Experiencing the angle properties in a circle

Jamhari
Institute for Innovative Learning
Mahidol University, Thailand
<jamhari.kawan@gmail.com>

Wararat Wongkia
Institute for Innovative Learning
Mahidol University, Thailand
<wararat.won@mahidol.edu>

This article presents some paper-folding activities for students to explore a different way to prove some of the angle properties in a circle.

Introduction

Geometry, one of the fundamental aspects of learning mathematics, is not only concerned with the study of shapes but also analyses the relationships and properties of the shapes (Luneta, 2015). By learning geometry, students have opportunities to develop their spatial thinking, visualisation skills, deductive reasoning, and proving (Battista, 2007). Circles are an elementary figure in geometry and are used to model physical phenomena (Brown et al, 2011). To learn the angle properties of a circle, students are expected to not only understand the properties but also to prove them (Robitaille, Wheeler, & Kieran, 1994). Angle properties in a circle have been included in secondary school mathematics curriculums of many countries, including Australia. The following four properties and their proofs were introduced:

Property 1: The angles at the centre and at the circumference of a circle subtended by same arc.
Property 2: Angles at the circumference subtended by a diameter.
Property 3: Angles at the circumference of a circle subtended by same arc.
Property 4: Angles in the cyclic quadrilateral.

Teaching activities for angle properties in a circle include paper folding and cutting (Central Board of Secondary Education, 2005), a scientific approach (Ministry of Education and Culture of Indonesia, 2013), and computer-assisted instruction (Akyuz, 2014). However, students have difficulties with angle properties in a circle (Jeopardy, 2013; Akyuz, 2014). Figure 1 shows two different appearances of angles in a circle. Even though both figures demonstrate the angles at the centre and at the circumference of the circles subtended by the same arc, some students do not realise that these figures are related to the same property of a circle. When students look at these two figures, the left-hand figure is obviously recognisable while the right-hand one is more difficult to recognise.

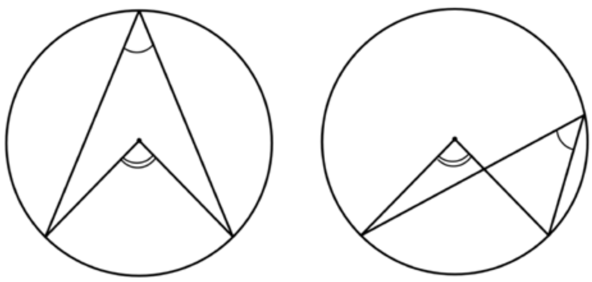


Figure 1. Different appearances of angles at the centre and at the circumference.

Another misconception is related to a cyclic quadrilateral. Figure 2 shows that a quadrilateral *ABCD* is a cyclic quadrilateral, however, students occasionally assume that quadrilateral *ABCO* is also a cyclic quadrilateral (Akyuz, 2014).

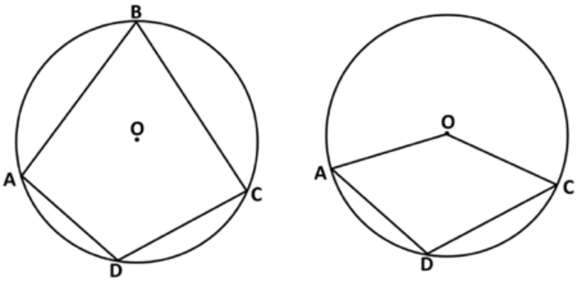


Figure 2. Quadrilateral *ABCD* and quadrilateral *ABCO* in circles.

A concrete manipulative is a physical object that allows students to see, touch, and manipulate it. It is particularly designed to support learning mathematical concepts with hands-on experiences (Cope, 2015). With the help of the manipulative, students are directly involved in an activity, so it helps to reduce the cognitive load produced by the activity that is not associ ated with the learning purposes (Bujak et al, 2013). Additionally, the use of the manipulatives helps students enhance their reasoning, problem-solving, and visualisation skills (Cass et al., 2003; Olkun, 2003; Baki, Kosa, & Guven, 2009). The concrete manipulative allows students to master the use of mathematical tools, for example a ruler, protractor, and compasses (Kilgo & White, 2014). Therefore, we propose an alternative concrete manipulative to support secondary students experience the angle properties in a circle.

Concrete manipulatives for angle properties in a circle

We originally created two sets of concrete manipulatives. One is called CircleBoard-Pro and helps students experience the angle properties in a circle. CircleBoard-Pro is made from a wooden circular board with movable circular protractors and elastic ropes. The circular board represents a circle while elastic ropes replace connected lines between points on the circle. We used circular protractors as the points on the circle for easier measuring of the angles between the elastic ropes in any direction (Figure 3).

Another manipulative is a paper-folding activity. The folding steps are created to facilitate students in proving the four properties. For each angle property, students were provided with paper-folding sheets and a cardboard circle (Figure 4).

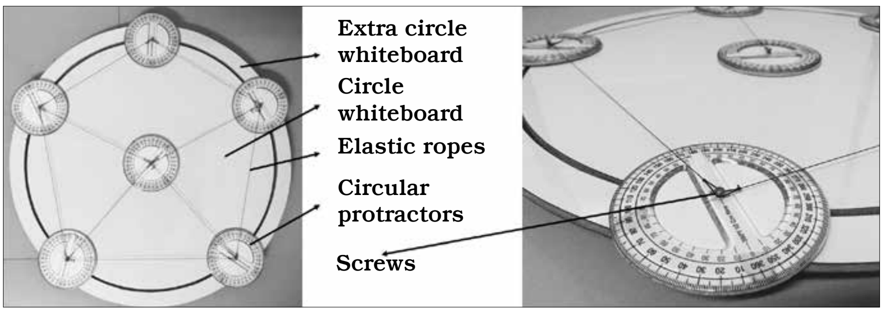


Figure 3. CircleBoard-Pro.

Paper-folding sheets can be folded. The Circle-cardboard is marked with lines and points related to the angle property. The paper-folding sheets and the Circle-cardboard can be fitted together as a puzzle. In each angle property, students explore CircleBoard-Pro to explore the relationship among angles and perform paper-folding activities to prove the property.

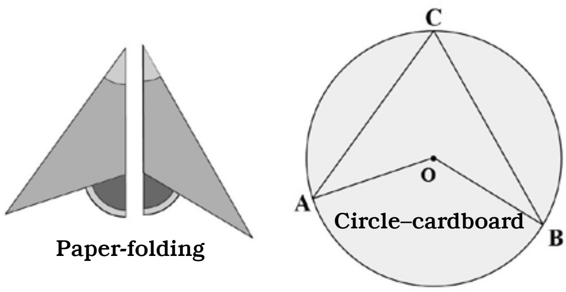


Figure 4. Paper-folding and circle-cardboard for Property 1.

Property 1: The angles at the centre and at the circumference of a circle subtended by same arc

To review students' prior knowledge, firstly have students put three circular protractors on the board; one at the centre O, and two at two points, A and B, on the circumference as shown in Figure 5.

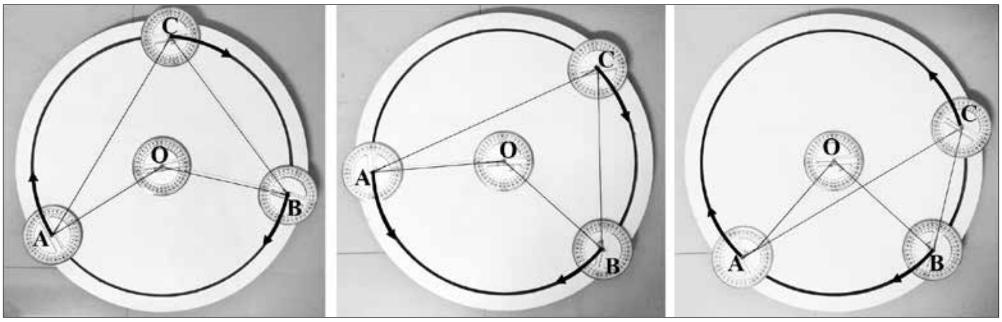
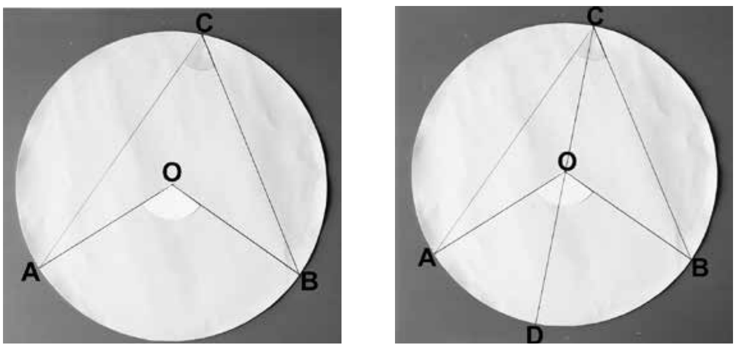


Figure 5. Exploring the relationship between angles at the centre and at the circumference.

Students should recall what the angle at the centre, $\angle AOB$, is. The question "What is the subtended arc of this angle?" is asked to let students notice the subtended arc, AB, of $\angle AOB$. Then, let students use another protractor, called C, to construct the angle at the circumference which is subtended by the arc AB. Then, have students drag the circular protractors along the circumference, and measure $\angle AOB$ and $\angle ACB$ as often as they want. Ask them to focus on what happens with those two angles when the arc AB is fixed and it is changed. After they explore and measure several pairs of those angles, they should make conjectures about the relationship between $\angle AOB$ and $\angle ACB$. The question "How do you make sure that all paired angles at the centre and at the circumference subtended by the same arc behave like your conjecture?" is given to motivate students to prove their conjecture.

Next, provide Circle-cardboard I as shown in Figure 6(a), and guide students to join the radius CO and then extend the line to meet the circumference at the point D (Figure 6(b)).



(a) (b)

Figure 6. (a) Circle-cardboard I (b) Drawing the line for proof.

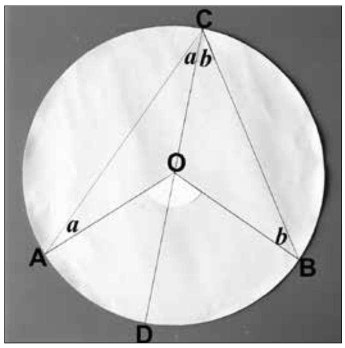
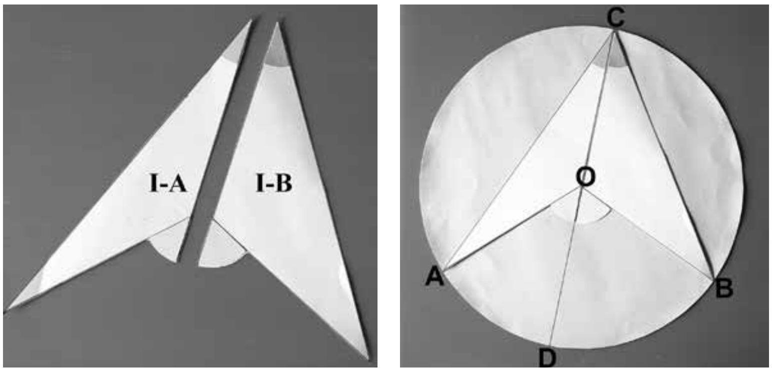


Figure 7. Labeling angles on Circle cardboard *I*.

Then, the questions "What did you see after constructing the line CD?" and "Why?" are asked to let students notice that the triangles OAC and OBC are isosceles triangles. The sides OA and OC of the triangle OAC are equal in length since they are radii of the circle. Similarly, the sides OB and OC of the triangle OBC are also equal in length.

Then, to give arbitrary constant angles, let students label $\angle OAC$ and $\angle OBC$, as a and b (Figure 7).

Now, we know that $\angle ACB$ is $a + b$ by the property of an isosceles triangle and the sum of two adjacent angles. Next, provide students with two sheets, Paper-folding *I–A* (Figure 8(a)) and Paper-folding *I–B* (Figure 8(b)), which are separated by the line *CD*. These two sheets can be put on Circle-cardboard *I* like a puzzle (Figure 8(c)).



(a) (b) (c)

Figure 8. (a) Paper-folding *I–A*, (b) Paper-folding *I–B*, and (c) making a puzzle with Circle-cardboard *I*.

Firstly, we should let students consider either Paper-folding *I–A* or *I–B*. To review the property of an isosceles triangle again, have students fold Paper-folding *I–A* in half (Figure 9).

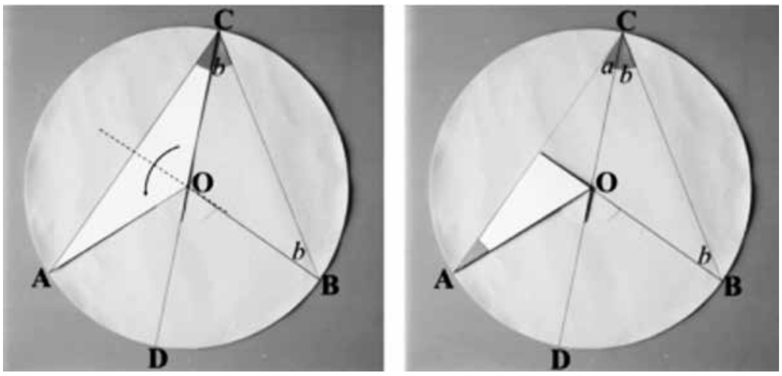


Figure 9. Folding Paper—folding *I–A* in half.

The question "What did you see after folding the paper in half?" is asked to let students observe that those two angles are equal because they are two equal angles of the isosceles triangle *OAC*. Next, have students fold Paper-folding *I–A* to place the vertex of $\angle OCA$ to coincide with the vertex of $\angle DOA$ (Figure 10).

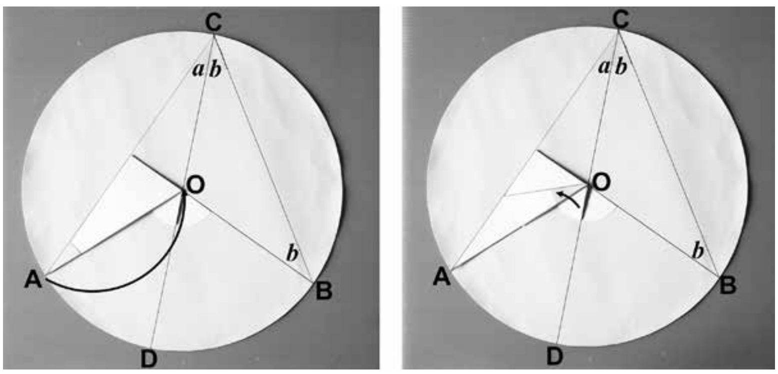


Figure 10. Folding $\angle DOA$ to be coincide with $\angle OCA$.

Then, fold the small part representing $\angle DOA$ along the line OA. Thus, $\angle DOA$ is on top of $\angle OCA$. Next, fold the overlay part back to meet the line OA again. Finally, let students compare $\angle DOA$ and $\angle OCA$ (Figure 11).

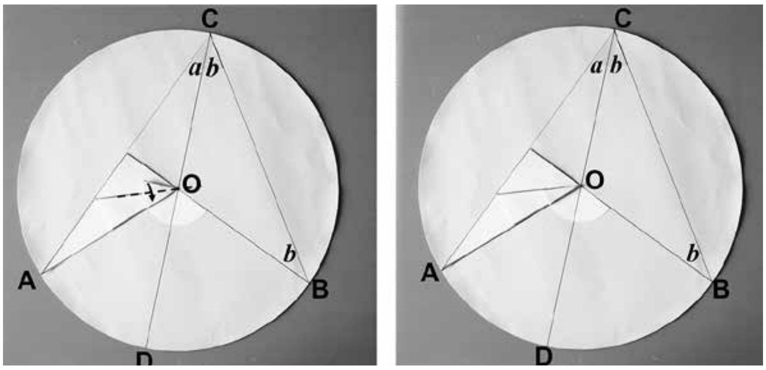


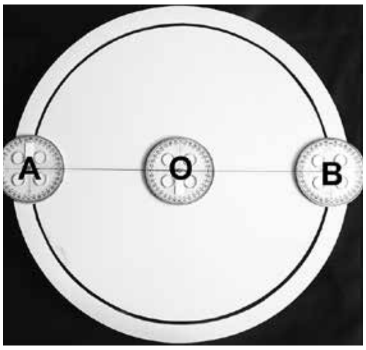
Figure 11. Folding Paper-folding *I–A* to compare the angles.

The purpose here is to let students see that half of $\angle DOA$ is an exterior angle of the triangle and equal to $\angle OCA$. We can say that $\angle DOA$ = twice $\angle OCA$ or $2a$. Similarly, Paper-folding *I–B* can also be folded. Thus, students should find that a half of $\angle DOB$ is equal to $\angle OBC$, or $\angle DOC$ = twice $\angle OBC$ or $2b$.

After students have all of the required statements and reasons, they should conclude that the angle at the centre $\angle AOB = \angle DOA + \angle DOB = 2a + 2b$ while the angle at the circumference $\angle ACB = a + b$. Therefore, the angle at the centre is twice the angle at the circumference if those angles are subtended by the same arc.

Property 2: Angles at the circumference subtended by a diameter

Have students use circular protractors, A, O, and B, to form a diameter of a circle (Figure 12).



Then, have students use another circular protractor, C, to form an angle at the circumference. At this point, students should see that $\angle ACB$ is subtended by the arc AB or the diameter of a circle. Next, let students move the protractor C and measure $\angle ACB$ until they make their conjectures (Figure 13).

Figure 12. CircleBoard-Pro representing a diameter of a circle.

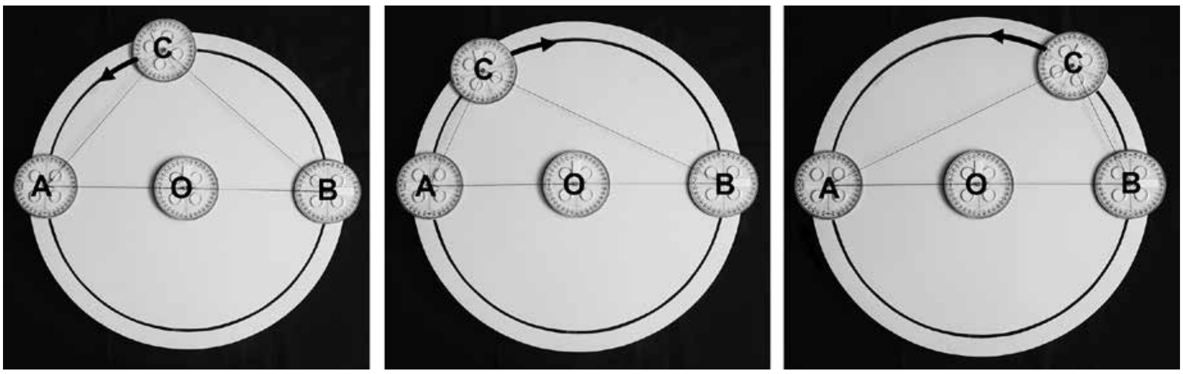


Figure 13. Exploring angles at the circumference subtended by a diameter.

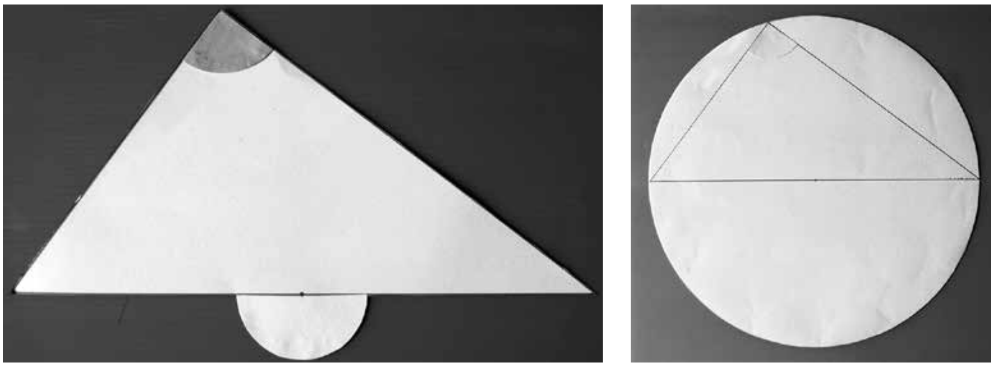


Figure 14. Paper-folding *II* and Circle-cardboard *II* used for Property 2.

Next, have students fold Paper-folding *II* into half of $\angle AOB$ (Figure 15).

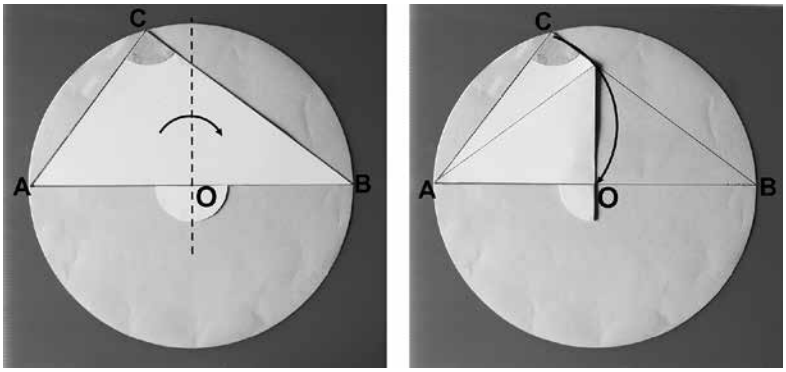


Figure 15. Folding paper—folding *II* in half of $\angle AOC$.

Then, fold to place the vertices of $\angle ACB$ and $\angle AOB$ to coincide, and fold to compare $\angle ACB$ and $\angle AOB$ (Figure 16).

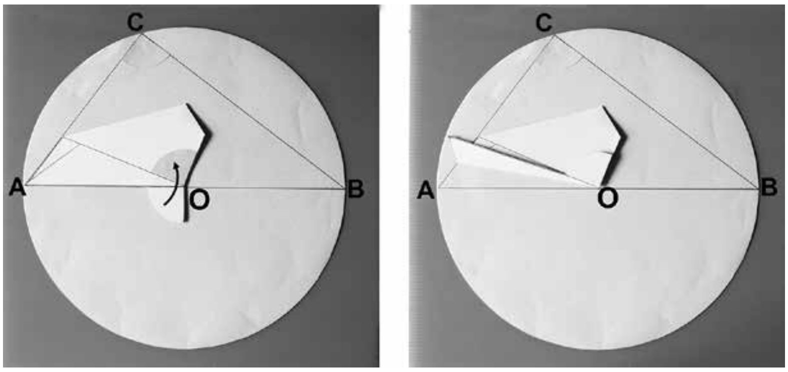


Figure 16. Folding paper—folding *II* to compare $\angle AOC$ and $\angle ABC$.

Here, students can review Property 1 in which $\angle ACB$ is half of $\angle AOB$ because the angle at the centre ($\angle AOB$) and at the circumference ($\angle ACB$) are subtended by the same arc. Next, students are asked the question "What is the magnitude of $\angle AOB$?" and "Why?". With the magnitude of $\angle AOB$ and Property 1, students can figure out that $\angle ACB$ is a right angle.

Property 3: Angles at the circumference of a circle subtended by same arc

Have students set CircleBoard-Pro to form an angle at the circumference ($\angle ABC$), then construct another angle at the circumference ($\angle ADC$) subtended by the arc AC.

Firstly, have students move the protractors B and D and measure $\angle ABC$ and $\angle ADC$ by fixing the protractors A and C. Secondly, have students change the arc AC by moving the protractors A and C. Then, move the protractors B and D and measure $\angle ABC$ and $\angle ADC$ (Figure 17).

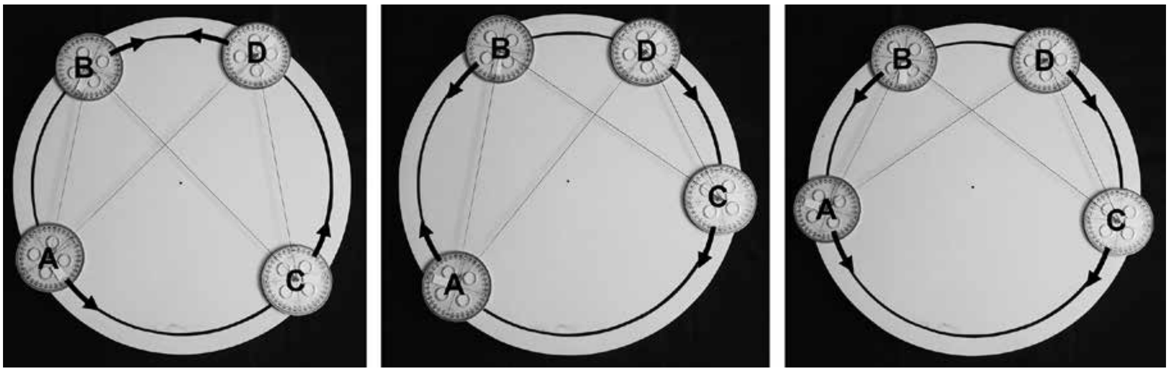


Figure 17. Exploring angles at the circumference subtended by the same arc.

They can repeat the first and second steps many times. They should find the relationship between the two angles at the circumference subtended by the same arc. Next in Figure 18, ask students to set another protractor to represent a new angle at the circumference ($\angle AEC$).

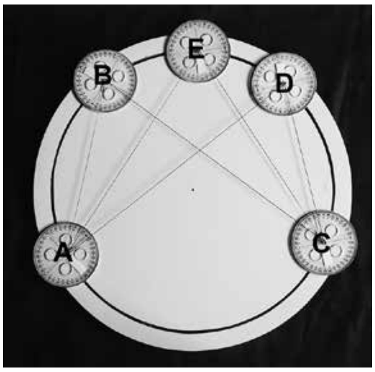
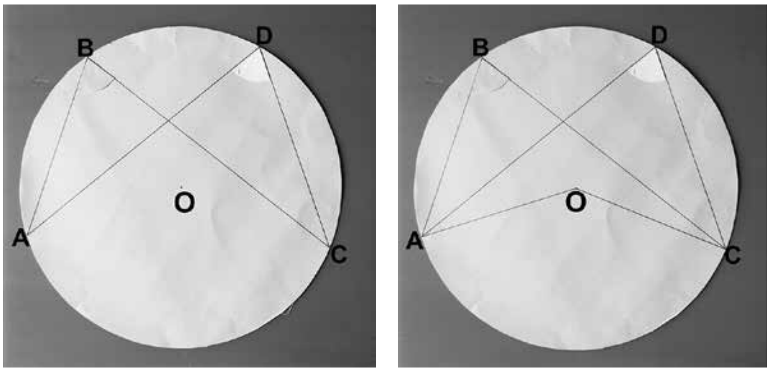


Figure 18. Setting a new angle at the circumference.



(a) (b)

Figure 19. (a) Circle-cardboard *III* and (b) joining *OA* and *OC*.

Then, have students move protractors B, D, and E and measure $\angle ABC$, $\angle ADC$, and $\angle AEC$. Here, they can crosscheck the relationship among angles at the circumference subtended by the same arc. To prove their conjecture, remind students about Property 1 and provide students with Circle-cardboard *III* (Figure 19(a)). Here, they should construct the angle at the center ($\angle AOC$) subtended by the arc AC (Figure 19(b)).

Then, ask the question "What is the relationship between the angle at the centre ($\angle AOC$) and those angles at the circumference $\angle ABC$ and $\angle ADC$?". To help students visualise and think, Circle-cardboard *III* and Paper-folding *III* are provided. Afterward, have students fold Paper-folding *III* in half at $\angle AOC$ (Figure 20).

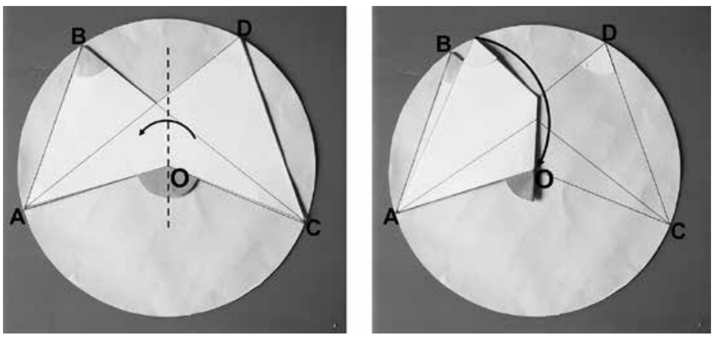


Figure 20. Folding paper—folding *III* in half at $\angle AOC$.

Then, fold to place the vertices of $\angle ABC$ and $\angle AOC$ to coincide, and compare $\angle ABC$ and $\angle AOC$ (Figure 21).

Students can recall Property 1 in which $\angle ABC$ is half of $\angle AOC$ because those angles are subtended by the same arc. Then, ask students to unfold Paper-folding *III* and then fold the other side to compare $\angle ADC$ with $\angle AOC$. Here, they should find that $\angle ADC$ is half of $\angle AOC$ as well. Then, students should conclude why $\angle ABC$ and $\angle ADC$ are equal.

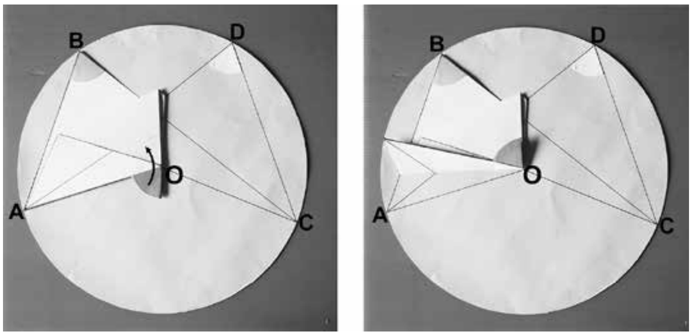


Figure 21. Folding paper—folding *III* to compare the angles.

Property 4: Angles in a cyclic quadrilateral

Have students set CircleBoard-Pro to form a quadrilateral inscribed a circle, and give the question "Where are the vertices of the quadrilateral?" to remind students about a cyclic quadrilateral. Then, have students move all the protractors and measure each angle until they can determine the relationship of angles in a cyclic quadrilateral (Figure 22).

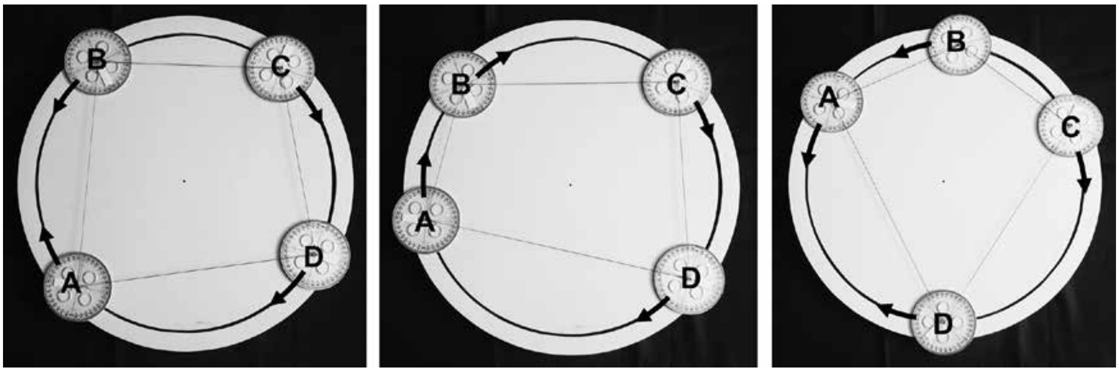


Figure 22. Exploring angles in a cyclic quadrilateral.

If students seem to struggle, they should be asked to add up the magnitudes of each pair of angles and observe what happens. Guide students to add $\angle A$ and $\angle B$, $\angle A$ to $\angle C$, and so on to help them make the conjectures related to Property 4. Then, provide students with Circle-cardboard *IV* (Figure 23).

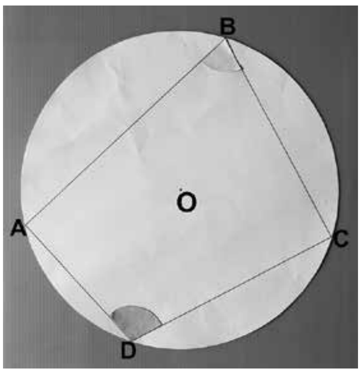


Figure 23. Circle-cardboard *IV* for Property 4.

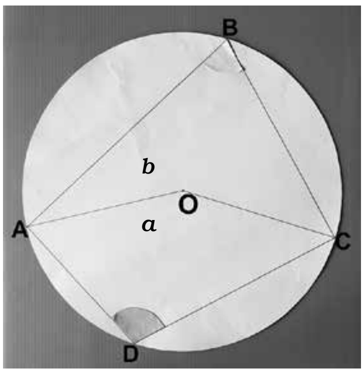


Figure 24. Joining radii *OA* and *OC* and labelling minor and major angle *AOC*.

Here, they should construct the angle at the centre, $\angle AOC$, by joining the radii OA and OC, and then label the minor and major angle AOC, a and b (Figure 24).

To encourage students, ask them to think what they know about the relationship between the minor and major angles AOC and ABC. To help students see the relationship between those angles, provide them Paper-folding *IV*, and then have students fold Paper-folding *IV* to show half of $\angle AOC$ (Figure 25).

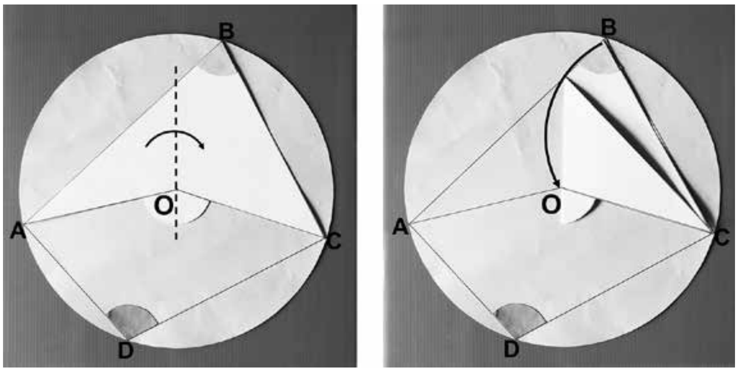


Figure 25. Folding Paper—folding *IV* to show half of $\angle AOC$.

Then, fold to make the vertices of $\angle ABC$ and the minor angle AOC to coincide, and compare $\angle ABC$ and the minor angle AOC (Figure 26).

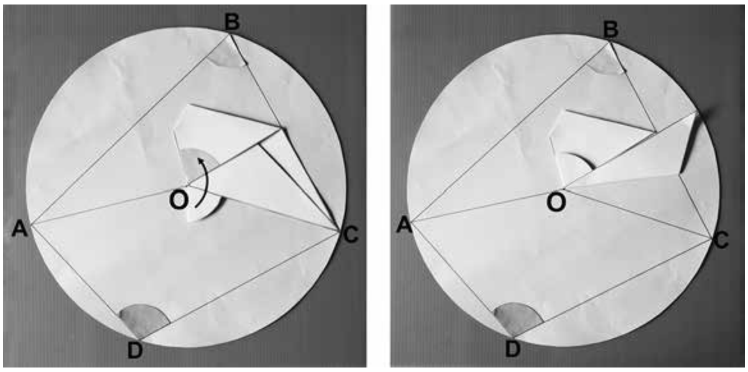


Figure 26. Folding Paper—folding *IV* to compare the angles.

Here, students can notice that $\angle ABC$ is half of the minor angle AOC ($ABC = \frac{1}{2}a$) and review Property 1. Without supporting another paper-folding, they should find that $\angle ADC$ is also half of the major angle AOC ($ADC = \frac{1}{2}$b) because of Property 1. Then, the question "What is the magnitude of $a + b$? And why?" is to let students conclude that the sum of $\angle ABC$ and $\angle ADC$ is 180° by considering $a + b = 360°$ and $\angle ABC + \angle ADC = \frac{1}{2}a + \frac{1}{2}b$. Then, repeat the steps to prove the other pair angles in the cyclic quadrilateral ($\angle BAD$ and $\angle BCD$) without using paper-folding.

Conclusion

The ideas of geometry are derived from phenomena on physical objects and can be used to explain the world. They are not only able to be applied in solving problems but also to be proved. Then, learning geometry assists students to improve their visualisation, problem-solving, reasoning, logical argument, and proof (Jones, 2002). In this activity, the concrete manipulative, CircleBoard-Pro, is utilised to facilitate students' visualisation of circle properties and then construct their mathematical conjectures through exploring the relationships between angles in a circle. Additionally, the paper-folding activities, which are easy to prepare, are utilised to help students construct logical arguments in proving their own conjectures. We do not expect students to write a formal proof of each property, however, we hope to see students' logical thinking. It affords opportunities for students to develop skills of investigating, reasoning, and proving. Therefore, learning geometry will be more understandable, attractive, and interesting.

References

Akyuz, D. (2016). Mathematical practices in a technological setting: a design research experiment for teaching circle properties. I*nternational Journal of Science and Mathematics Education, 14*(3), 549–573.

Baki, A., Kosa, T., & Guven, B. (2011). A comparative study of the effects of using dynamic geometry software and physical manipulatives on the spatial visualisation skills of pre-service mathematics teachers. *British Journal of Educational Technology, 42*(2), 291–310.

Battista, M. T. (2007). The development of geometric and spatial thinking. In: F. Lester (Ed.), Second Handbook of Research on Mathematics Teaching and Learning. Charlotte, NC: NCTM/Information Age Publishing.

Brown, P., Evans, M., Hunt, D., McIntosh, J., Pender, B., & Ramagge, J. (2011). *Circle Geometry: A guide for teachers–Years 9–10.* Melbourne: International Centre of Excellence for Education in Mathematics.

Bujak, K. R., Radu, I., Catrambone, R., MacIntyre, B., Zheng, R., & Golubski, G. (2013). A psychological perspective on augmented reality in the mathematics classroom. *Computers and Education, 68*, 536–544.

Cass, M., Cates, D., Smith, M., & Jackson, C. (2003). Effects of manipulative instruction on solving area and perimeter problems by students with learning disabilities. *Learning Disabilities Research and Practice, 18*(2), 112–120.

Central Board of Secondary Education. (2005). *Guidelines for Mathematics Laboratory in Schools.* Delhi: Preet Vihar

Cope, L. (2015). Math Manipulatives: Making the abstract tangible. *Delta Journal of Education, 5*(1), 10–19.

Jeopardy (2013, June 3). *Flipped classroom: Circle theorems and common misconceptions.* Retrieved from https://danpearcymaths.wordpress.com

Jones, K. (2002), Issues in the Teaching and Learning of Geometry. In: Linda Haggarty (Ed), *Aspects of Teaching Secondary Mathematics: perspectives on practice.* London: Routledge Falmer. Chapter 8, pp 121–139.

Kilgo, R. W., & White, A. A. (2014). *Virtual versus physical: manipulatives in the mathematics classroom.* Paper presented at 26th International Conference on Technology in Collegiate Mathematics, 2014 March 20–22, San Antonio, Texas.

Luneta, K. (2015). Understanding students' misconceptions: An analysis of final Grade 12 examination questions in geometry. *Pythagoras, 36*(1), 1–11.

Ministry of Education and Culture of Indonesia. (2014). *Book: Mathematics for Secondary School, Grade VIII, Semester 2.* Jakarta: Centre of Curriculum and Book, Balitbang.

Olkun, S. (2003). Comparing computer versus concrete manipulatives in learning 2D geometry. *Journal of Computers in Mathematics and Science Teaching, 22*(1), 43–56.

Robitaille, I. D. F., Wheeler, D. H., Hrsg, C. K., & Lectures, S. (1994). *On the appreciation of theorems by students and teachers.* In the 7th International Congress on Mathematical Education (pp. 353–363). Sainte-Foy: Université Laval.