangle (Figure 4a), three perpendicular triangles form in between them the three dimensional equivalent—the truncated tetrahedron (Figure 4b). Truncated means that the edge of the element was clipped at a third of its length. Therefore, if in the transition from one to two dimensions, a hypotenuse of a triangle is formed by two lines perpendicular to each other, from two to three dimensions, the new 'hypotenuse' is formed by the diagonal lines of three triangles perpendicular to each other, which is the hexagon. One is inside the other, and the two dimensional theorems in all three orthogonal planes drive all together in a group movement the three dimensional theorem. The Pythagorean gear is an ensemble of all one, two and three dimensional theorems, all in one. While two isosceles right triangles opposing each other form a square, two truncated tetrahedra opposing each other form a cube. Hence, the hypotenuse is the diagonal of a square, while the hexagon (composed of three hypotenuses) is the diagonal of a cube.

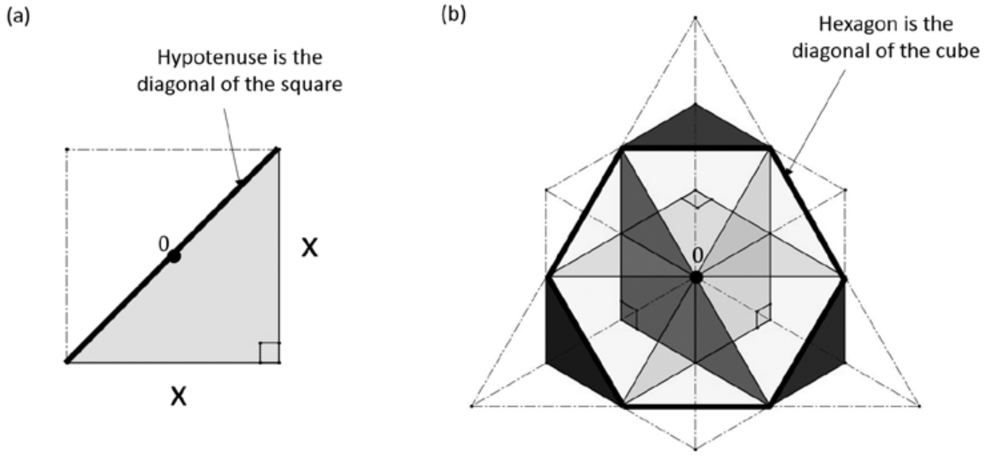

Figure 4. Diagonals of: (a) a square is the hypotenuse and (b) cube is a hexagon.

The proof that the diagonal face of a cube is a hexagon is an exercise in geometry for students and teachers alike to do as an assessed assignment. This exercise develops skills in how to solve pure geometrical problems using computer-aided design (CAD) programs in the context of the Pythagorean Theorem which is relevant to the Australian Curriculum (ACARA, n.d.). Whilst this example was created with the free academic version of Solidworks, students can also use other equivalent software existing in their school/ university (like Catia, Ansys DesignModeler, Siemens NX, etc). The layout of these programs may be different, but they all have the same functions described in this exercise (e.g., sketches, extrusion, planes, etc). The solution to this exercise is described in the following steps:

1. To create a cube of side 1, first select an orthogonal plane *XY*, *YZ* or *XZ* and sketch a square of side length 1. Then extrude boss (this is a technical term used in CAD that means to add material based on a sketch) the square in a perpendicular direction by 1 (Figure 5a).
2. Create a 3-D sketch (that is, a sketch that is not bound by a plane) and draw a line from one opposing corner of the cube to the other (Figure 5b). Add a midpoint to this diagonal line (tip: always rotate the model to make sure that indeed the point is on the line and is the midpoint, as often certain perspectives can cause optical illusions and the point may actually be floating somewhere else). Add a midplane by selecting the midpoint and making it perpendicular to the diagonal line (Figure 5c).
3. Intersecting the plane with the cube, results in a hexagon as the diagonal face of the cube (Figure 5d). This completes the proof and the exercise.

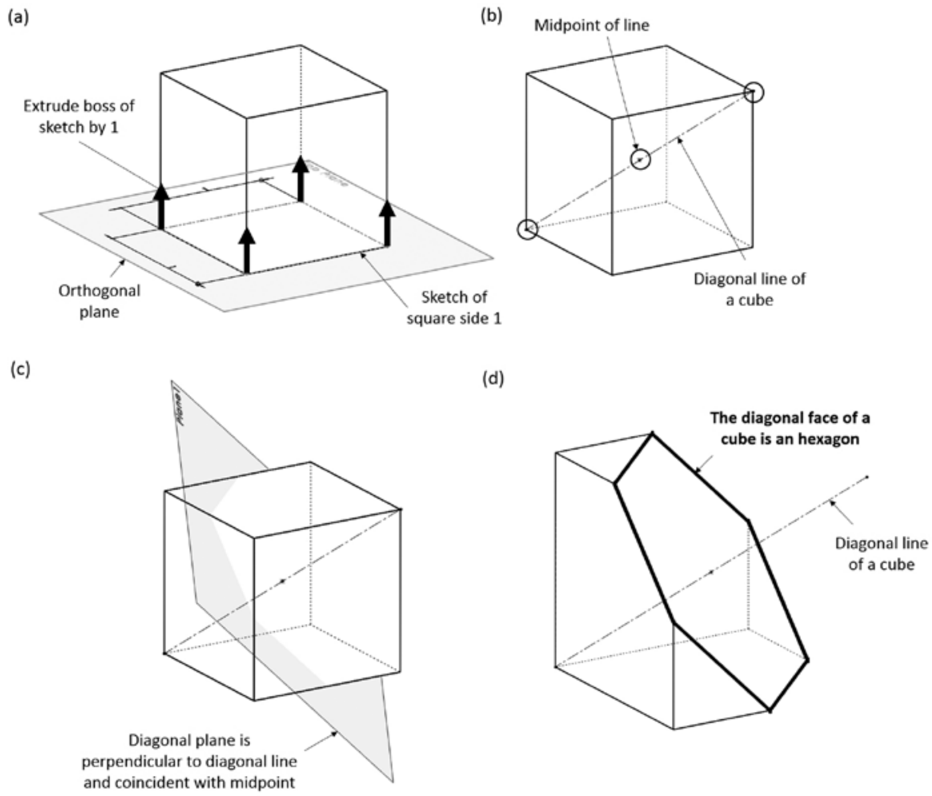

Figure 5. Exercise on how to create the diagonal of a cube: (a–d) steps involved in the solution.

Revolving the isosceles right triangle gives the now familiar shape of a square inside a square, where the inside square has side length hypotenuse $\sqrt{2}$ (Figure 6a). Similarly, revolving the truncated tetrahedron around all three axis gives a truncated octahedron (Figure 6b). A truncated octahedron is composed of 6 square faces and 8 hexagonal faces.

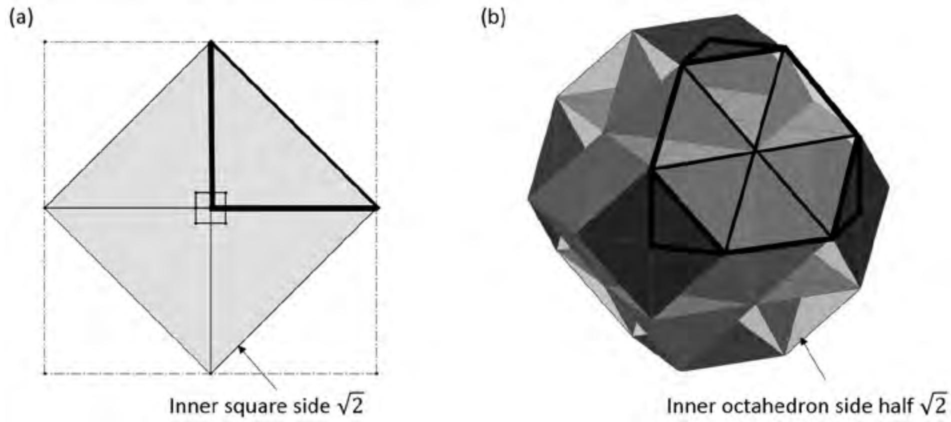

Figure 6. Inner elements side $\sqrt{2}$: (a) square and (b) octahedron.

The special case $x = y$ of the two-dimensional Pythagorean gear relates the inner square side $\sqrt{2}$ with the outer square side 2, as $2(\sqrt{2}\,)^2 = 2^2$. The special case of the three-dimensional Pythagorean gear relates the inner truncated octahedron with the outer cube side 2, and the relationship is defined later in equation (6).

A square is the dual of itself, and it is well known since the times of Kepler that the octahedron is the dual of the cube (Kepler, 1619). However, in the context of the special case of the Pythagorean Gear, it is the special version of the octahedron—the truncated octahedron—that is the dual of the cube. A transition of the inner square (side length $\sqrt{2}$) to its dual (the outer square of side 2) is achieved by adding four additional isosceles right triangles (Figure 7a). Similarly, the transition from the inner truncated octahedron (side length half $\sqrt{2}$) to its dual (the outer cube of side 2) is achieved by adding 8 additional truncated tetrahedrons (Figure 7b).

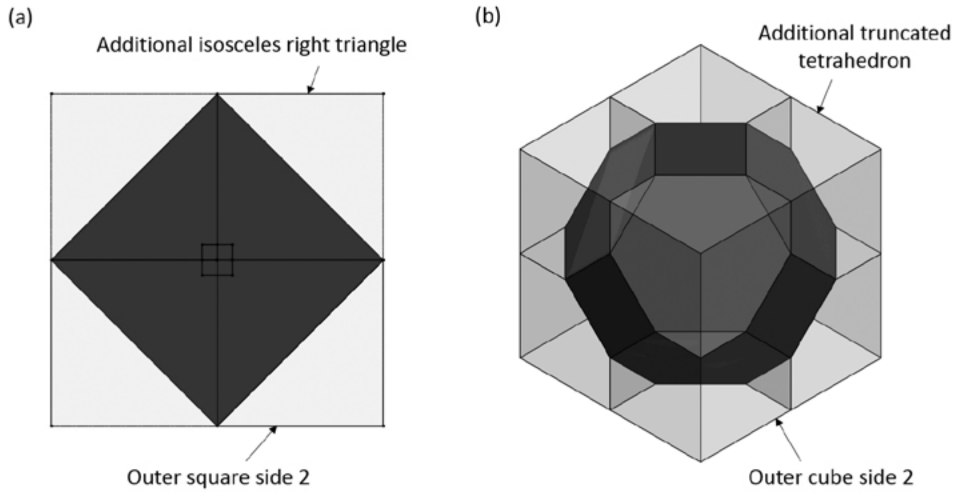

Figure 7. Outer (dual) elements side 2: (a) square and (b) cube.

Construction using volumes

Start with the simplified special case of the Pythagorean gear in Figure 8a. Extruding all surfaces by 0.5 in both directions (up and down) transits the theorem to three dimensions (Figure 8b). Mathematically, multiplying the original equation by 1 gives the same equation. The subtlety lies in the re-arrangement of the terms of that expression.

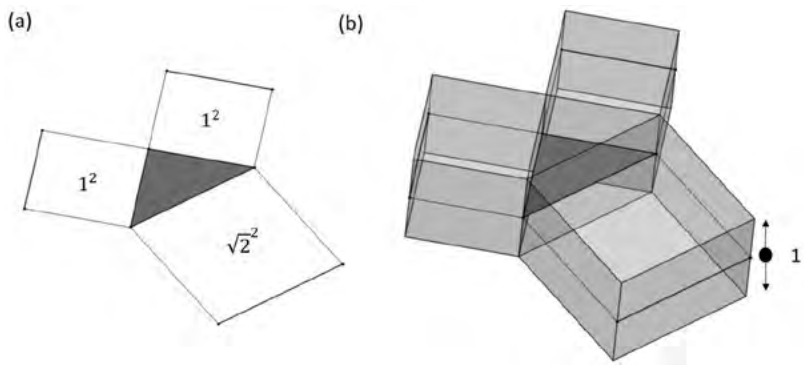

Figure 8. (a) A simplified view of the Pythagorean Gear and (b) its extrusion into three dimensions.

It is the reorganisation of this volume (none is added or subtracted) that gives the volumetric variant of the gear. Extending lines from the nodes of the larger square $ABCD$ (of side $\sqrt{2}$) to the smaller square $EFGH$ (with half the width $\sqrt{2}$, and centred at the top of the parallelepiped), and extending in turn its edges to the projected larger square $A'B'C'D'$ gives the wireframe presented in Figure 9a. Sectioning the volume along these lines, and rotating each of the three pieces about the edges of the midplane square gives Figure 9b.

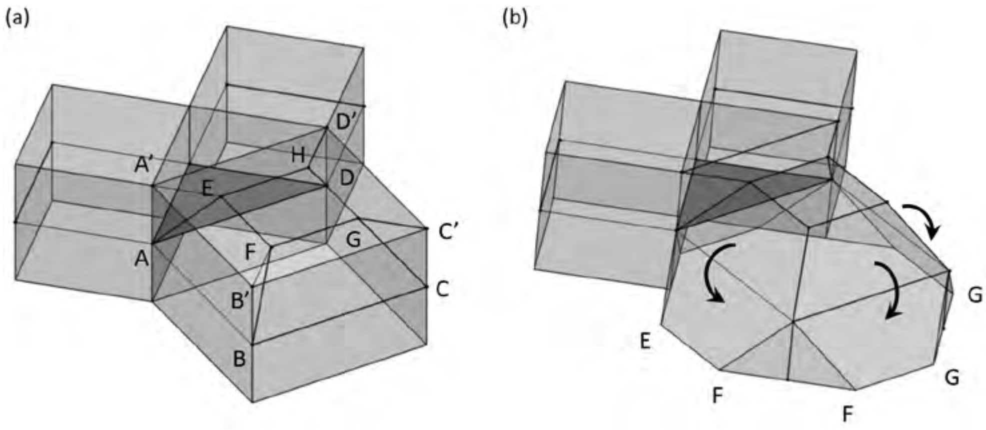

Figure 9. (a) Partition of the parallelepiped and (b) rotation of the four pieces.

Figure 9 is rotated and redrawn in Figure 10 to better visualise its shape. Since no volume was added or subtracted, the geometric assembly states that the sum of the volumes of two cubes side 1 is linked to the volume of half a truncated octahedron (or half $\sqrt{2}\left(\sqrt{2}(1)\right)^3$) via an intermediate element, which is the truncated tetrahedron. Note that Figure 10 also holds in itself both the two dimensional version of sum of squares, and the one dimensional version of sum of lengths.

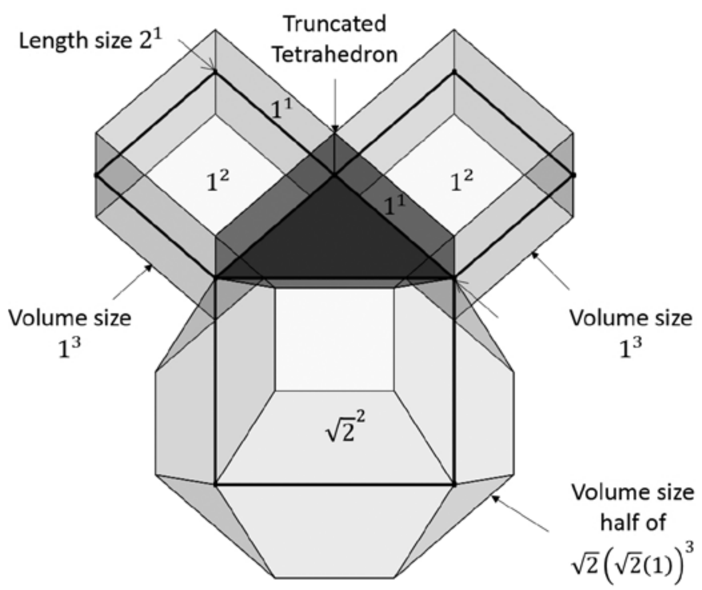

Figure 10. The geometry of the one, two, and three dimensional equations.

However, the two dimensional Pythagorean gear needs to be satisfied in all three orthogonal planes (Figure 10 shows only one plane) in order to be volumetrically correct. Therefore, following the orientation of the triangles shown previously in Figure 3, the three orthogonal Pythagorean theorems take the disposition in Figure 11 as follows. Aligning the two dimensional gear first along the *XY* plane (Figure 11a), then along the *YZ* plane (Figure 11b), along the *XZ* plane (Figure 11c), and finally superimposing the volumetric content gives Figure 11d. By inspection, we can see that Figure 11d is the expression of the three dimensional Pythagorean gear as a function of three orthogonal two-dimensional Pythagorean gears (shown initially in Figure 2a). This is valid for the special case when $x = y$. The general case where $x \neq y$, termed by the author the 'universal gear', will be presented in a future publication.

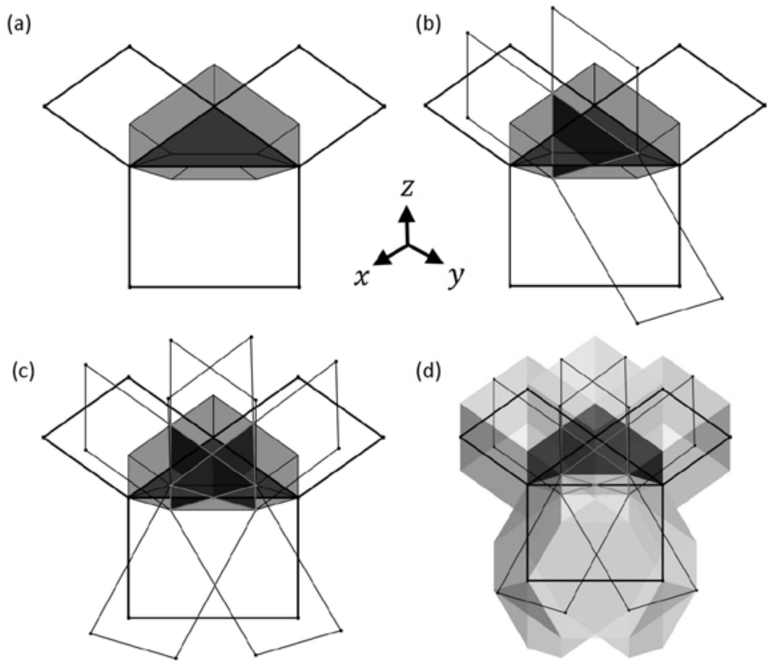

Figure 11. Integration of the 2-D Pythagorean gear: (a) along XY plane, (b) along YZ plane, (c) along XZ plane and (d) the volume superimposed becoming the 3D Pythagorean gear.

The same cube side 2 can be split into two principal ways: as a sum of 8 cubes of side 1 (as shown in Figure 12a, this is the orthogonal way), or as a sum of an inner and outer central truncated octahedron (as shown in Figure 12b, this is the diagonal way). The meaning of orthogonal and diagonal way will be explained later in the section that refers to grids.

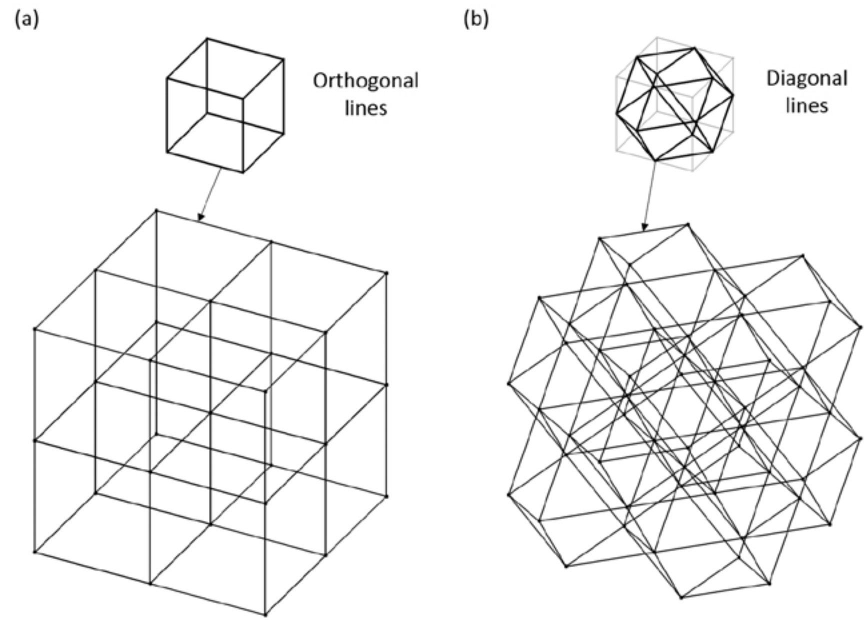

Figure 12. Grid lines: (a) the orthogonal way and (b) the diagonal way.

Mathematics of the Pythagorean gear

Consider the shaded three-dimensional version of the gear in Figure 13 (redrawn from Figure 7b). In one dimension, the balance within the gear is that of the sum of lengths. That is, the addition of two lines length 1 gives another line length 2, resulting in equation (1):

$$1 + 1 = 2 \tag{1}$$

And in being mathematically more accurate, the exponent 1 can be added to denote an equation that varies along one dimension, resulting in equation (2):

$$1^1 + 1^1 = 2^1 \tag{2}$$

This one dimensional balance repeats itself all around the cube.

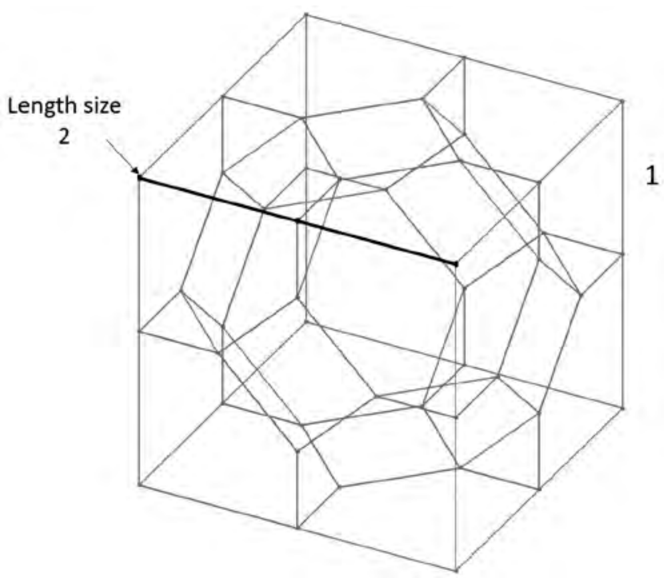

Figure 13. The one dimensional balance in the sum of lengths.

In two dimensions, the balance within the gear is that of the sum of areas. That is, the addition of two areas size 1 give another area size 2. This is again seen in the gear in Figure 14. The transition from a sum of lines (equation 1) to a sum of squares (equation 3) is denoted by a multiplication factor of 2.

$$2 + 2 = 4 \tag{3}$$

In essence, this 2 is in fact the product $\sqrt{2}\,\sqrt{2} = 2$, and equation (3) can be reorganised to express its two dimensionality as equation (4):

$$\left(\sqrt{2}(1)\right)^2 + \left(\sqrt{2}(1)\right)^2 = 2^2 \tag{4}$$

This two dimensional balance also repeats itself all around the cube.

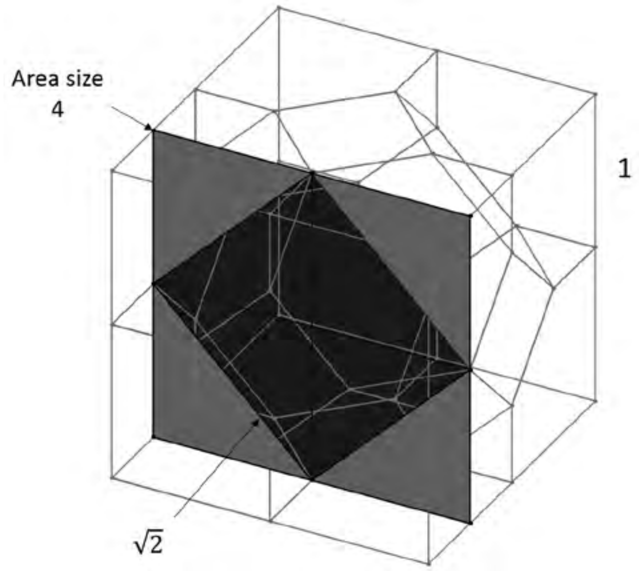

Figure 14. The two dimensional balance in the sum of areas.

In three dimensions, the balance within the gear is that of the sum of volumes. That is, the addition of two volumes of size 4 (the truncated octahedrons) gives another volume of size 8 (the cube). This is again highlighted on the gear in Figure 15. The transition from a sum of areas (equation 3) to a sum of volumes (equation 5) is denoted again by a multiplication factor of $2 = \sqrt{2}\,\sqrt{2}$.

$$4 + 4 = 8 \tag{5}$$

Therefore, equation 5 can be reorganised into equation 6:

$$\sqrt{2}\left(\sqrt{2}(1)\right)^3 + \sqrt{2}\left(\sqrt{2}(1)\right)^3 = 2^3 \tag{6}$$

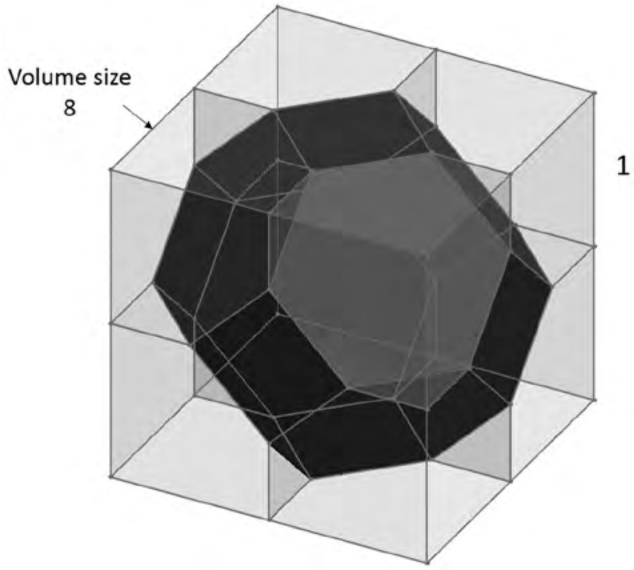

Figure 15. The three dimensional balance in the sum of volumes.

Note that if $\sqrt{2}$ is an operator that converts information from one orthogonal directional path to the diagonal, and vice versa, then the various multipliers $\sqrt{2}$ appearing in equations (4) and (6) express a series of movements of conversion of information between the orthogonal and diagonal grids.

The one-, two- and three-dimensional grid

A gear can be seen as an individual balance, or a network balance of many gears. In fluid dynamics, a network of geometrical elements (tetrahedral and prismatic) are used to study the behaviour of a fluid within or around a body (Anderson Jr, 1995). Figure 16a shows how the volume around a computer-aided design (CAD) YF-17 fighter jet geometry was split into a variable size network of elements (often called 'cells' in the aerospace world). Each cell holds in itself a set of equations that expresses the state of physics (e.g., pressure, temperature, density, etc.) in that particular cell. Each cell communicates with its neighbour in a network of information. In the computational fluid dynamics (CFD) world, a 'converged' solution to a field refers to the situation when the numerical balance between all those cells and equations has reached a sufficiently stable state. There is software that can collect and plot in a meaningful way the field of the generated data. A typical combination package of software used for analysis is *Solidworks* for CAD, ICEM for meshing, *Fluent* for solving and *Tecplot* for post-processing. Figure 16b shows an example of the computed surface pressure distribution on an F-16 XL aircraft in flight. Such information is then used in a multitude of applications, such as redesigning the aircraft.

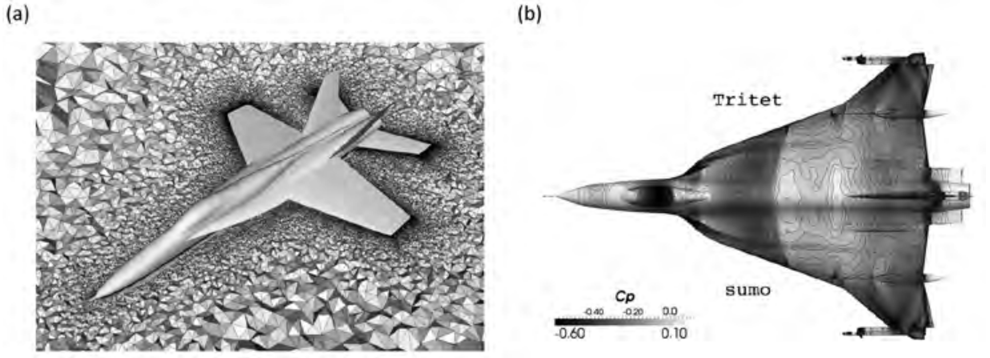

Figure 16. Computational fluid dynamics: (a) grid mesh around a YF-17 fighter jet and (b) pressure distribution around a F-16 XL (Tomac & Eller, 2014).

While CFD is a very complex topic, the fundamental principle on which it operates is the same as that for the Pythagorean gear. One gear can be seen as coupled to its neighbour in a network of information. This network transmits conservation of length and area (as the arrows in the Pythagoras tree of triples shown in Figure 17a show). These arrows imply that four squares of area 1 transit into the diagonal grid as two squares of area 2, and return to the orthogonal grid as one square of area 4, and also, as shown by the present paper, volume information (as shown in Figure 17b). Similarly, eight cubes of volume 1 in the orthogonal grid transit into two truncated octahedrons of volume 4 in the diagonal grid, and return to the diagonal grid as one cube of volume 8.

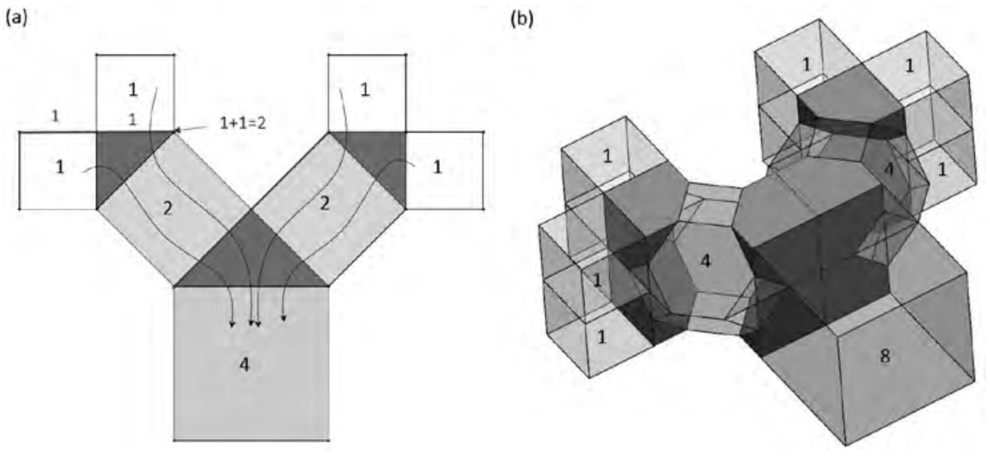

Figure 17. Network of Pythagorean Gears: (a) in two dimensions and (b) in three dimensions.

Figure 18a shows the various one dimensional grids (sequence of unit line segments connected in chain) disposed both orthogonally and diagonally, forming the two dimensional grid of the Pythagorean gears (as shown previously in Figure 2a). Information flows from the orthogonal to the diagonal grid via the intermediate nodes, and back again. This process is fractal, that is, it happens at different scales of the grid simultaneously. Aligning many of these two dimensional grids as per Figure 11d gives a three dimensional grid composed of many three dimensional Pythagorean gears (Figure 15).

The particularity of this truncated octahedron where $x = y$ is that its square and hexagonal surfaces are a perfect match to other neighbouring truncated octahedrons, thus forming an infinite grid occupying all space (Figure 18b). Each truncated octahedron is coupled to its neighbour. Ultimately, the general case $x \neq y$ of the Pythagorean gear—the 'universal gear'—is like a clock governed by the jewel movement of the inner orthogonal and diagonal grids composed of variable size and shape right-angled triangles. This general case is beyond the scope of the present article and will be published in a future article.

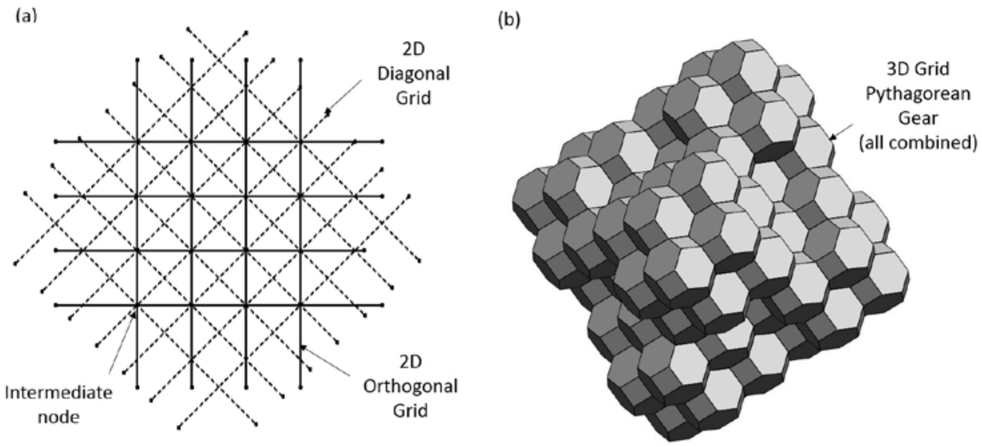

Figure 18. The geometrical grids: (a) orthogonal and diagonal in 2-D, and (b) cubic and octahedral in 3-D.

Conclusion

The nature of spatial reality is at least three dimensional. Following on the footsteps of a previous publication, this article extends the dynamics of the Pythagorean gear to its three dimensionality. A good starting point for the examination of any theory is the inspection of its exceptions. Therefore, the special case $x = y$, and the role of Pythagoras' constant $\sqrt{2}$, is investigated. The hypothesis is that the three dimensional Pythagorean gear is built from orthogonal versions of its two dimensional expression. As a result, it is shown that just as two lines perpendicular to each other define a hypotenuse (and overall a right-angled triangle), three triangles aligned along the orthogonal planes having as a common centre point define the new three-dimensional 'hypotenuse' that is a hexagon (and overall, they form a truncated tetrahedron). Just as the hexagon is formed by three diagonal hypotenuses, so is the truncated tetrahedron formed by three isosceles right triangles. Therefore, the truncated tetrahedron is the three dimensional equivalent of the two-dimensional isosceles right triangle. Ultimately, the Pythagorean gear is an ensemble of all one-, two- and three-dimensional theorems, all in one, defining a perfect balance between lengths, areas and volumes. This balance of the gear decomposes into orthogonal and diagonal directions forming two distinct grids. This topic is practical in nature, and its implications are very real. In modern aerospace industry, in particular in computational fluid dynamics, a network of geometrical elements (tetrahedral and prismatic) termed 'cells' are used to form grids that allow the study of the behaviour of a fluid within or around a body (like an aircraft). The Pythagorean gear can be regarded in a similar manner. A gear can be seen as holding an individual balance, or a network balance of many gears. Neighbouring truncated octahedrons, geared onto each other, form a network that fills space. Pythagoras' constant $\sqrt{2}$ can be seen as an operator that converts information (i.e., length, areas and volumes) back and forth between the orthogonal and diagonal grids in a fractal way. Since the present three dimensional Pythagorean gear—the subject matter of research of this article—applies to the special case $x = y$, the general case $x \neq y$ was termed by the author the 'universal gear' and will be the subject of discussion of a future publication.

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