A unified geometrical approach for trigonometric angle sum and difference identities

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Introduction

Two important pedagogical techniques for developing deeper mathematical understanding are to prove a given theorem in different ways and to unify the proofs of different theorems. Trigonometric angle sum and difference identities are introduced in Unit 2 of Specialist Mathematics in the Australian Curriculum (Australian Curriculum, Assessment and Reporting Authority, n.d.). Typically, students first see these identities derived using an admixture of geometrical and algebraic methods, as exemplified in Weisstein (2018). In later (generally early tertiary) mathematics courses, complex exponential forms of the trigonometric functions and the Euler formula provide a more consistently algebraic approach to these identities. On the other hand, several geometric constructions are available for the various cases (Kung, 2008; Ren, 1999; Smiley, 1999; Smiley & Smiley, 2018) while Nelsen (2000) provides an excellent although apparently not widely known figure from which all six trigonometric angle sum and difference identities (for acute angles) may be derived geometrically.

In this paper an alternative, unified and simplified geometrical development of these identities is presented based on what is also in essence a single geometric construction. Allowing students already familiar with the traditional derivations of the trigonometric angle sum and difference identities to work through this alternative approach provides opportunities to deepen and consolidate their understanding of both the identities themselves and the geometrical techniques involved.

Tangent identities

In Figure 1(a) and (b), $OC = 1$, α and β are positive and acute, $\alpha \geq \beta$ (w.l.o.g.) and ΔDCE is non-trivial when $\alpha + \beta < \frac{\pi}{2}$. Also note that these diagrams are effectively equivalent as one may be obtained from the other by sliding B horizontally past O. Then, in *both* diagrams:

- $\alpha = \angle DCE = \angle DBA$ by opposite angles and angle sums of right triangles;
- the labelled sides are found from the definition of the tangent function; and
- $\Delta DBA \sim \Delta DCE$.

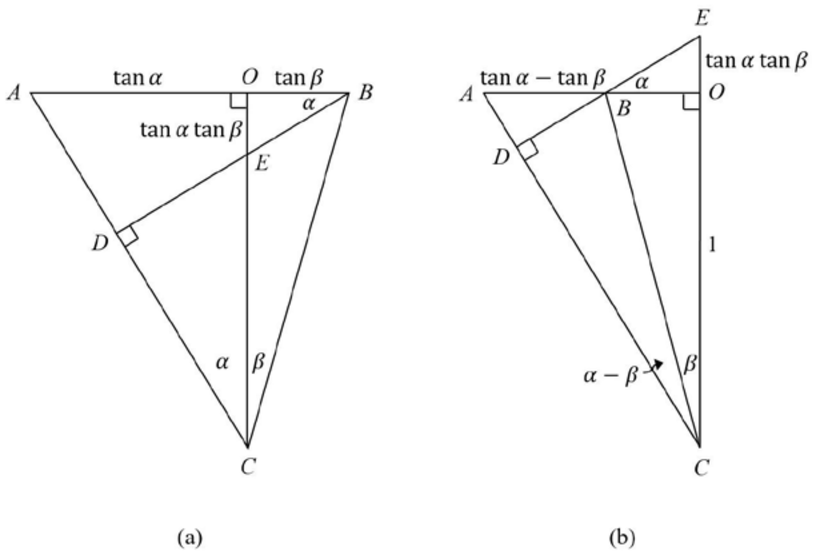

Figure 1

Comparing corresponding sides of ΔDBA and ΔDCE in Figure 1(a) gives

$$\frac{DB}{DC} = \frac{BA}{CE} \Rightarrow \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

while the identical comparison in Figure 1(b) gives

$$\frac{DB}{DC} = \frac{BA}{CE} \Rightarrow \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Sine and cosine identities

Figure 2 may be obtained from Figure 1 by multiplying all sides of the latter by the factor $\cos\alpha\cos\beta$, so Figure 2 is simply a scaled version of Figure 1. In particular, $OC = 1$ in Figure 1 becomes $OC = \cos\alpha\cos\beta$ in Figure 2. The other correspondingly labelled sides are also found without separate geometrical proofs simply by multiplying each side of Figure 1 by this factor. CB is then found directly from the definition of $\sin\beta$ in ΔOCB, i.e.:

$$\sin\beta = \frac{OB}{CB} \Rightarrow CB = \frac{\sin\beta\cos\alpha}{\sin\beta} = \cos\alpha$$

with $\sin\beta \neq 0$ by the general conditions stated previously.

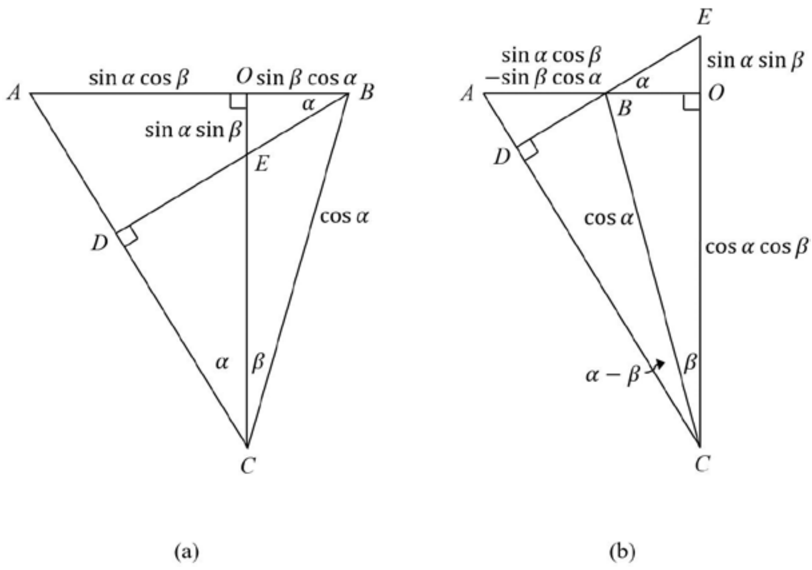

Figure 2

Then, in Figure 2(a), for BA, CB, $CE \neq 0$, combining the definition of $\cos\alpha$ in ΔDBA with the result above, rearranging then finding the indicated sides in the diagram gives

$$\frac{DB}{BA} = \cos\alpha = CB \Rightarrow \frac{DB}{CB} = BA \Rightarrow \sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

and, similarly,

$$\frac{DC}{CE} = \cos\alpha = CB \Rightarrow \frac{DC}{CB} = CE \Rightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

while the identical logic applied to Figure 2(b) gives

$$\frac{DB}{CB} = BA \Rightarrow \sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

and

$$\frac{DC}{CB} = CE \Rightarrow \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Further exploration

It should be noted that the proofs given above only apply to the restricted range of angles indicated, a detail often not made explicit in geometrical proofs provided to students. However, the underlying geometry of the diagrams unifies a broader range of diagrams and angles than displayed here. Using the intervals BA and CE to define axes with origin O, students can explore the cases that arise when point B is allowed to move along the entire horizontal axis. For instance, the standard negative angle identities such as $\tan(-\theta) = -\tan(\theta)$ may be derived by comparing results for angles $\alpha - \beta$ and $\beta - \alpha$ (without formally defining positive and negative angle directions) and the cases of $\alpha + \beta \geq \frac{\pi}{2}$ may also be considered. Such explorations could be assisted by use of suitable geometrical software. Finally, a version of Figure 1(b) has appeared in a 'proof without words' format (Ren, 1999) and this approach for the other identities also provides interesting student exercises.

Conclusion

One of the attractions of Figures 1 and 2 is that all four diagrams are effectively the same construction. The six standard trigonometric angle sum and difference identities are then seen to be different expressions of the same underlying geometry, a fact not obvious to students from the identities themselves or their typical 'mixed-method' derivations. The component parts of the identities are readily located in the diagrams and why they combine as they do in the identities becomes clearer. A wider range of angles in these identities than is available in alternative geometric proofs can also be explored.

It is hoped that this unified and purely geometrical approach will promote deeper mathematical insights by students already familiar with the trigonometric angle sum and difference identities.

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