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PLANE AND SPACE FIGURATE NUMBERS: VISUALIZATION WITH THE GEOGEBRA´S HELP

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Abstract: The study and interest of figurate numbers can be observed even in ancient Greece. On the other hand, it becomes very important the historical understanding about an evolutionary process and the particular generalization of such 2D, 3D and m-D figurate numbers and they remain the interest of current scientific investigations. On the other hand, in Brazil, when we talk about Mathematics Education, the component of visualization acquires more and more relevance for the teaching. In this way, in the present work, we present a proposal of discussion of the figure numbers, with idea that the understanding of arithmetic, algebraic and geometric properties can be facilitated with the use of GeoGebra software and it´s use by the mathematical teacher.

Key words: Figurate numbers 2D and 3D, Visualization, Mathematics Teaching, Mathematics Education.

1. Introduction

The mathematical historian Eves (1969: 53) explains that "the ancient Greeks distinguished between the study of conceptual relations between numbers and the practical art of counting." The author recalls that the Pythagoreans spoke about the numbers that allegedly possessed mystical properties, such as perfect numbers, deficient numbers, and abundant numbers; however, in this section, we will focus on the study of figurate which, according to Eves (1969, 54) originated with the first members of society Pythagorean. In essence, a figurate number presents the links between Arithmetic and Geometry, and other aspects are commented by Eves (1969, 1983), Dickson (1928) and Katz (1998).

In it´s turn, Popper (1972, p. 106) mentions without further explanation that Greek reasoning applied to forms geometrically, was extended to solids, despite the difficulties of identifying the 3D configurations which suggests the discussion of pyramidal numbers, or more specifically, pyramidal numbers triangles, square pyramidal numbers, pentagonal pyramidal numbers (Koshy, 2007).

Deza & Deza (2012) comment on the interest of ancient Greek mathematicians about the figurate numbers. The theory of figurate numbers does not belong to the central domains of Mathematics, but the beauty of these numbers attracted the attention of many scientists over the years. In this sense, the authors comment afterward on an extensive list of mathematicians who have worked in this field.

> The list (not full) of famous mathematicians, who worked in this domain, consists of Pythagoras of Samos (ca. 582 BC–ca. 507 BC), Hypsicles of Alexandria (190 BC–120 BC), Plutarch of Chaeronea (ca. 46–ca. 122), Nicomachus of Gerasa (ca. 60–ca. 120), Theon of Smyrna (70–135), Diophantus of Alexandria (ca. 210–ca. 290), Leonardo of Pisa, also known as Leonardo Fibonacci (ca. 1170 –ca. 1250), Michel Stifel (1487–1567), Gerolamo Cardano (1501–1576), Claude Gaspard Bachet de M´eziriac (1581–1638), René Descartes (1596–1650), Pierre de Fermat (1601–1665), John Pell (1611–1685), Blaise Pascal (1623–1662), Leonhard Euler (1707–1783), Joseph-Louis Lagrange (1736–1813), Adrien-Marie Legendre (1752–1833), Carl Friedrich Gauss (1777–1855), Augustin-Louis Cauchy (1789–1857), Carl Gustav Jacob Jacobi (1804–1851),Wac_law Franciszek Sierpinski (1882–1969), Barnes Wallis (1887–1979). (Deza & Deza, 2012, p. Xvii).

In addition, several other formal theorems can be formulated from the figure numbers and they are closely related to special classes of other numbers and numerical sequences, such as Fibonacci-Lucas numbers, Mersenne and Fermat numbers (Wenchang, 2007). Thus, in the field of Number Theory, we

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can find innumerable generalizations. On the other hand, we observe the relationships of figurate numbers with important theorems still in the present day, as expressed below Deza & Deza (2012).

> Figurate numbers were studied by the ancients, as far back as the Pythagoreans, but nowadays the interest in them is mostly in connection with the following *Fermat's polygonal number theorem*. In 1636, Fermat proposed that every number can be expressed as the sum of at most *m* *m*-gonal numbers. (Deza & Deza, 2012, p. Xvii).

To exemplify, let's look at the examples given by Deza & Deza (2012). Starting from a point, add to it two points, so that to obtain an equilateral triangle. Six-points equilateral triangle can be obtained from three-points triangle by adding to it three points; adding to it four points gives ten-points triangle, etc. So, by adding to a point two, three, four etc. points, then organizing the points in the form of an equilateral triangle and counting the number of points in each such triangle, one can obtain the numbers $1,3,6,10,15,21,28,36,45,55,\ldots$ which are called triangular number. Similarly, by adding to a point three, five, seven etc. points and organizing them in the form of a square, one can obtain the numbers $1,4,9,16,25,36,49,64,81,100,\ldots$ which are called square numbers.

On the other hand, the authors also describe the pentagonal, hexagonal numbers and the method for their construction and, correspondingly, the visualization in the plane. In fact, by adding to a point four, seven, ten etc. points and forming from them a regular pentagon, one can construct pentagonal numbers $1,5,12,22,35,51,70,92,117,145,\ldots$. Following this procedure, we can also construct hexagonal numbers $1,6,15,28,45,66,91,120,153,190,\ldots$. In figure 1 we visualize some particular examples, provided by Deze & Deza (2012) about the 2D figurate numbers. Kline (1972, p. 30) comments that the Pythagoreans knew the behavior of the following finite sum for the triangular figure numbers $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$ (Hoggat, 1974; Reuiller, 2008).

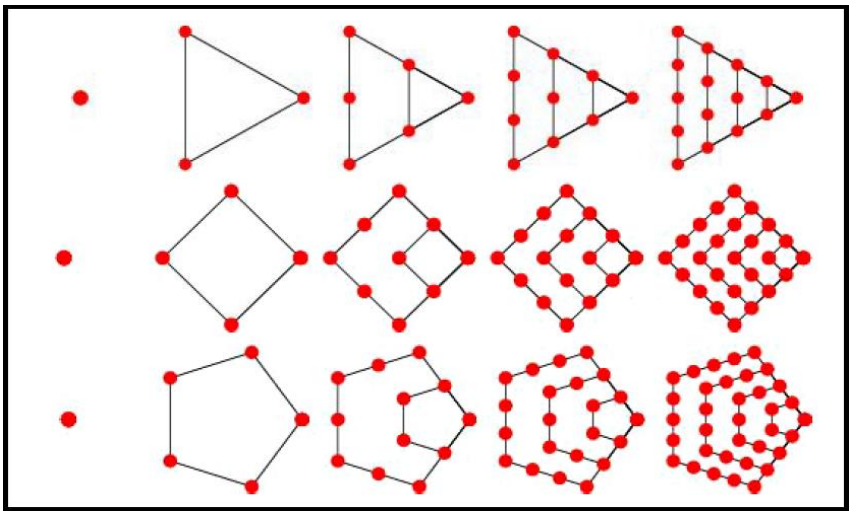

Figure 1. *Deza & Deza (2012) describe the construction of some examples of 2D figurate numbers.*

In figure 1, we can see some static constructions that correspond to the initial 2D figurate numbers. Deza & Deza (2012, pp. 2 - 3) further explain several other examples of 2D figurate numbers when they indicate that:

> Heptagonal numbers 1, 7, 18, 34, 55, 81, 112, 148, 189, 235, . . .(Sloane's A000566), Octagonal numbers 1, 8, 21, 40, 65, 96, 133, 176, 225, 280, . . . (Sloane's A000567), Nonagonal numbers 1, 9, 24, 46, 75, 111, 154, 204, 261, 325, . . . (Sloane's A001106), Decagonal numbers 1, 10, 27, 52, 85, 126, 175, 232, 297, 370, . . . (Sloane's A001107), Hendecagonal numbers 1, 11, 30, 58, 95, 141, 196, 260, 333, 415, . . . (Sloane's A051682), Dodecagonal numbers 1, 12, 33, 64, 105, 156, 217, 288, 369, 460, . . . (Sloane's A051624), etc. (Deza & Deza, 2012, p. 2 – 3).

We have thus seen the description of several 2D figurate and other supplementary information that can be found, for example, in the *On-line encyclopedia of the integer sequence*, founded in 1964, by the

mathematician Neil James Alexander Sloane (1939-?). Thus, in the subsequent sections, we will present some preliminary and mathematical properties on the 2D and 3D figurate numbers. Then we will see how the software GeoGebra can provide didactical situations for the use of visualization and the dynamic capacity of the constructions allowed by this software and the identification and understanding of the numerical and geometric properties, derived from these special numbers. Thus, with regard to the development of ideas and conceptions about Mathematics Education in Brazil, we have the possibility to identify that the use of technology for teaching of Mathematics allows the description of new scenarios for learning and teaching (Alves; Borges Neto; Duarte Maia, 2012).

2. Some mathematical aspects about the 2D figurate numbers

Deza & Deza (2012, p. 4) explain that the general rule for enlarging the regular polygon to the next size is to extend two adjacent sides by one point and then to add the required extra sides between those points. So, to transform *n*-th *m*-gonal number into the (*n* + 1)-th *m*-gonal number, one adjoins (*m* − 2)*n* + 1 elements. Thus, let us consider the n-th m-gonal number $S_m(n)$ is the sum of the first n elements of the arithmetic progression $1, 1+(m-2), 1+2(m-2), 1+3(m-2), 1+4(m-2), \ldots, m \geq 3$. In a simplified way, we can write the general mathematical formula indicated by the finite sum $S_m(n) = 1+(1+1\cdot(m-2))+(1+2\cdot(m-2))+(1+3\cdot(m-2))+\ldots(1+(m-2)\cdot(n-1))$.

In particular, we get some particular expressions that we listed below:

$$S_3(n) = 1+(1+(3-2))+(1+2(3-2))+(1+3(3-2))+\ldots(1+(3-2)(n-1)) = 1+2+3+\cdots+n;$$

$$S_4(n) = 1+(1+(4-2))+(1+2(4-2))+(1+3(4-2))+\ldots(1+(4-2)(n-1)) = 1+3+5+\cdots+2n-1;$$

$$S_5(n) = 1+(1+(5-2))+(1+2(5-2))+(1+3(5-2))+\ldots(1+(5-2)(n-1)) = 1+4+7+\cdots+3n-2;$$

$$S_6(n) = 1+(1+(6-2))+(1+2(6-2))+(1+3(6-2))+\ldots(1+(6-2)(n-1)) = 1+5+9+\cdots+4n-3;$$

$$S_7(n) = 1+(1+(7-2))+(1+2(7-2))+(1+3(7-2))+\ldots(1+(7-2)(n-1)) = 1+6+11+\cdots+5n-4;$$

$$S_8(n) = 1+(1+(8-2))+(1+2(8-2))+(1+3(8-2))+\ldots(1+(8-2)(n-1)) = 1+7+13+\cdots+6n-5;$$

Above expression implies the following recurrent formula for *m*-gonal numbers indicated by $S_m(n+1) = S_m(n)+(1+(m-2)n), S_m(1) = 1$. In particular, we get that $S_3(n+1) = S_3(n)+(n+1)$, $S_4(n+1) = S_4(n)+(2n+1)$, $S_5(n+1) = S_5(n)+(3n+1)$, $S_6(n+1) = S_6(n)+(4n+1)$, $S_7(n+1) = S_7(n)+(5n+1)$, $S_8(n+1) = S_8(n)+(6n+1)$, etc.

The following theorem describes a general formula for the description of any 2D figurate number. The reader will note that we provide more than one demonstration or proof concerning the same formula.

Theorem 1: The general formula for *n*-th *m*-gonal number is $S_m(n) = \frac{(m-2)}{2}\left(n^2-n\right)+n$ or $S_m(n) = \frac{n((m-2)n-m+4)}{2} = \frac{1}{2}m\left(n^2-n\right)-n^2+2n$, for every integer $n \geq 1$ and $m \geq 3$.

Proof 1. We consider the set $1, (1+1\cdot(m-2)), (1+2\cdot(m-2)), (1+3\cdot(m-2)), \ldots, (1+(m-2)\cdot(n-1))$ and since the finite sum of the first *n* elements of an arithmetic progression, we can write the fraction $S_m(n) = n\cdot\left(\frac{1+(1+(m-2)\cdot(n-1))}{2}\right) = n\cdot\left(\frac{1+(1+(m-2)n-m+2)}{2}\right) = n\cdot\left(\frac{(m-2)\cdot n-m+4}{2}\right) =$

$$=\frac{(m-2)n^2-(m-4)n}{2}=\frac{(m-2)n^2-(m-2)n+2n}{2}=\frac{(m-2)\left(n^2-n\right)+2n}{2}=\frac{(m-2)}{2}\left(n^2-n\right)+n.$$

We can still write $S_m(n)=n\cdot\left(\frac{(m-2)n-m+4}{2}\right)=\frac{(m-2)n^2-(m-4)n}{2}=\frac{mn^2-2n^2-mn+4n}{2}=$

$$=\frac{m\cdot(n^2-n)-2n^2+4n}{2}=\frac{m}{2}\left(n^2-n\right)-n^2+2n\,\square$$

Proof 2. Moreover, we can take the special finite summation and we can group to match the second terms we observe in the two finite sums below. Next, let's consider the expression $2\cdot S_m(n)$.

$$\begin{cases} S_m(n)=0 \quad + \quad 1 \quad + 1+(m-2) \quad + \cdots+\cdots+\cdots+\cdots\cdots+1+(m-2)(n-1) \\ S_m(n)=1+(m-2)(n-1)+\ 1+(m-2)(n-2)\ +\cdots \quad + \quad 1 \end{cases}$$

. In this way, we will find that $2\cdot S_m(n)=(2+(m-2)\cdot(n-1))+\cdots+((2+(m-2)\cdot(n-1))=n\cdot\left(2+(m-2)\cdot(n-1)\right)=$

$$=n\left((m-2)n-m+4\right)\therefore S_m(n)=\frac{n\left((m-2)n-m+4\right)}{2}\,\square$$

Proof 3. Another way to derive the general formula for $S_m(n)$ is to use the first three *m*-gonal numbers to find the coefficients *A*, *B*, and *C* of the general 2-nd degree polynomial $p(n)=\mathrm{A}n^2+Bn+C$. For $n=1\therefore p(1)=\mathrm{A}+B+C=1$ and for $n=2\therefore p(2)=4\mathrm{A}+2B+C=m$. Finally, for $n=3\therefore p(3)=9\mathrm{A}+3B+C=3m-3$. Then we will solve the following system $\begin{cases}\mathrm{A}+B+C=1\\ 4\mathrm{A}+2B+C=m\\ 9\mathrm{A}+3B+C=3m-3\end{cases}$. This leads to $\mathrm{A}=\frac{m-2}{2},B=\frac{4-m}{2},C=0$. In this way, we find that

$$p(n)=\left(\frac{m-2}{2}\right)n^2+\left(\frac{4-m}{2}\right)n+0=\frac{n\left((m-2)n-m+4\right)}{2}\,\square$$

The formulas for m-gonal numbers with $3\le m\le 30$ are described in the in the *Sloane's On-Line Encyclopedia of Integer Sequences* are given in the table below. We can determine the particular cases of the formulas that we can identify in figure 2, in the left column. For example, if we consider the triangular numbers 2D, we can consider that we deal with a polynomial function of the second degree An^2+Bn+C and, we can get that $\begin{cases}A+B+C=1\\ 4A+2B+C=3, n=1,2,3\\ 9A+3B+C=6\end{cases}$. This leads to $A=\frac{1}{2}=B,C=0$.

Moreover, for the square numbers, we can get the system $\begin{cases}A+B+C=1\\ 4A+2B+C=4, n=1,2,3\\ 9A+3B+C=9\end{cases}$. This leads to $A=1,B=0,C=0$. In both cases, we determine that we can write $S_3(n)=\frac{n^2+n}{2}$ and $S_4(n)=n^2$.

In Figure 2, Deza & Deza (2012, p.6) indicate some first few elements of the corresponding sequences and the Figure Numbers numbers of these sequences in *Sloane's On-Line Encyclopedia of Integer Sequences*. Based on the table below we can analyze the values indicated for the *On-Line Encyclopedia of Integer Sequences* which is an extensive database, which records sequences of integers freely available on the *Internet*.

Name	Formula												Sloane
Triangular	$\frac{1}{2}(n^2+n)$	1	3	6	10	15	21	28	36	45	55	66	A000217
Square	$\frac{1}{2}(n^2-0\cdot n)$	1	4	9	16	25	36	49	64	81	100	121	A000290
Pentagonal	$\frac{1}{2}(3n^2-1\cdot n)$	1	5	12	22	35	51	70	92	117	145	176	A000326
Hexagonal	$\frac{1}{2}(4n^2-2n)$	1	6	15	28	45	66	91	120	153	190	231	A000384
Heptagonal	$\frac{1}{2}(5n^2-3n)$	1	7	18	34	55	81	112	148	189	235	286	A000566
Octagonal	$\frac{1}{2}(6n^2-4n)$	1	8	21	40	65	96	133	176	225	280	341	A000567
Nonagonal	$\frac{1}{2}(7n^2-5n)$	1	9	24	46	75	111	154	204	261	325	396	A001106
Decagonal	$\frac{1}{2}(8n^2-6n)$	1	10	27	52	85	126	175	232	297	370	451	A001107
Hendecagonal	$\frac{1}{2}(9n^2-7n)$	1	11	30	58	95	141	196	260	333	415	506	A051682
Dodecagonal	$\frac{1}{2}(10n^2-8n)$	1	12	33	64	105	156	217	288	369	460	561	A051624
Tridecagonal	$\frac{1}{2}(11n^2-9n)$	1	13	36	70	115	171	238	316	405	505	616	A051865
Tetradecagonal	$\frac{1}{2}(12n^2-10n)$	1	14	39	76	125	186	259	344	441	550	671	A051866
Pentadecagonal	$\frac{1}{2}(13n^2-11n)$	1	15	42	82	135	201	280	372	477	595	726	A051867
Hexadecagonal	$\frac{1}{2}(14n^2-12n)$	1	16	45	88	145	216	301	400	513	640	781	A051868
Heptadecagonal	$\frac{1}{2}(15n^2-13n)$	1	17	48	94	155	231	322	428	549	685	836	A051869
Octadecagonal	$\frac{1}{2}(16n^2-14n)$	1	18	51	100	165	246	343	456	585	730	891	A051870
Nonadecagonal	$\frac{1}{2}(17n^2-15n)$	1	19	54	106	175	261	364	484	621	775	946	A051871
Icosagonal	$\frac{1}{2}(18n^2-16n)$	1	20	57	112	185	276	385	512	657	820	1001	A015872
Icosihenagonal	$\frac{1}{2}(19n^2-17n)$	1	21	60	118	195	291	406	540	693	865	1056	A015873
Icosidigonal	$\frac{1}{2}(20n^2-18n)$	1	22	63	124	205	306	427	568	729	910	1111	A015874
Icositrigonal	$\frac{1}{2}(21n^2-19n)$	1	23	66	130	215	321	448	596	765	955	1166	A015875
Icositetragonal	$\frac{1}{2}(22n^2-20n)$	1	24	69	136	225	336	469	624	801	1000	1221	A015876
Icosipentagonal	$\frac{1}{2}(23n^2-21n)$	1	25	72	142	235	351	490	652	837	1045	1276	
Icosihexagonal	$\frac{1}{2}(24n^2-22n)$	1	26	75	148	245	366	511	680	873	1090	1331	
Icosiheptagonal	$\frac{1}{2}(25n^2-23n)$	1	27	78	154	255	381	532	708	909	1135	1386	
Icosioctagonal	$\frac{1}{2}(26n^2-24n)$	1	28	81	160	265	396	553	736	945	1180	1441	
Icosinonagonal	$\frac{1}{2}(27n^2-25n)$	1	29	84	166	275	411	574	764	981	1225	1496	
Triacontagonal	$\frac{1}{2}(28n^2-26n)$	1	30	87	172	285	426	595	792	1017	1270	1551	

Figure 2. *Deza & Deza (2012) describe the construction of some examples of 2D figurate numbers.*

We then observe the explanations of Deza & Deza (2012) concerning the study of centered polygonal numbers or the *n*-th centered *m*-gonal number, which we denote by $CS_m(n)$.

> The centered polygonal numbers (or, sometimes, polygonal numbers of the second order) form a class of figurate numbers, in which layers of polygons are drawn centered about a point. Each centered polygonal number is formed by a central dot, surrounded by polygonal layers with a constant number of sides. Each side of a polygonal layer contains one dot more than any side of the previous layer, so starting from the second polygonal layer each layer of a centered *m*-gonal number contains *m* more points than the previous layer. (Deza & Deza, 2012, p. 48).

In the figure 3 we see a centered triangular number, a centered square number, a centered pentagonal number and a centered hexagonal number, which represents a hexagon ith a dot in the center and all other dots surrounding the center dot in a hexagonal lattice. In figure 3 we see some of these planar configurations. We can compare their numerical values with the 2D figurate numbers indicated in the predecessor paragraphs. We will then deal with the dimensional increase for such numbers.

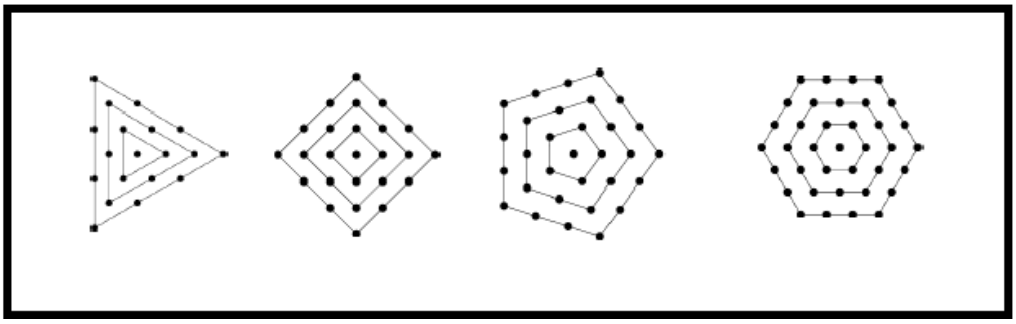

Figure 3. *Deza & Deza (2012) describe the construction of some examples of 2D figurate numbers.*

In the following section, we will cover some mathematical properties about the 3D figurate numbers.

3. Some mathematical aspects about the 3D figurate numbers

The *n*-th *m*-pyramidal number $S_m{}^3(n)$ is defined as the sum of the first *n* *m*-gonal numbers: $S_m{}^3(n) = S_m(1)+S_m(2)+S_m(3)+\cdots+S_m(n)$. So, it holds the following recurrent formula for the *m*-pyramidal numbers $S_m{}^3(n) = S_m{}^3(n-1)+S_m(n), S_m^3(1)=1$. The following theorem describes a general formula for the description of any 3D figurate number.

Theorem 2: The general formula for *n*-th *m*-gonal pyramidal number has the form $S_m{}^3(n) = \dfrac{n(n+1)\cdot\left((m-2)n-m+5\right)}{6}$.

Proof. Let us prove it by Mathematical induction. In fact, for the initial case, we consider that $n=1\therefore$ $S_m{}^3(1)=1=\dfrac{6}{6}=\dfrac{1\cdot 2\cdot(3)}{6}=\dfrac{1\cdot 2\cdot(-2+5)}{6}=\dfrac{1\cdot 2\cdot(m-2-m+5)}{6}=\dfrac{1\cdot 2\cdot((m-2)\cdot 1-m+5)}{6}$. Going from *n* to *n* + 1, one obtains $S_m{}^3(n+1)=S_m{}^3(n)+S_m(n+1)=\dfrac{n\cdot(n+1)\cdot\left((m-2)\cdot n-m+5\right)}{6}+$

$$\frac{(n+1)((m-2)(n+1)-m+4)}{2}=\frac{(n+1)\cdot\left((m-2)n^2+n(5-m)\right)+3(n+1)(m-2)+3(4-m))}{6}=$$

$$=\frac{(n+1)\cdot\left((m-2)n^2+(n+1)\cdot n(5-m)+3n(m-2)+3(m-2)+3(4-m)\right)}{6}=$$

$$=\frac{(n+1)}{6}\left[\frac{(n^2+3n+3)(m-2)+n(5-m)+3(4-m))}{1}\right]=\frac{(n+1)}{6}\left[\frac{(n^2+3n+3)(m-2)+5n-nm+12-3m}{1}\right]$$

$$=\frac{(n+1)}{6}\left[\frac{(n^2+3n+3)(m-2)+(5n-nm+12-3m)+10-10-2-3m+3m}{1}\right]=$$

$$=\frac{(n+1)}{6}\left[\frac{(n^2+3n+3)(m-2)+5n-nm+10}{1}\right]=$$

$$=\frac{(n+1)}{6}\left[\frac{(n^2+3n+3)(m-2)+5n-nm+10-2m+m-2-10+2m+12-3m}{1}\right]=$$

$$=\frac{(n+1)}{6}\left[\frac{(n^2+3n+3)(m-2)+(n+2)(5-m)+(m-2)-2(5-m)+3(4-m)}{1}\right]=$$

$$=\frac{(n+1)(n+2)(mn+m-2n-2-m+5)}{6}=\frac{(n+1)(n+2)((m-2)(n+1)-m+5)}{6}$$

Deza & Deza (2012, p.89) comment that the formula $S_m{}^3(n)=\dfrac{n(n+1)\cdot\left((m-2)n-m+5\right)}{6}$ was known to Archimedes (287-212 BC) and it was used by him for finding volumes. On the other hand,

in current research, we can see the interest of researchers in the generalization process of this formula. This way, we can determine the 4D, 5D, 6D, etc. m-gonal figurate numbers (Arjun, 2013).

Algebraically, *n*-th centered *m*-gonal number $CS_m(n)$ is obtained as the sum of the first *n* elements of the sequence 1,*m*, 2*m*, 3*m*, ... So, by definition, it holds $CS_m(n) = 1 + m + 2m + 3m + \cdots + (n-1)m$. In particular, we get $CS_3(n) = 1 + 2 + 6 + 9 + \cdots + 3(n-1)$, $CS_4(n) = 1 + 4 + 8 + 12 + \cdots + 4(n-1)$, $CS_5(n) = 1 + 5 + 10 + 15 + \cdots + 5(n-1)$, $CS_6(n) = 1 + 6 + 12 + 18 + \cdots + 6(n-1)$, etc. Consequently, the above formulates the following recurrent formula for the centered m-gonal numbers is $CS_m(n+1) = CS_m(n) + n \cdot m, CS_m(1) = 1$. On the outher hand, we note that the finite sum $m + 2m + 3m + \cdots + (n-1) \cdot m = m \cdot \left(1 + 2 + 3 + \cdots + (n-1)\right) = m \cdot \frac{n(n-1)}{2}$. We obtain the following general formula for *n*-th centered *m*-gonal number: $CS_m(n) = 1 + m \cdot \frac{n(n-1)}{2} = \frac{m \cdot n^2 - m \cdot n + 2}{2}$.

In particular, we have: $CS_3(n) = \frac{3 \cdot n^2 - 3 \cdot n + 2}{2}$, $CS_m(n) = 2 \cdot n^2 - 2 \cdot n + 1$, $CS_5(n) = \frac{5 \cdot n^2 - 5 \cdot n + 2}{2}$, $CS_6(n) = 3n^2 - 3 \cdot n + 1$, $CS_7(n) = \frac{7 \cdot n^2 - 7 \cdot n + 2}{2}$, $CS_8(n) = 4 \cdot n^2 - 4 \cdot n + 1$, etc, etc. b

In the last sections, we discuss some formal properties and theorems about the 2D and 3D figurate numbers. Now, with the help of the GeoGebra software, we will see that they can be represented in a dynamic way with the software and the visualization constitutes an important component for the essential properties of this class of numbers.

4. Visualization of 3D figures figures with GeoGebra software

In the current section, we present some examples involving the use of Geogebra software in order to visualize certain numerical, geometric (2D and 3D) properties involving relations derived from the figurate numbers. In figure 2, to the right side, with the aid of Geogebra software, we can visualize the dynamic description of the increasing composition and determination of the 2D square numbers. On the other hand, on the right side, in the same figure, we can visualize the corresponding configuration of the pyramidal numbers based on the square numbers. From the last section, we know the following formula $S_4{}^3(n) = \frac{n(n+1)(2n+1)}{6}$. With the help of the software GeoGebra, we can adjust the mobile selector, in our case, we set the values are determined by $1 \le k \le 15$.

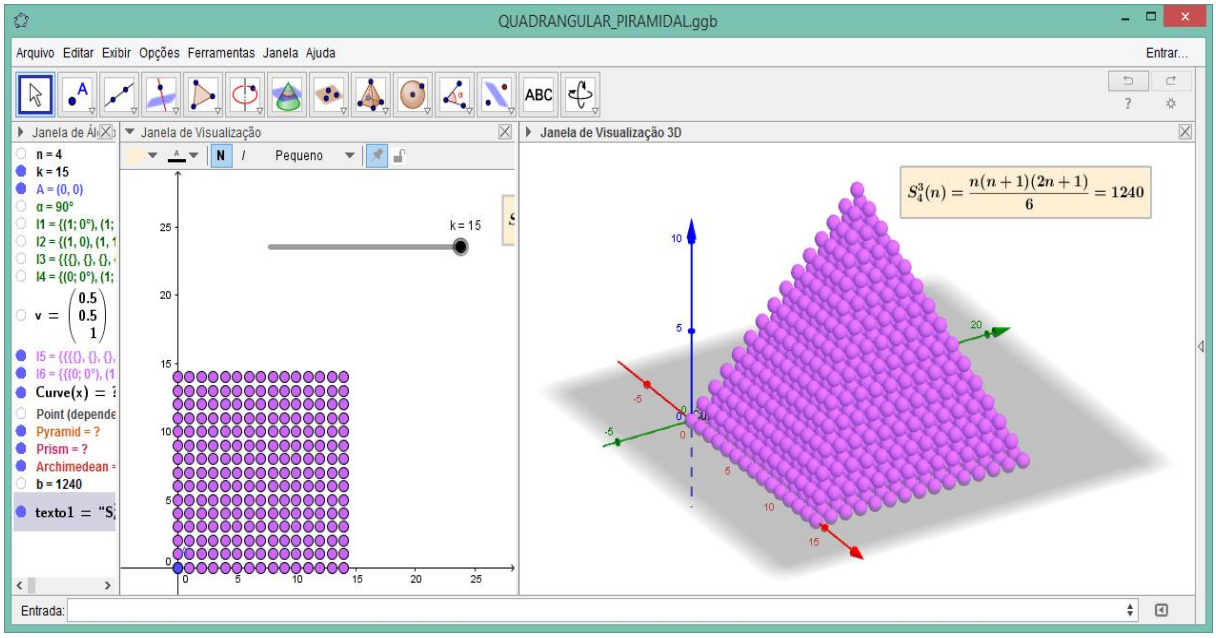

Figure 4. *Visualization of the 2D square numbers and the square pyramidal numbers of square base 3D with the aid in GeoGebra software.* (Source: Authors' elaboration)

On the left side, in figure 4, we can identify the description of the 2D figure numbers and that, when we relate to the 3D figure on the right side, in the same figure, we can understand that they are precisely the base of the 3D pyramid, for different values of $1 \leq k \leq 15$. The dynamic properties of GeoGebra software allow the immediate and precise calculation of algebraic expression $S_4^{\,3}(n) = \frac{n(n+1)(2n+1)}{6} = 1240$, for $n = 15$. In figure 4, we can identify another angle for the 3D visualization and the identification of arithmetic and geometric spatial properties derived from the configuration indicated by a square pyramidal number. The software GeoGebra enables the exploration of various viewing angles and the acquisition of an mathematical understanding of the growth and numeraical properties of the figures 4 and 5.

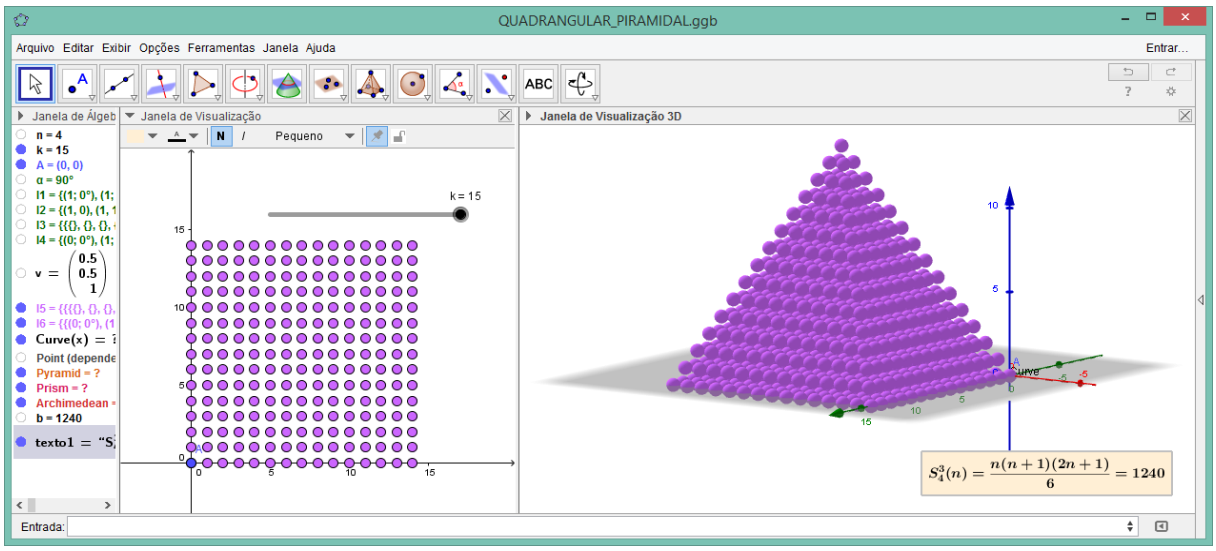

Figure 5. *Visualization of the 2D square numbers and the pyramidal numbers of square base 3D with the aid in GeoGebra software.* (Source: Authors' elaboration)

We know that a pentagonal pyramidal number corresponds to a pentagonal pyramid. Similarly, we know that a hexagonal pyramidal number corresponds to a hexagonal pyramid. In figure 5, on the left side, a dynamic configuration that can grow and decrease, in the dependence of the delimiting control that we defined as follows and values $1 \leq k \leq 15$. We know the formula $S_6^{\,3}(n) = \dfrac{n(n+1)(4n-1)}{6}$ and, the software allows to explore several values and to perceive the corresponding variations for both windows that we find in figures 6 and 7. Different viewing angles can be explored immediately with the use of GeoGebra software.

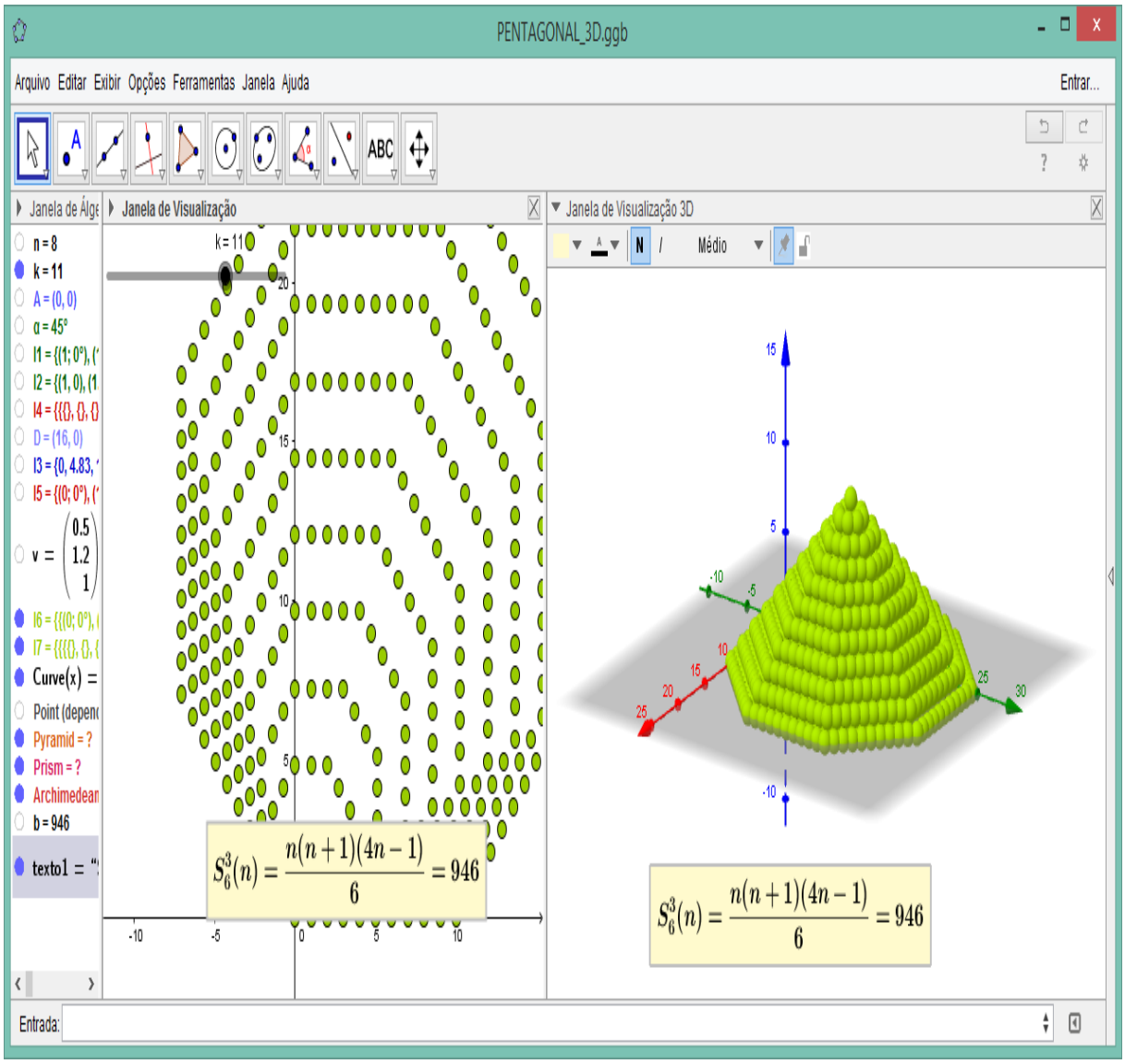

Figure 6. *Visualization of the 2D hexagonal numbers and the Hexagonal pyramidal numbers of Pentagonal base 3D with the aid in GeoGebra software.* (Source: Authors' elaboration)

In the following case, we bring to the reader a learning scenario for the analysis of 2D and 3D behavior corresponding to the heptagonal pyramidal number that corresponds to a heptagonal pyramid. From the previous section, we know that $S_7^{\,3}(n) = \dfrac{n(n+1)(5n-2)}{6}$. In a similar way to the previous cases, from the variations of the selector command, we can verify and predict the behavior of several cases and, above all, examine them in the planar and spatial configuration for this numbers. Let us now see Figures 6 and 7. In Figure 6 we can identify that the software Geogebra allows the

handling and manipulation with a large set of different configuration and dynamic configuration of Heptagonal pyramidal numbers. We can choose a more suitable angle for the visualization!

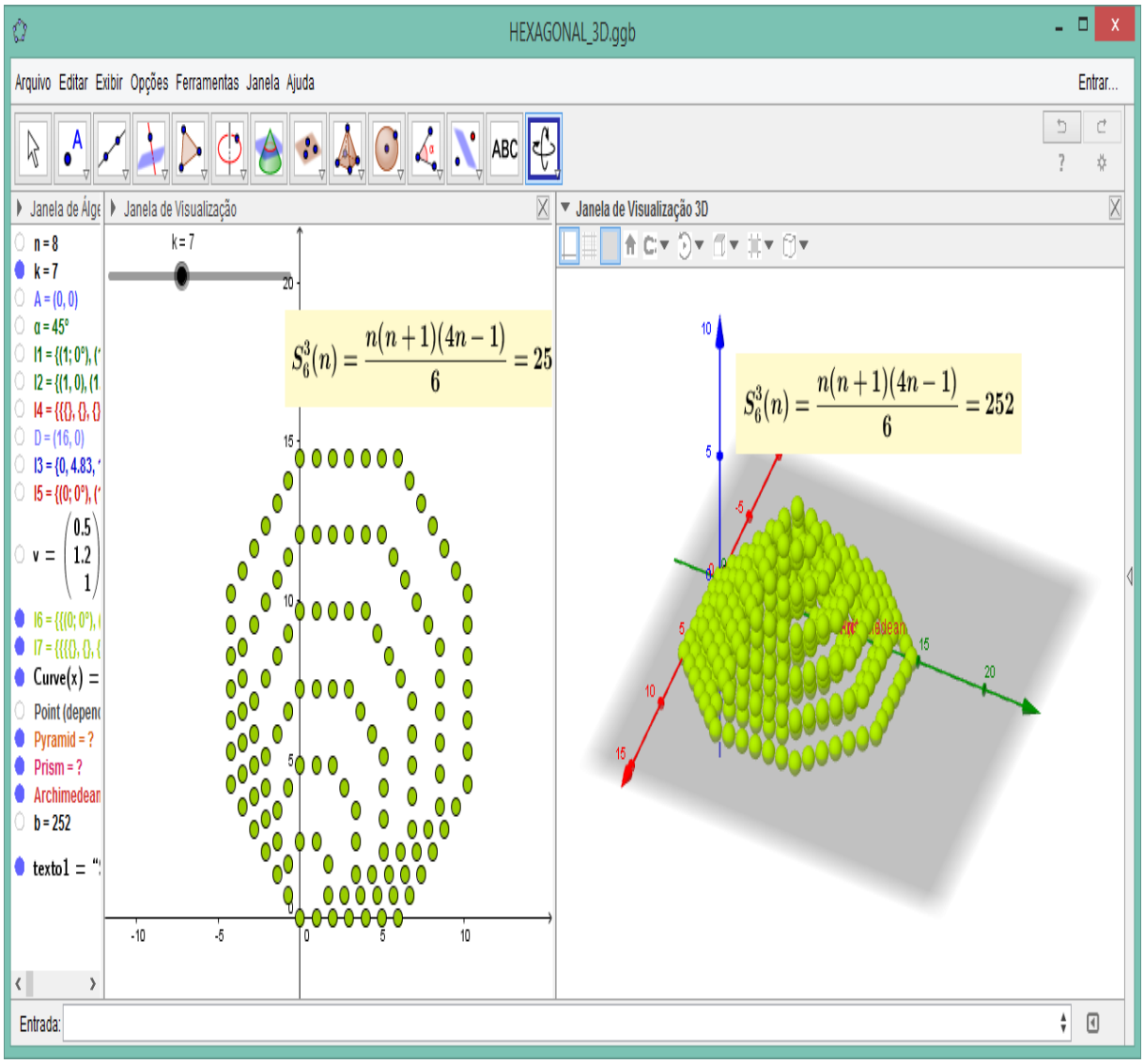

Figure 7. *Visualization of the 2D pentagonal numbers and the pentagonal pyramidal numbers of pentagonal base 3D with the aid in GeoGebra software.* (Source: Authors' elaboration)

For example, in the figure 6, we can easily evaluate the numerical behavior of the following expression $S_7^{\,3}(n) = \frac{n(n+1)(5n-2)}{6} = 7931$ for $k = 15$ and $S_7^{\,3}(n) = \frac{n(n+1)(5n-2)}{6} = 880$ for $k = 10$. In figure 7 we visualize different angles of analysis and manipulation of the 3D construction corresponding to a heptagonal pyramidal numbers 3D. We can conclude that by applying some properties of GeoGebra software we can explore and identify some arithmetic, algebraic and geometric properties, in a non-static way, when compared with the constructions we present in Figure 1. For example, when constructing constructs that have a 2D representation and 3D, we can visualize the corresponding qualitative relations, as we visualized in the previous figures. In addition, as we can verify, with the use of technology we can determine a differentiated didactic or a didactical transposition for the study and mathematical investigation about the 2D and 3D figurate numbers. This ability, more and more, is essential for the mastery of the Mathematics teacher in Brazil.

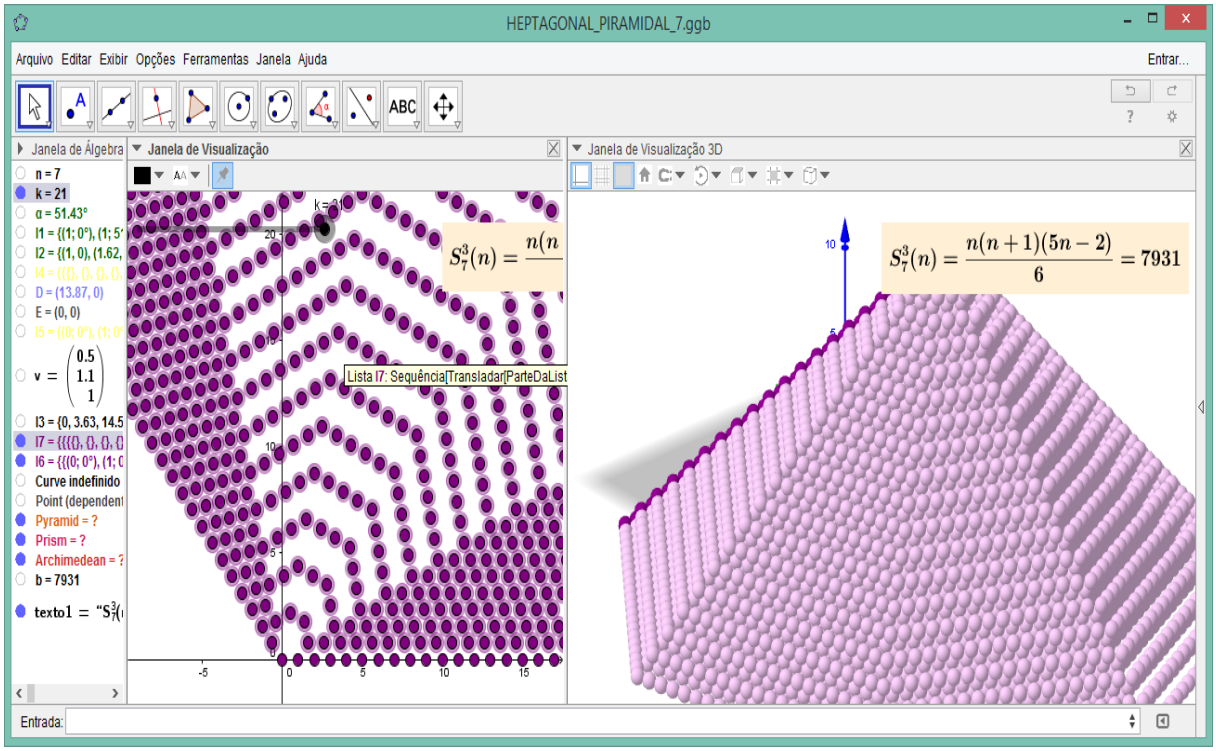

Figure 8. *Visualization of the 2D heptagonal numbers and the heptagonal pyramidal numbers base 3D with the aid in GeoGebra software.* (Source: Authors' elaboration)

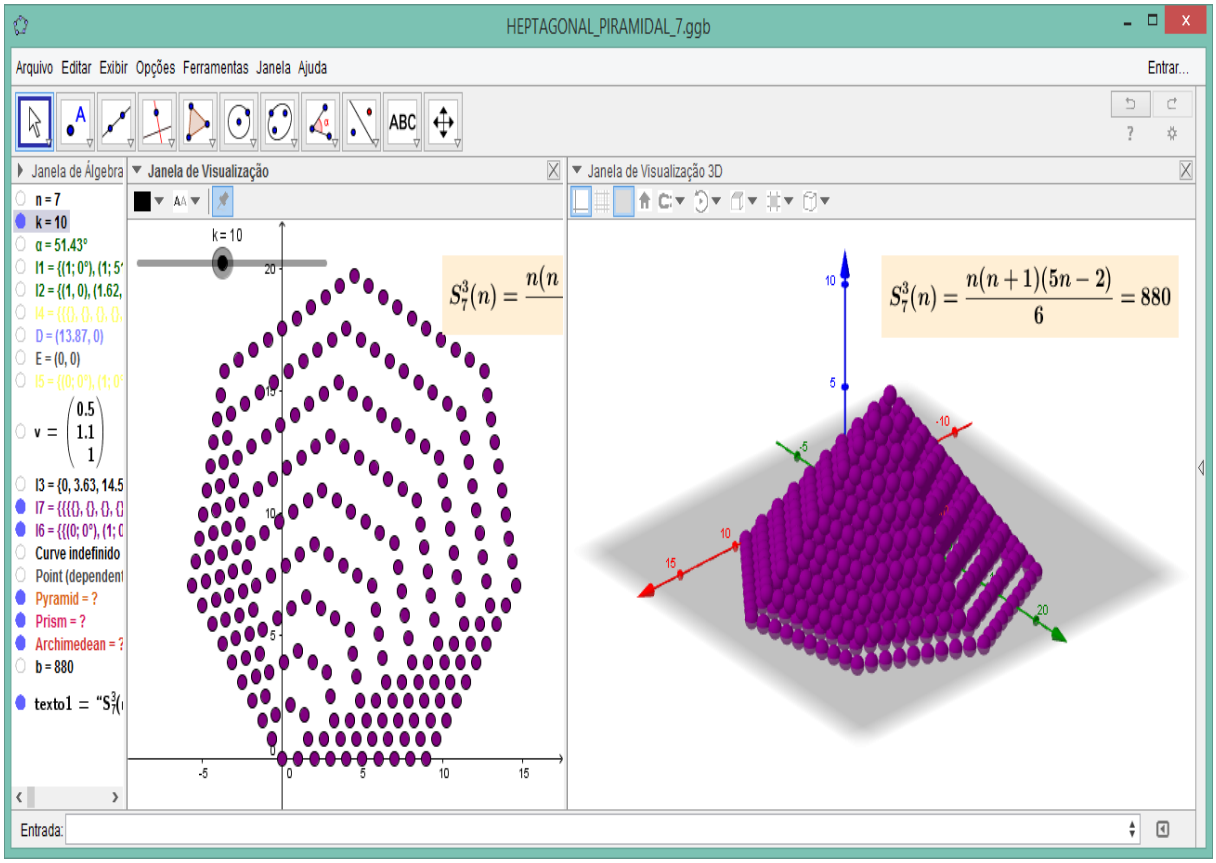

Figure 9. *Visualization of the 2D heptagonal numbers and the heptagonal pyramidal numbers 3D with the aid in GeoGebra software.* (Source: Authors' elaboration)

5. Some reflections on Mathematics Education in Brazil and the use of Technology

In the past sections, we have identified that the technology and, in particular, the GeoGebra software allows the exploration and determination of a learning scenario that presents the mathematical knowledge through the use of perception and visualization, through a dynamic description of arithmetical relations that are related to other 2D and 3D geometric properties. In this sense, we have identified an important component for researches in Mathematics Education in Brazil that usually value visualization as a catalyzing element of mathematical intuition and, consequently, meaningful learning. In the table I below we indicate some information about the 3D figurate numbers and the corresponding algebraic formulas for the sum of the numbers. In the right column, in the same table, we look at the formulas of extension to the field of integers.

Table I. *Comparative table and properties of 3D figure numbers and the process of extension to negative integer indices for the 2D figurate numbers*

Pyramudal numbers	The sum of the first *n* consecutive m gonal numbers	The generalized m-gonal numbers with negative índices $S_m(-n)$
Triangular pyramidal number	$S_3^{\,3}(n)=\frac{n(n+1)(n+2)}{6}$	$-S_3(n)=S_3(-n)=\frac{(n-1)n}{2}$
Square pyramidal number	$S_4^{\,3}(n)=\frac{n(n+1)(2n+1)}{6}$	$-S_4(n)=S_4(-n)=n^2$
Pentagonal pyramidal number	$S_5^{\,3}(n)=\frac{n^2(n+2)}{2}$	$-S_5(n)=S_5(-n)=\frac{3n^2+n}{2}$
Hexagonal pyramidal number	$S_6^{\,3}(n)=\frac{n(n+1)(4n-1)}{6}$	$-S_6(n)=S_6(-n)=2n^2+n$
Heptagonal pyramidal number	$S_7^{\,3}(n)=\frac{n(n+1)(5n-2)}{6}$	$-S_6(n)=S_6(-n)=2n^2+n$
⁝⁝⁝⁝	⁝⁝⁝⁝	⁝⁝⁝⁝

Source: Authors' elaboration

Now, in table II we describe some properties that must be discovered by the teacher of Mathematics and properly explained to the students.

Table II. *Some properties and differences on 2D figurate numbers*

2D Figure	Formula	Difference
Triangular Number	$S_3(n)=\frac{1n^2-(-1)n}{2}$	$S_4(n)-S_3(n)=\frac{n^2-n}{2}$
Square Number	$S_4(n)=\frac{2n^2-(0)n}{2}$	$S_5(n)-S_4(n)=\frac{n^2-n}{2}$
Pentagonal Number	$S_5(n)=\frac{3n^2-(1)n}{2}$	$S_6(n)-S_5(n)=\frac{n^2-n}{2}$
Hexagonal Number	$S_6(n)=\frac{4n^2-(2)n}{2}$	$S_7(n)-S_6(n)=\frac{n^2-n}{2}$
Heptagonal Number	$S_7(n)=\frac{1n^2-(-1)n}{2}$	$S_8(n)-S_7(n)=$
⁝⁝⁝	⁝⁝⁝	⁝⁝⁝

Source: Authors' elaboration

From the data in Table II, students should be encouraged formulating the following conjecture and showing another demonstration for Theorem 1. Here is another example of mathematical research that can be presented to students in Brazil.

Conjecture: Every arithmetic series for two-dimensional figurate numbers can be generated by the formula $\frac{(m-2)n^2-(m-4)n}{2}, m,n \in IN, m \geq 3$.

Proof. By mathematical induction, we assume that $S_m(n)=\frac{(m-2)n^2-(m-4)n}{2}$ and, in the next step, we take the difference $S_{m+1}(n)-S_m(n)=\frac{n^2-n}{2} \therefore S_{m+1}(n)=\left[S_m(n)+\frac{n^2-n}{2}\right]$. In this way, we can write $S_{m+1}(n)=S_m(n)+\frac{n^2-n}{2}=\frac{(m-2)n^2-(m-4)n}{2}+\frac{n^2-n}{2}=\frac{(m-1)n^2-(m-3)n}{2}=$

$$\frac{((m+1)-2)n^2-((m+1)-4)n}{2} \square$$

Let's look at some more examples that can be the object of mathematical research, with the support of technology. Let's look at Cubic numbers and Octahedral numbers. Here we have other categories of figures that acquire a dynamic representation with the use of technology, aiming at the teaching of Mathematics.

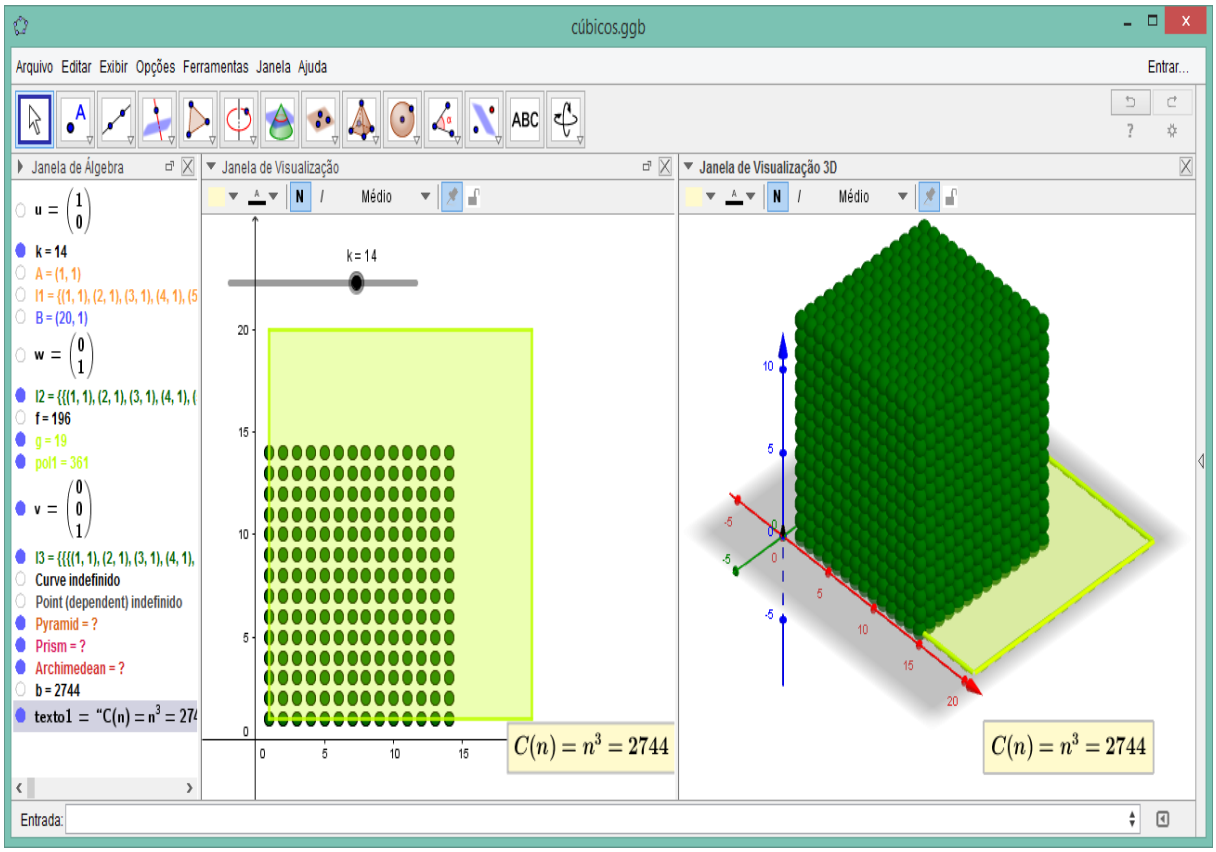

Figure 10. *Visualization of a cubic cubic number (or perfect cube).* (Source: Authors' elaboration)

"A cubic number (or perfect cube) is a space figurate number corresponding to a cube constructed from balls." (Deza & Deza, 2012, p. 99). The *n*-th cubic number $C(n)$ is the sum of n copies of *n*-th

square number $S_4(n)$, and has the form $\mathrm{C}(n) = n^3$. The series of perfect cubes starts as follows: 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . . (Sloane's A000578). In the following section, the authors comment on the interest of Indian mathematicians in the study about the cubic numbers (see figure 10).

> Determination of cube of large numbers was very common in many ancient civilizations. Aryabhatta, the ancient Indian mathematician, in his famous treatise *Aryabhatiya* explains the mathematical meaning of cube as *the continuous product of three equals as also the (rectangular) solid having 12 equal edges are called cube*. Similar definitions can be found in ancient texts such as *Brahmasphuta Siddhanta* (XVIII. 42), *Ganitha Sara Sangraha* (II. 43), and *Siddhanta Sekhara* (XIII. 4). It is interesting that in modern Mathematics too, the term *cube* stands for two mathematical meanings just like in Sanskrit, where the word *Ghhana* means a factor of power with the number, multiplied by itself three times and also a cubical structure. (Deza & Deza, 2012, p. 99).

It should be noted that the Greek perspective is emphasized by Gundlach (1969, 33) when he emphasizes that the Greek in general "glimpsed Mathematics with more than Geometry and Arithmetic. And since its first the Greeks considered the numbers as a whole, and we are not surprised that they endeavored to represent numbers as geometric forms. " In addition, some authors record the repercussions of this integrative vision inaugurated by the Hellenic, in this sense, Aleksandrov (1956, p. 30) stresses that "in the interaction between arithmetic and geometry we can see that the development of mathematics is a conflicting process between several contrasting elements. "

Moreover, we can also recall the study of the Platonic solids which have a corresponding non-trivial representation as 3D figures. In the section below Deza & Deza (2012) explain a little more about the subject.

> Also well-known are octahedral numbers, corresponding to the next, after tetrahedron and cube, Platonic solid, octahedron. They are equal to the sum. two consecutive square pyramidal numbers. Dodecahedral and icosahedral numbers correspond to the last two Platonic solids, dodecahedron and icosahedron. Their construction is more complicate and these numbers rarely appear in special literature. (Deza & Deza, 2012, p. 87).

An octahedral number is a space figurate number that representes an octahedron, or two pyramids placed together, one upsidedown underneath the other. Therefore, *n*-th octahedral number $O(n)$ is the sum of two consecutive square pyramidal numbers: $\mathrm{O}(n) = S_n^3(n-1) + S_n^3(n) = \dfrac{n(2n^2+1)}{3}$. In the figure below, with the help of the software GeoGebra, we observe some examples of an octahedral number. We can extract important information about the numerical and geometric variations when we change the corresponding valves of the mobile selector. The first few octahedral numbers are 1, 6, 19, 44, 85, 146, 231, 344, 489, 670, . . . (Sloane's A005900).

In the figures below, with the help of the software GeoGebra, we observe some examples of an octahedral number. We can extract important information about the numerical and geometric variations when we change the corresponding valves of the mobile selector. For a professor of Mathematics in Brazil, besides the need to master the mathematical model that involves each situation, it is important to understand the potentialities and limitations of GeoGebra software. In the case of our discussion and in the context of the research in Mathematical Education, it is necessary to continue the investigation regarding other categories and type of special figurate numbers and, a non-trivial challenge corresponds to verify the correlate construction in the software. This task presents itself as a challenge to the mathematics teacher in Brazil.

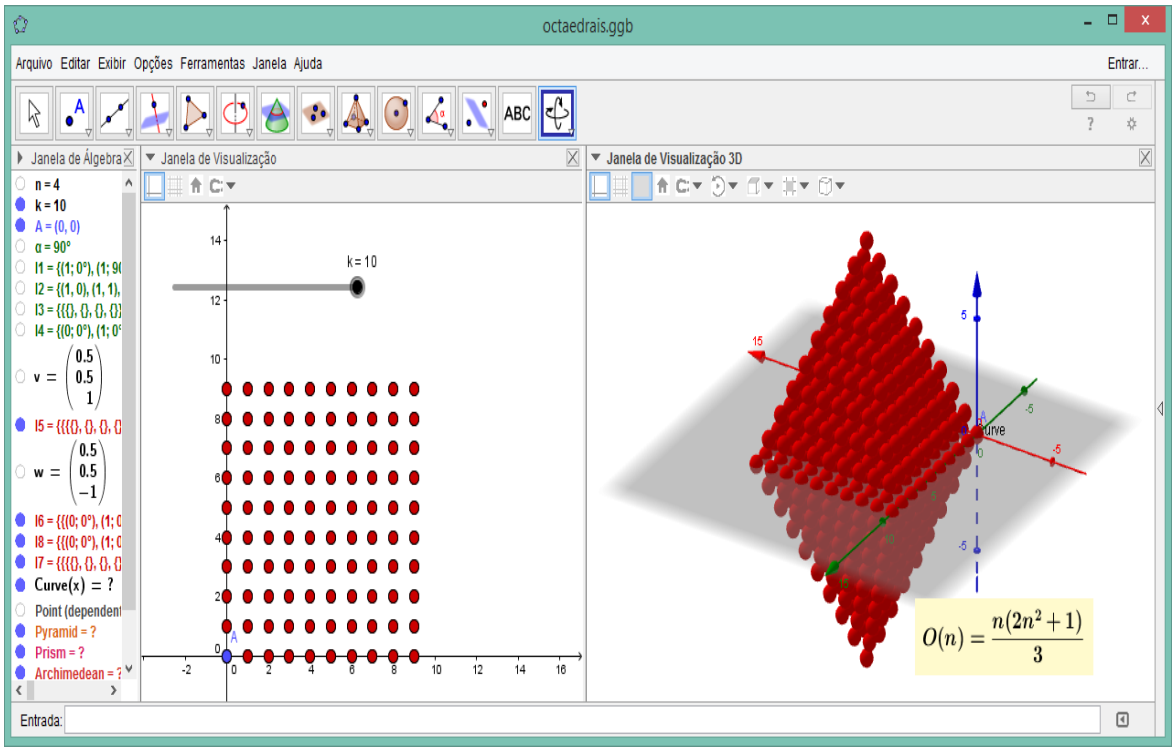

Figure 11. *Dynamic visualization of an octahedral number with the help of GeoGebra software..* (Source: Authors' elaboration)

In figure 12, we visualize an octahedral number.

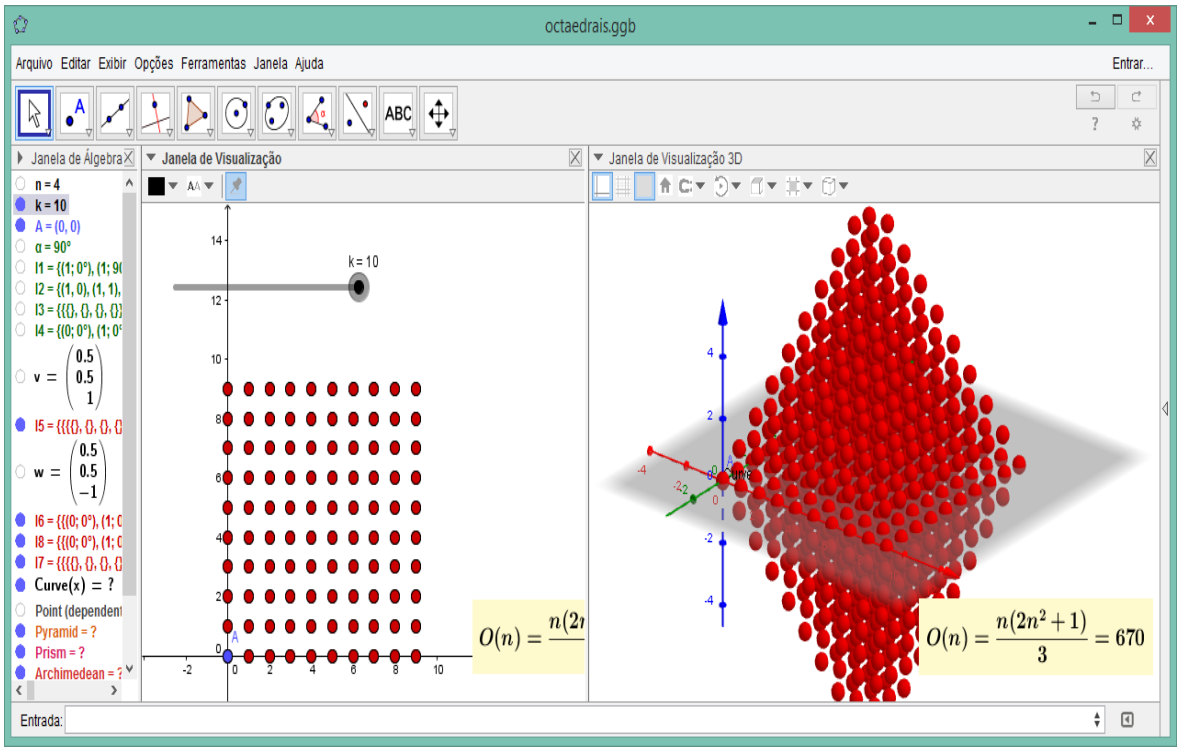

Figure 12. *Dynamic visualization of an octahedral number with the help of GeoGebra software.* (Source: Authors' elaboration)

To conclude this section, we recall a last and interesting property of the figures. In this sense, Deza and Deza (2012, pp. 76-77) define the generalized plane figures are defined as all values of the standard formulas for a given class of the plane figures numbers, taken for any integer value of the

index. According to the authors, we can say that "the generalized polygonal numbers or, specifically, the generalized *m*-gonal numbers, are defined as all values of the formula $S_m(n) = \frac{n((m-2)n-m+4)}{2} = \frac{(m-2)}{2}(n^2-n)+n$ for *n*-th *m*-gonal number, taken for any integer value of *n*". (Deza & Deza, 2012, p. 77). For exemplo, the generalized triangular numbers are $\ldots,10,6,3,1,0,0,1,3,6,10,15,\ldots,$ for $n=0,\pm1,\pm2,\pm3,\ldots.$ They have values 2, 7, 15, 26, 40, ... So, the generalized pentagonal numbers are $\ldots,40,26,15,7,2,0,1,5,12,22,35,\ldots,\ldots,\ldots.$ for $n=0,\pm1,\pm2,\pm3,\ldots.$

3. Conclusion

In the past sections, we have discussed and discussed some mathematical and numerical properties about 2D figurate numbers and 3D figurate numbers (Castillo, 2016). We show. by means of several examples, that visualization constitutes an important component in teaching and, from the point of view of Mathematics Education developed and disseminated in Brazil, the use of GeoGebra software, allows a differentiated script for the teaching of Mathematics, with the possibility of stimulate the mathematical intuition and the perception of the Brazilian students.

From the point of formation of Mathematics teachers in Brazil, it is essential to establish knowledge involving the use and application of technology in order to provide a differentiated didactic transposition for mathematical knowledge. In the previous sections of the paper, we present a form of extension or generalization of the figure numbers, insofar as we can describe them from integer indices (positive or negative) (see table II). In this case, we have another itinerary to investigate the behavior of these conceptual entities, given the potential of dynamic visualization with GeoGebra software.

In recent works (Chahal; Griffin & Nathan, 2018; Gopalan; Somanath & Geetha, 2013a; 2013b), we can verify that the study and specialized research around the other kind of figurate numbers are vigorous and enable the teacher of mathematics to understand the process of generalization of certain notions. We have, for example, the case of the Generalized Trapezoidal Numbers numbers (Berana; Montalbo & Magpantay, 2015, Overtone, 2016, Jitman & Phongthaim, 2017).

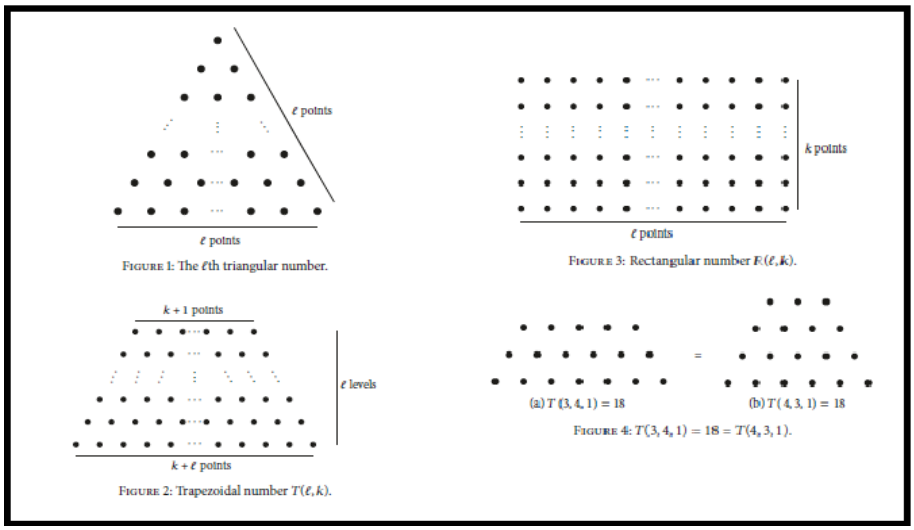

Figure 13. *The Generalized Trapezoidal Numbers numbers studied by* Gamer; Roeder & Watkins (1985)

To conclude, we rely on Choquet's (1963, p. 43) perspective when he warns that "the main goal is to provide our students with some tools and teach them how to apply them"; however, in the teaching of Mathematics the great questioning falls on the problem regarding which "methodology" appropriates, in the period of academic formation, that makes possible the proper implementation and "application" of mathematical concepts, however, as we indicated at the beginning of this article, this

methodological option will be related to the elements acquired from his own experience as a student in the course of his graduation (Laforest & Ross, 2012).

Finally, with the present work, we have the opportunity to disseminate certain ideas and assumptions assumed by Mathematical Education and History of Mathematics research in Brazil (Fauvel & Maanen, 2002), from a special role dedicated to the intuitive process of understanding mathematical scientific concepts, through visualization. In this case, the teaching of mathematics requires the ability of the Mathematics teacher in Brazil and that does not transform the learning process in a tiring moment and supported by the formal and repetitive operations and tasks. Otherwise, visualization, with the use of technology, may enable new scripts and possibilities for teaching (Reuiller, 2008).

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