ELPSA learning design to develop conceptual understandings of algebraic equivalence: The use of ribbons

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This project describes a way to help students recognise that the expressions on both sides of the equals sign are the same. It was developed to support teachers in disadvantaged communities to help their students make sense of mathematics.

Introduction

In previous editions of AMT, difficulties experienced by students learning algebra have been highlighted. For example, Baroudi (2006) maintained that students experienced difficulties with symbolic manipulation, the concept of equivalence, and the interpretation of the equals sign. Elsewhere, Wagner and Parker (1993) claimed that students do too much computing of answers in arithmetic which limits students' understanding of equivalence.

This article describes an approach to teaching algebra which was developed within a Government Partnership for Development (GPFD) project in West Nusa Tenggara, a province in Indonesia, funded by the Department of Foreign Affairs and Trading (DFAT). This approach helps students to recognise that the expressions on both sides of the equals sign are the same. In addition, the approach provides opportunities for the students to be flexible in their mathematical reasoning associated with symbolic representations of algebraic concepts. The activities foster opportunities for students to understand the use of the equals sign as a symbol of equivalence in algebraic equations. This is critical since students who consider the equals sign as a "do something" signal usually cannot see the fundamental relationship between the equals sign and the notion of equivalence (Kieran, 1981, p. 317).

The approach provides opportunities for students to make a meaningful transition from common arithmetical knowledge to algebra through making verbal and symbolic generalisations. Banerjee and Subramaniam (2012) identified that students' difficulties in algebraic symbol manipulation was due to inadequate understanding of arithmetic expressions. This approach uses pictorial representations to help students gain access to algebraic language and encourages the students to observe the pictures (ribbons arrangement) to search for relationships. This approach also presents opportunities for students to understand that variables can be represented by a specific number, or many numbers simultaneously. This understanding is critical as 'variable' is one of the most difficult concepts to grasp in algebra. Furthermore, the approach helps students make sense of the rule of transposing terms from one side of the equation to the other.

The mathematics content associated with the teaching ideas presented aligns to the Year 7 content descriptions in the *Australian Curriculum: Mathematics* (ACARA, 2018): “Introduce the concept of variables as a way of representing numbers using letters (ACMNA175)”; “Create algebraic expressions and evaluate them by substituting a given value for each variable (ACMNA176)”; “Extend and apply the laws and properties of arithmetic to algebraic terms and expressions (ACMNA177)”; and “Solve simple linear equations (ACMNA179)”.

ELPSA learning design using ribbons

The Experience, Language, Pictorial, Symbolic, and Application (ELPSA) learning design is used to sequence the tasks for this lesson so students are guided to make better sense of the concepts. ELPSA affords opportunities for teachers better understand the way in which pedagogical practices and learning experiences can be effectively presented to students in ways that reflect students’ concept development (Lowrie & Patahuddin, 2015). The ELPSA framework is described in detail in another article within this edition of the journal (Lowrie, Logan, Patahuddin, 2018).

This learning design ensures the students have adequate related experiences, provides opportunities for the students to talk about the ideas using their own language and mathematics terminologies, engages them with various pictorial representations and leads them to symbolic representations. By doing this, students are expected to develop a deep understanding of the concepts which will allow them to apply their knowledge to solve new or more complex problems.

Experience

The first component of the framework—Experience—is concerned with the knowledge and understandings students possess, drawn from both in- and out-of-school engagement. It is necessary to develop a relationship between the new content being introduced, and students’ existing knowledge.

The algebra equality lesson commences by finding out what students know about arithmetic equality using ribbon problems, as illustrated in Figure 1. It is important to establish whether the students can recognise the relationship between the length of the ribbons. For this purpose, Figure 1 is provided with measures displayed on the ribbons. This allows students to use simple arithmetic knowledge (e.g., 2 + 4 = …) or daily life experiences (e.g., demonstrates that the length of the ribbons can be compared and cutting a ribbon would produce a shorter ribbon).

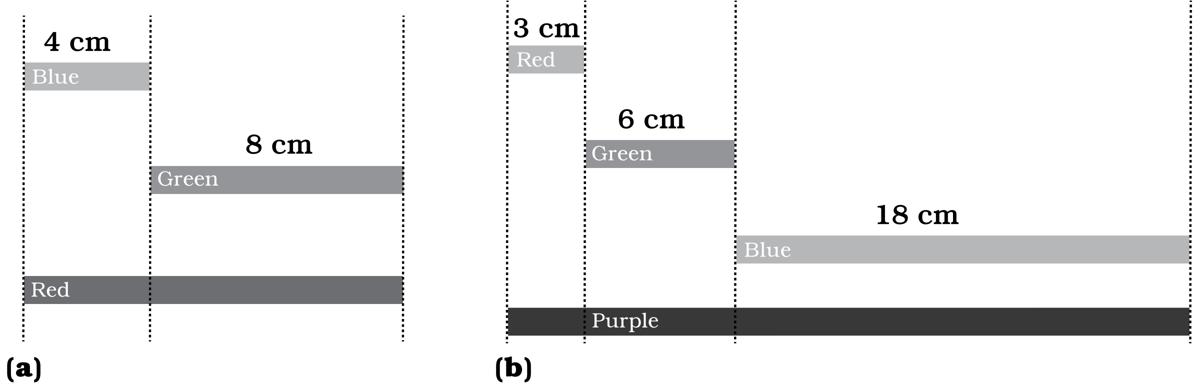


Figure 1. Ribbons’ display with known measures.

With Figure 1(a), the students could be led to generate some equivalent expression both verbally and symbolically to develop a stronger understanding of arithmetical knowledge. The students could then be encouraged to present their ideas both in sentences and mathematical expressions.

The students might state:

- The length of the blue ribbon combined with the length of the green ribbon is equal to the length of the red ribbon, that is 4 + 8 = 12.
- If you cut the red ribbon by the length of the blue ribbon, then you will have a ribbon that is the same length as the green ribbon, that is 12 – 4 = 8.
- Similarly, if you cut the red ribbon by the length of the green ribbon, then you will have a ribbon that is the same length as the blue ribbon, that is 12 – 8 = 4.

Following this demonstration, students are asked to work with another example using four ribbons, such as in Figure 1(b) to produce verbal expressions together with each of the following equivalent mathematical expressions.

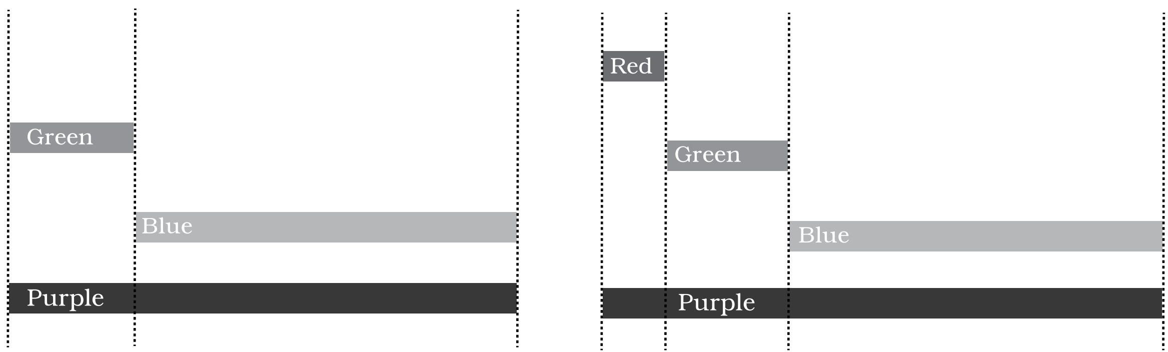
3 + 6 + 18 = 27	3 + 18 = 27 – 6	3 = 27 – 6 – 18	18 = 27 – 3 – 6
3 + 6 = 27 – 18	6 + 18 = 27 – 3	6 = 27 – 3 – 18	27 – 6 = 18 + 3

Through this activity, the teachers can assess the extent to which the students recognise equivalent relationships. This also encourages the students to think flexibly to produce arithmetical equations differently.

Language

The Language component of the framework has two purposes, namely: (1) to introduce mathematical terminology in ways that build upon the student's own experiences; and (2) to provide a bridge that connects the Experiences to the following Pictorial component. It focuses on the specific terminology the teacher models to present the mathematical concepts. In this component, the classroom teacher pays attention to the generic and specific language used to help clarify ideas about the concept.

This algebra lesson requires the development of a number of critical mathematical terms in order to establish understanding of the concept, such as add, subtract, equal to, and the length or measure of the ribbon. If the students were to state "red ribbon plus green ribbon", this misconception should be addressed immediately with the teacher emphasising that mathematically the measures are added, rather than the ribbons. By emphasising this, the misconception which could happen in the future, such as treating it as names (Wagner & Parker, 1993) would be anticipated (e.g., $a + b$ for apple plus banana). Probing questions could be posed to encourage the students to verbalise their thinking. For this purpose, Figure 2 can be used to stimulate the students' language and responses.



(a) **(b)**

Figure 2. Ribbons' display with unknown measures.

Figure 2 displays arrangements of differently coloured ribbons without reference to size. This is intended to develop further flexibility of thinking, and to create opportunities for the students to express their ideas without focusing on numbers. Implicitly, the students are introduced to variables without naming them as variables. The focus should be on the relationship of the length of the ribbons. Students could be invited to observe Figure 2(a) and questions could be posed such as:

- What do you notice about this picture?
- If the length of the ribbons is given, excluding the length of the green ribbon, how would you find the length of the green ribbon? What information do I need to find the length of the green ribbon?
- How would you find the length of the blue ribbon?

If the students use the symbols G, B and P for green, blue and purple ribbons, ask them to be explicit about what these symbols represent. After the students are clear about Figure 2(a), set up small groups for them to explain their insight about Figure 2(b). With this figure, the students are expected to produce various expressions including the following examples.

- The length of the green ribbon is equal to the length of the purple ribbon if you subtract the length of the red and blue ribbons. It can be written as $G = P - B - R$.
- The length of the blue ribbon is equal to the length of the purple ribbon if you subtract the length of the red and green ribbons. It can be written as $B = P - G - R$.

Pictorial

The Pictorial component is associated with the representations that help support mathematics understandings. These representations include concrete materials, visualisations, graphical displays and diagrammatical pictures. The pictorial representations can be modelled by the teacher, presented as a scaffold or encoded by the students (Samson, 2012).

Here, the students could be provided with grid paper and invited to draw a configuration of ribbons, such as in Figure 1 and 2, based on the instructions below. Students can use the grid to help them decide about the measure of each ribbon, as exemplified in Figure 3.

- Draw an arrangement of four ribbons which is associated with this equality $2 + 3 + 5 = 10$
- Produce other equalities based on your picture.
- Draw another arrangement of four ribbons which is associated with this equality $2 + 5 = 15 - 8$.

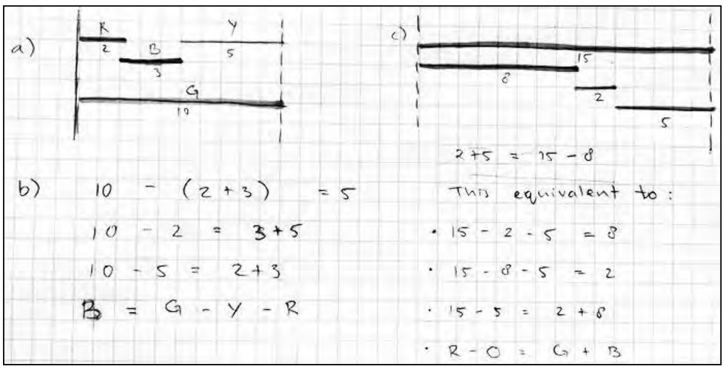


Figure 3. An example of student work.

A follow-up activity for the Pictorial component is to ask the students to work in pairs. One student is to draw an arrangement of ribbons (with or without measures) and the other student is to verbalise and produce various equivalent mathematical expressions. Another activity is for a student to write an arithmetic equality which represents the relationship between the lengths of the ribbons with another student drawing the associated arrangement of ribbons. From an ELPSA perspective, these pictorial activities lead to the production of symbolic representations.

Symbolic

The Symbolic component is associated with using symbols to represent mathematical ideas with accuracy and efficiency of representation. The Symbolic component always follows the Pictorial so that understanding and meaning of the symbols' representation is clear.

In the previous components, the notion of variables might be developed from students' responses. By contrast, the idea that each variable represents the length of each ribbon must be introduced explicitly as something that represents an unknown measure in this component of the framework. Figure 4(a) can be displayed on the board to introduce this notion.

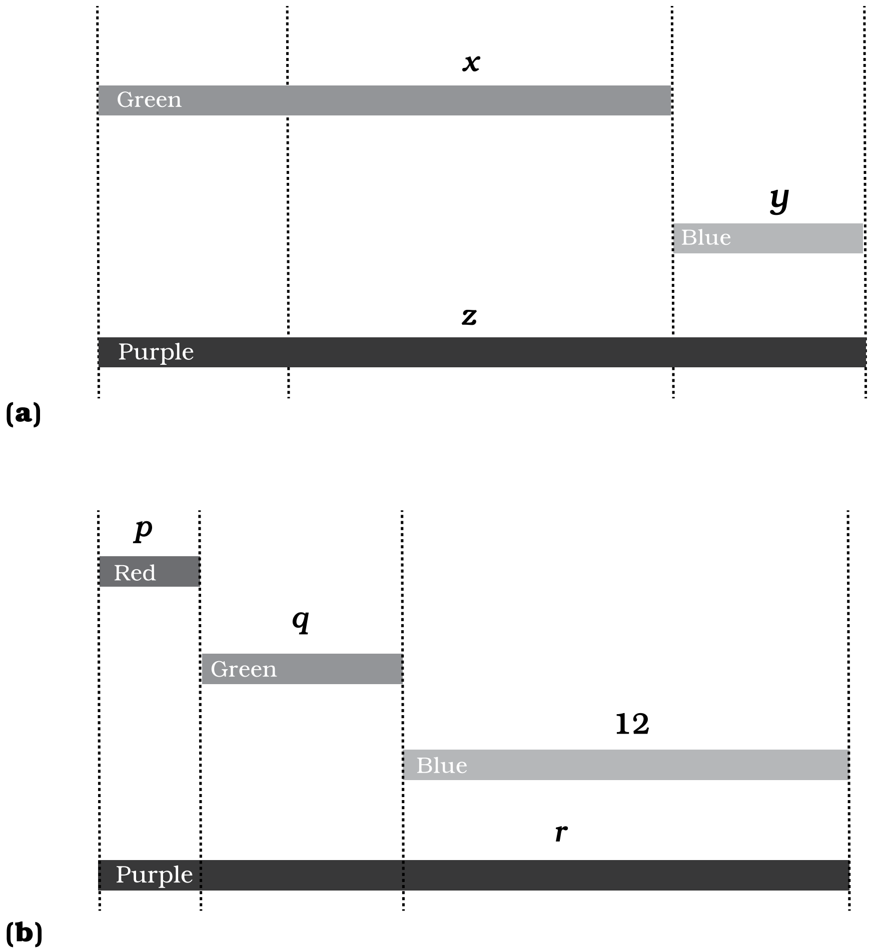


Figure 4. Ribbons' display with unknown measures represented as variables.

At this point, the students can be encouraged to find x in relation to y and z. Similarly, for other variables the students should be encouraged to explain their reasoning while producing the following expected symbolic expressions.

$$x + y = z \qquad x = z - y \qquad y = z - x$$
$$z = x + y \qquad z - y = x \qquad z - x = y$$

Small groups of students could be set up to explain to each other their insights about Figure 4(b) and together to produce the following algebraic symbolic representations.

$$p + q + 12 = r \qquad p + 12 = r - q \qquad p = r - q - 12 \qquad 12 = r - p - q$$
$$p + q = r - 12 \qquad q + 12 = r - p \qquad q = r - p - 12 \qquad r - 12 = q + p$$

During the Symbolic component, we expect the students to rely less on the pictures when manipulating the symbols. Therefore, it is important to help the students notice patterns from various equivalent algebraic expressions. For example, the terms on the left side of the equals sign use the opposite operation to the terms on the right side. In the symbolic representation of Figure 4(a), the x variable is added to the y variable on the left side (i.e., $x + y = z$), whereas the x variable is subtracted from the z variable on the right side (i.e., $y = z - x$).

Regarding the mathematical expressions generated from Figure 4(b), invite the students to notice the following:

- Each mathematical expression contains four terms (four ribbons).
- Symbol q is located on the left or right of the equals sign.
- If q is on the left side of the equals sign, it indicates an addition, and if q is on the right side of the equals sign, it indicates a subtraction.

The next step is to provide students with the algebraic form without pictures and ask them to generate as many equivalent equations as possible. "Write at least two other forms of the equation that are equivalent to each of the following equations".

$$A + B = C + D \qquad p + 4 = q + 6 \qquad x = y + 2 - z$$
$$2m + 2n - p = q \qquad -2k + 1 - m - n = 0 \qquad 2F = -G - H$$

It is also important to note some students will still need to draw their own pictures because they are having difficulty thinking symbolically. They should be permitted to do this but encouraged to move to the symbolic form when the student's understanding is sufficiently developed. It is important for teachers to facilitate students moving from an over-reliance on concrete objects or pictures to a mastery of symbolic manipulation or computation (Wu, 2001). In contrast, students who can do the tasks quickly should be provided with more complex algebraic expressions. By the end of this component, we expect students to be more flexible in manipulating symbolic representations using the concept of equivalence.

Application

The Application component is where students use concepts and ideas developed during the unit and apply these in other meaningful ways and to new situations.

The Application of this lesson could be challenging the students to find the value of the variables in Figure 5. This requires the students to transform the pictorial representations into symbolic form then manipulate the symbolic forms to find the value of A and p. Figure 5(a) can be considered a scaffold to solve the problem in Figure 5(b).

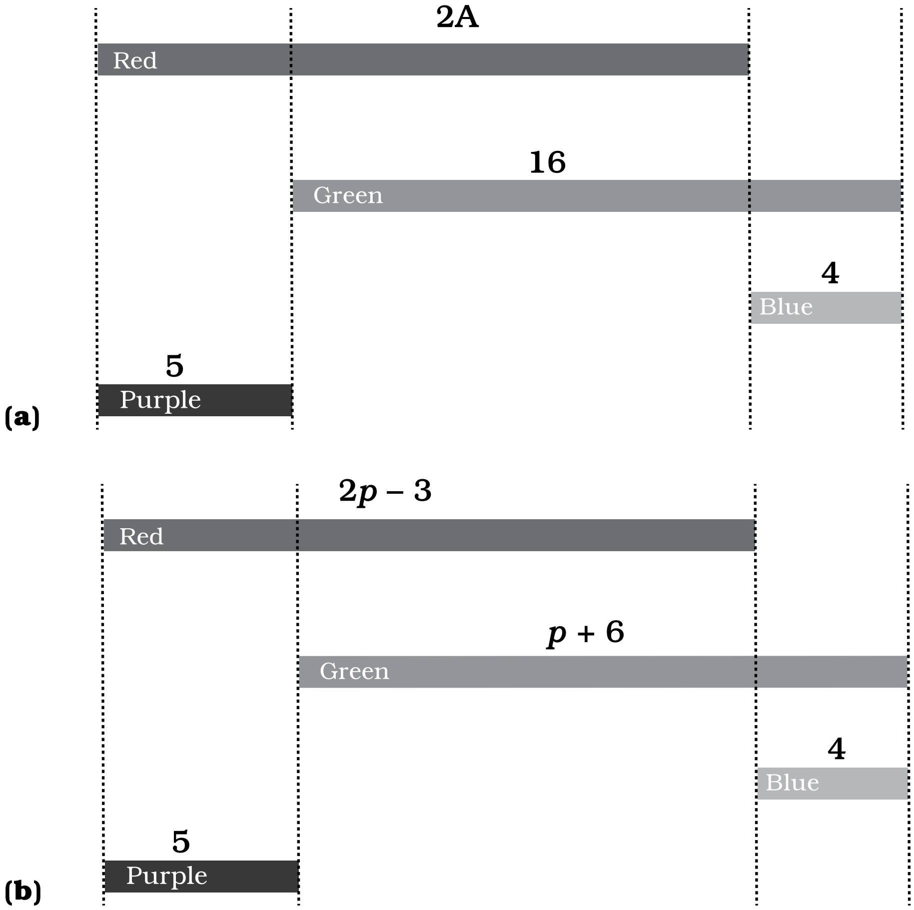


Figure 5. Ribbons' display to find the value of variables.

Figure 5(a) allows different ways to model the problem depending on how the students interpret the picture. This also provides opportunities for students to understand the pictorial representation while at the same time practise manipulating the symbolic representation (e.g., moving across the equals symbol changes the addition or subtraction sign).

Regarding the problem in Figure 5(a) two examples have been provided (Figure 6 and Figure 7). Figure 6 includes a retrospective interview from Student 1 whilst Figure 7 shows Student 2's strategies on the working-out paper.

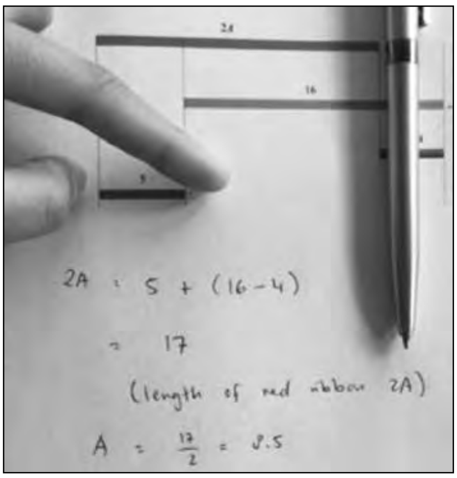


Figure 6. Student 1's strategy.

Teacher: What can you tell me about the length of each ribbon?

Student: The purple ribbon is five units, the green ribbon is sixteen units, the blue ribbon is four units and the red ribbon is 2A.

Teacher: How do you find the length of the red ribbon?

Student: [student wrote the answer, see the working out Figure 6]

Teacher: Can you please explain what you have written?

Student: I immediately noticed that the purple ribbon was underneath the red ribbon and its length was a part of the length of the red ribbon. Then, I wanted to add on the length of the green ribbon but it was too long for what I needed, so I cut the length of it by as much as the length of the blue ribbon [student indicated 'cutting the ribbon' by placing a pen on the third line in the diagram]. Finally, I added the length of the purple ribbon with the length of the cut green ribbon. So, this is the length of the red ribbon.

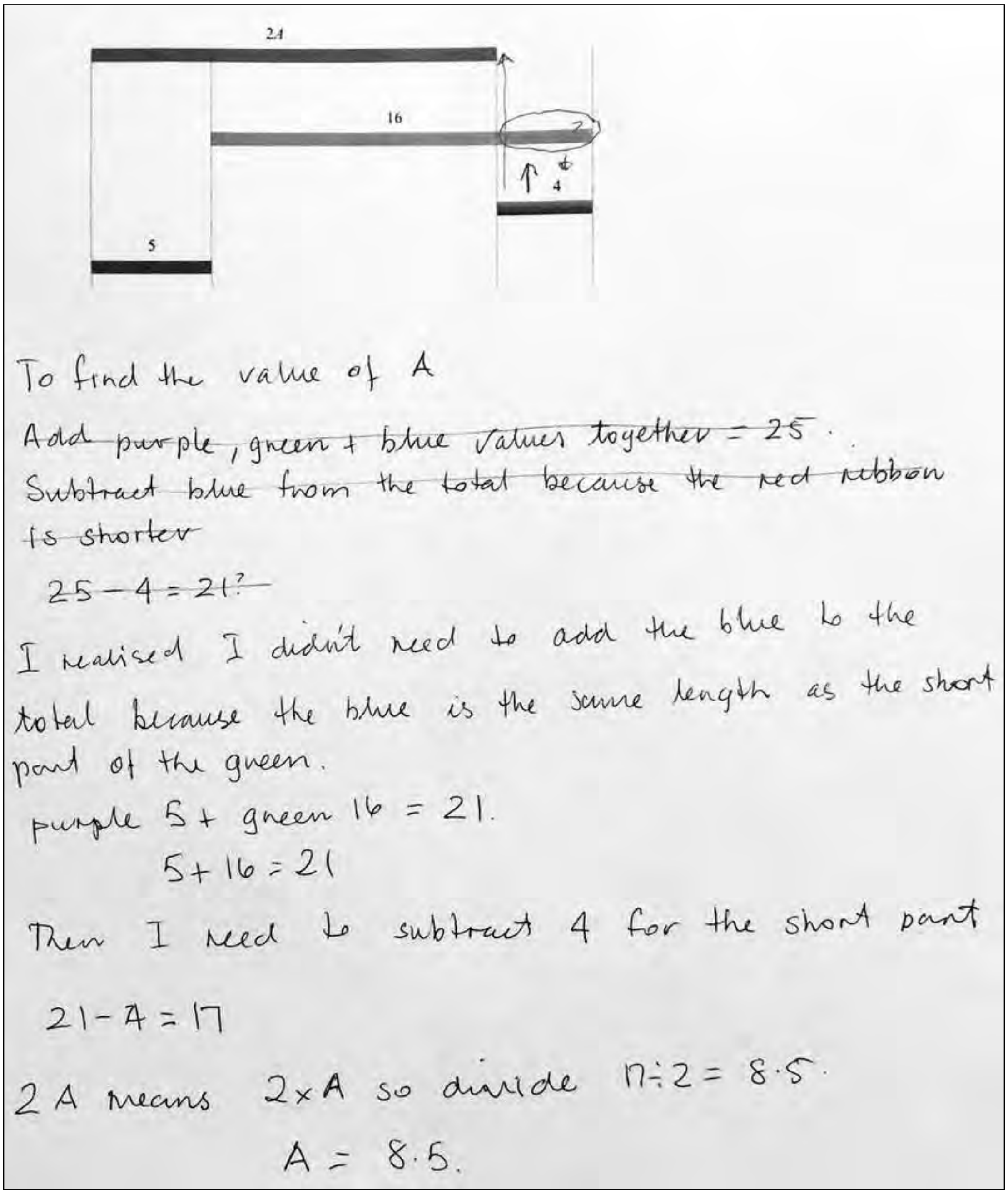


Figure 7. Student 2's strategy.

The Application component also provides opportunities for students to see how mathematics can be used in and out of school contexts. With respect to algebra equality, there are many applications in daily life, for example, modelling rates of changes in industries such as chemistry, medicine, agriculture, business, and accountancy.

Concluding comments

The task of identifying equal expressions from an arrangement of ribbons without requiring computations (all measures are removed) was designed to encourage students to focus more on the structure of the expressions. The first task was simple, to help the students easily rearrange the terms of equality and to easily check the value of each side of the equation. A more complex task followed by adding one more ribbon, to include a variety of equivalent expressions. Attaching symbols to the ribbons allowed the students to understand the meaning of the symbols. The Application provides students with a new problem, finding the value(s) of the variable for which the left- and right-hand sides are equal where the students need to focus on the relational aspect of an equation.

We argue that the ELPSA framework can be applied to any mathematics topic. The ELPSA design affords opportunities for the students to experience learning opportunities in a sequence that aligns to typical concept development. It is important to note that particular attention was given to attaching meaning to the students' understanding of equivalence by using pictorial representations. Those representations allowed the students to gain an understanding of the symbolic form, which provides a solid foundation for extensions of algebraic thinking.

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