



## **COMPARING MODEL-BUILDING PROCESS: A MODEL PROSPECTIVE TEACHERS USED IN INTERPRETING STUDENTS' MATHEMATICAL THINKING**

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### **Abstract**

Mathematical thinking is an important aspect of mathematics education and, therefore, also needs to be understood by prospective teachers. Prospective teachers should have the ability to analyze and interpret students' mathematical thinking. Comparing model is one of the interpretation models from Wilson, Lee, and Hollebrands. This article will describe the prospective teacher used the model of the building process in interpretation students' mathematical thinking. Subjects selected by considering them in following the students' strategies in solving the Building Construction Problem. Comparing model is a model of interpretation in which a person interprets student thinking based on student work. There are two types comparing model building process prospective teacher use in interpreting students' mathematical thinking ie. comparing work and comparing knowledge. In comparing works, prospective teachers use an external representation rubric. This is used to analyze student activities in order to provide an interpretation that is comparing the work of students with their own work. In comparing knowledge, prospective teachers use internal representation rubrics to provide interpretation by comparing the students' work with their knowledge or thought.

**Keywords:** Comparing Model, Interpretation, Students' Mathematical Thinking

### **Abstrak**

Pemikiran matematis merupakan aspek penting dalam pendidikan matematika dan, oleh karena itu, juga perlu dipahami oleh calon guru. Calon guru harus memiliki kemampuan untuk menganalisis dan menafsirkan pemikiran matematika siswa. Model Membandingkan adalah salah satu model interpretasi dari Wilson, Lee, dan Hollebrands. Artikel ini akan menjelaskan bagaimana calon guru menggunakan model proses membangun dalam menafsirkan pemikiran matematis siswa. Subjek dipilih dengan mempertimbangkan bagaimana mereka mengikuti strategi siswa dalam menyelesaikan Masalah Konstruksi Bangunan. Model Membandingkan adalah model interpretasi di mana seseorang menginterpretasikan pemikiran siswa berdasarkan pada pekerjaan siswa. Ada dua jenis proses membangun Model Membandingkan yang digunakan calon guru dalam menafsirkan pemikiran matematis siswa yaitu Membandingkan Karya dan Membandingkan Pengetahuan. Dalam Membandingkan Karya, calon guru menggunakan rubrik representasi eksternal. Ini digunakan untuk menganalisis aktivitas siswa dalam rangka memberikan interpretasi yaitu membandingkan pekerjaan siswa dengan pekerjaan mereka sendiri. Dalam Membandingkan Pengetahuan, calon guru menggunakan rubrik representasi internal untuk menyediakan interpretasi yaitu dengan membandingkan pekerjaan siswa dengan pengetahuan atau pemikiran mereka.

**Kata kunci:** Model Membandingkan, Interpretasi, Pemikiran Matematis Siswa

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One of the most fundamental goals of mathematics education is being able to use mathematical thinking in solving problems (Stacey, 2006; Shahrill, *et al.* 2018). Therefore, teachers should support the development of students' mathematical thinking. For this purpose, accessing students' thinking when students solve problems is essential to get information about students' potential and also

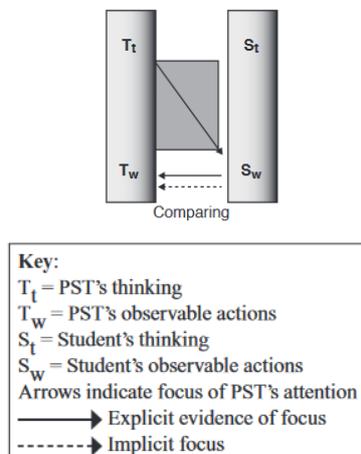
difficulties. Such information is important as a consideration for teachers to choose and prepare appropriate teaching strategies and/or materials. Despite the importance of accessing students' thinking process is not easy because what teachers can directly access are student learning written and oral activities. As mentioned by von Glaserfeld (1995), teachers do not have direct access to students' mathematical thinking. What teachers can do is accessing the evidences of students' mathematical thinking, such as students' works. Student works show students' mathematical activity that might indicate the process of students' thinking. In this respect, teachers can develop hypothesis of students' mathematical understandings by observing students' mathematical activities. Mathematical activities include hypothesizing what students know and understand (Cobb & Steffe, 1983; Steffe & Thompson, 2000). For prospective teachers (PTs), their understanding of students' mathematical thinking is determined much by what they understand or learn about textual theory or knowledge and little about the strategies that students undertake. Understanding students' strategies will lead prospective teachers to understand students' thinking. This relates to how to interpret students' mathematical thinking.

Interpreting student thinking is an important component of high-quality learning and assessment (NCTM, 2014; Teachingworks, 2012; Ahmad, *et al.* 2018). According to Sapti, Purwanto, Mulyati, and Irawan (2016), interpreting students' mathematical thinking is "giving impression, opinion, or a theoretical view towards mathematical information in the form of students' written work in solving problems". A key factor in interpreting students' work is the ability to see essential aspects of students' mathematical thinking (Jacobs, Lamb, & Philipp, 2010; Sherin, Jacobs, & Philipp, 2011; van Es & Sherin, 2002). Paying attention to students' mathematical thinking does not only require attention to children's strategies but also interpretation of mathematical understanding reflected in these strategies. Identifying the extent to which the teacher's evidence shows in interpreting children's understanding is not looking for the best single interpretation but the extent to which participant interpretation is consistent with the details of certain children's strategies and research on children's mathematical development (Jacobs, *et al.* 2010). Teachers can interpret students' mathematical thinking by identifying strategies that students might use in solving a problem. Teachers can also identify why certain problems become difficult and cause problems considering the characteristics of students' thinking (Widodo, Nayazik, & Prahmana, 2019). At the end, students' mathematical thinking can be connected to providing appropriate learning opportunities because different mathematical thinking might require different learning approaches.

Despite the importance of interpreting students' mathematical thinking, a number of studies show prospective teachers' sufficient skill in this respect. Jacobs, Lambs, and Phillip (2010) revealed that prospective teachers and beginner elementary school teachers experience difficulties in identifying and interpreting student strategies. Similarly, Shaughnessy, Boerst, and Ball (2014) and Sleep and Boerst (2012) also found that beginner teachers have difficulties in interpreting students' mathematical thinking and appropriate learning approaches. Due to their lack of teaching experiences, prospective teachers have limited knowledge and experiences about the various strategies of students

in solving the problem. Their understanding of student strategies might heavily rely on their theoretical experiences from lecturers and/or their own prior experiences as students. Prospective teachers will compare students' actions with their own actions, either implicitly or explicitly (Wilson, Lee, & Hollebrands, 2011). Such situation is called comparing model analysis. Based on that, to interpret students' mathematical thinking, prospective teachers analyze by comparing, i.e., equating or differentiating student strategy artifacts with their own work or concept. This allows the emergence of two types comparing model: comparing work and comparing knowledge.

Comparing is one of model-building process in interpreting students' mathematical thinking. This relates to the process of building interpretations of students' thinking based on student learning activities. Wilson, *et al.* (2011) depicted a comparing model in the process of constructing interpretations as two separately parallel boxes (see Figure 1). One box contains prospective teachers' (PTs) thinking (TT) and their written work (TW) and the other box contains a student's thinking (ST) and their written work (SW) (Wilson, *et al.* 2011). The arrows were used to represent a PST focusing on his or her attention. We used solid arrow to indicate explicit evidence of a PST's attention and dotted arrow to indicate an implicit of a PST's attention.



**Figure 1.** Comparing model to illustrate PTs' analysis of students' work (Wilson, *et al.* 2011)

There are two types of PTs's focus of attention, those are explicit and implicit attention. The explicitly attention is what PTs attend to the student's written work or what students might think based on student work. Otherwise, implicit attention is inferred based on the PST's work. To modeling how PTs analyze students' work, we follow Wilson, *et al.* (2011) in read the diagram from the upper-left corner and following the arrows indicating their focus of attention.

Considering the importance of developing prospective teachers' skills in interpreting students' mathematical thinking, the present study intend to examine how prospective teachers use the comparing model in interpreting students' mathematical thinking.

## **METHOD**

This qualitative study was conducted on third year graduate students of Mathematics Education Programs in one of a private university in Purworejo, Indonesia. Purworejo is surrounded by favorite public and private colleges in Central Java, Indonesia.

### ***Participant***

The participants of this study were 23 prospective teachers (PTs) consisting of 18 female and 7 male graduate students. Participants were asked to follow the strategies of students in completing the Building Construction Problem (BCP) as shown in the example of student work. There are four fundamental behaviour in attending to student's strategies i.e. making right written solution, making false written solution, not making solution, not making solution but understand the solution. This study selected purposively participant who make correct written solution and participant who did not make written solution but understand the solution and could attend to at least 2 of 4 examples of students' strategies in solving BCP.

### ***Data Collection***

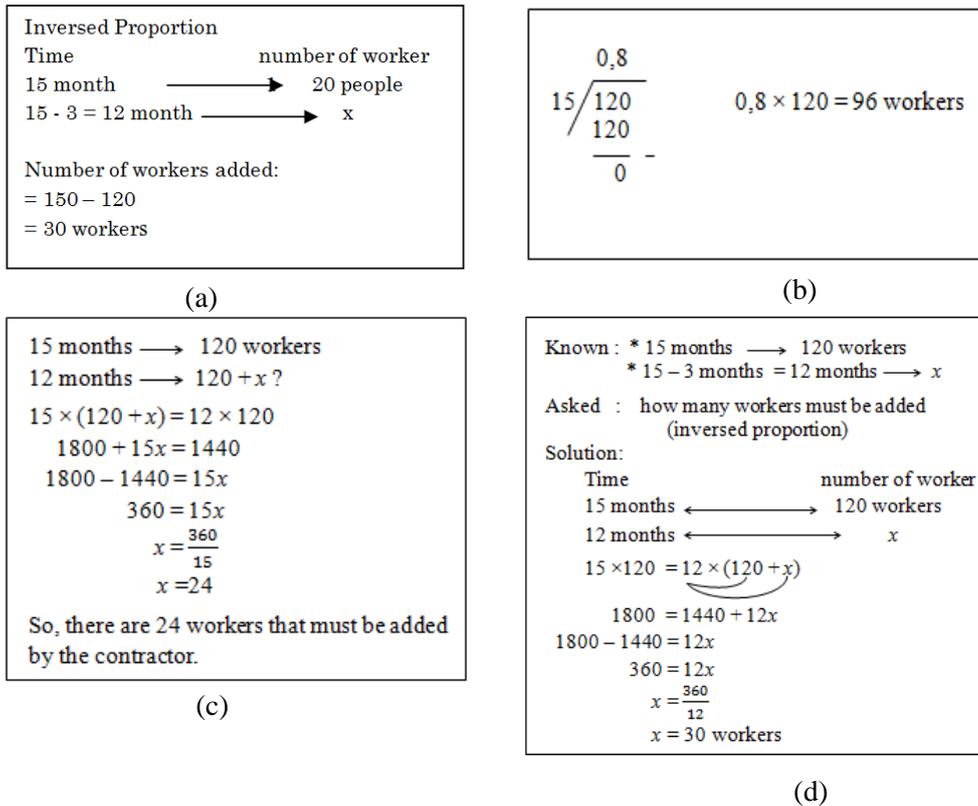
We assigned PTs to complete Task of Interpretation of Students' Mathematical Thinking (ToIoSMT). The ToIoSMT provide Building Construction Problem(BCP) and 4 different examples of students' written work about BCP. The following is a BCP that students have done and examples of students' written work.

Building Construction Problem:

In order to construct a building, the contractor takes 15 months with 120 workers. For a reason, the contractor wants a 3 month accelerated job. If the ability to work for each worker is the same and that the project can be completed on time, how many workers should be added?.

The examples of students' written work in solving BCP represent four characteristics. Student A completed BCP by writing down the method used that was inversed proportion as well as the linkage of information time and many workers. Student A did not write down the steps to get the number 150. He simply wrote that many of the added workers were  $150 - 120 = 30$  workers, whereas, student D wrote in detail the calculation of inversed proportion. Students B and C both completed the BCP by using direct proportion. Student B divided 12 by 15 and multiplies the result by 120. He multiplied the time ratio by the number of workers, while student C writes in detail the steps of BCP problem solving: known information, completion plans, direct proportion calculations, and conclusions.

We provided these four characteristics of students' written work and presented rewrite of examples of students' written work in solving BCP in the Figure 2.



**Figure 2.** The rewrite of examples of (a) Student A, (b) Student B, (c) Student C, and (d) Student D written works [translated and rewritten due to low resolution]

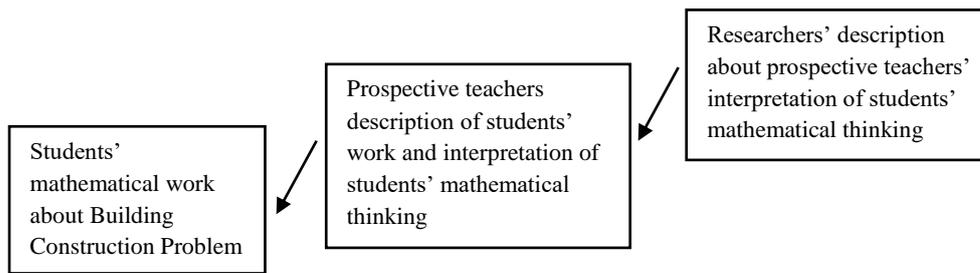
In The ToIoSMT, PTs assigned to explain what they understand about students' mathematical thinking based on student work. PTs asked to think out loud when completing the task. Furthermore, an interview was conducted to corroborate the PTs' think out loud and written work in interpreting the students' mathematical thinking. The artifact of PTs explanation of students' mathematical thinking named as PTs interpretation.

**Data Analysis**

Qualitative analysis was used to generate the description of the building process, model of PTs interpretation of students' mathematical thinking based on their written work and think aloud. We conducted data analysis following three stages of qualitative data analysis activities from Miles & Huberman (1999) and six stages of analysis and interpretation of qualitative data from Creswell (2012). This study used triangulation methods by examining the data with different methods: task; think out loud, and interview. Compared PTs written work, think out loud transcript, and interview transcripts resulting data consistency.

By focusing on important themes such as whether PTs make scribbles outside work, convey theories related to problems, this study provide description of how PTs interpret students' mathematical thinking, and characteristics of their interpretations. We used third-order models from Wilson, *et al.* (2011) to describe our description of PTs' interpretation of students' mathematical thinking (see Figure 3). This is similar to the point of view of Simon & Tzur (1999) in which they

explain “the teacher’s perspective from the researchers’ perspectives” (p. 254).

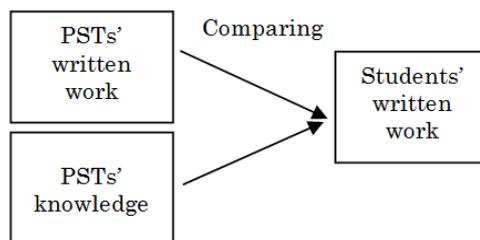


**Figure 3.** A representation of the process of researchers characterising PTs interpretation of students’ mathematical thinking

**RESULT AND DISCUSSION**

Looking at how PTs interpret student thinking is done by looking at how they see student work data and how they obtain interpretations. Of the 23 participants, nine participants show the characteristics of comparing models include: three participants doing comparing work, and six participants doing comparing knowledge. This study describes two participant for each comparing model.

These findings trigger the importance to study prospective teachers’ skills in interpreting students’ mathematical thinking. Consideration of what type of reasoning is involved when prospective teachers are asked to analyze student work. This study tends to describe comparing model as building process model prospective teachers used in interpreting student’s mathematical thinking. Comparing model is one of process that PTs used to construct models of students’ thinking(Wilson, *et al.* 2011). In comparing model, PTs compared students’ activity with their own activity, either implicitly or explicitly. The model-building processes in this category include how the PT searched for similarities and differences between theirs and those that the PTs noticed in the students’ work. Comparing between students’ observable action and PTs observable action can lead us to two different characteristic of interpretation. First, comparing work is a characteristic when PTs compare students’s action about the task with their own action. Their own action is their written work about the same task that student’s done. Second, comparing knowledge is the characteristics when PTs compare students’ action with his/her knowledge or theories about given problem. This situation is represented in Figure 4.



**Figure 4.** Type of Comparing

### ***Comparing Work Model***

The subjects in comparing work group are Ar and Sal. This group begin completing ToIoSMT by reading the BCP and attending to student's strategies. Ar and Sal showed evidence that they used comparing work model in interpreting students' mathematical thinking. Both of them completed the BCP to be compared with the students' written work artifacts. Ar completed the BCP first before attending to the student's work, while Sal completed the BCP after following the student's work. Although Sal solved BCP problems after attending to student work, both used their work explicitly to determine what was true among the four students i.e. belong to A and D.

Ar used his work to determine which of the four students' written work are correct. According to him, the number of workers to be added is 30 people and there are two correct answers. This is evident from the following aloud quotes.

*If according to my work, e... additional workers will be 30 people. But here are different. Two students answered correctly.*

He said "if according to my work", it shows that he compared the student's work with his own work. At the time of interpreting the mathematical thinking of student A, Ar showed evidence that by completing the BCP he could presume students' unwritten thoughts. Ar's interpretation of student A's mathematical thinking is that student A understands what is known and what to look for eventhough the student does not perform a detailed calculation to obtain 150. Although the calculation process for obtaining 150 was not written, Ar implicitly assumes that students use inversed proportion. It was apparent from what Ar submitted in the passage.

*... If I modeled the A student's work, A already knows that it used the concept of inversed proportion. .... Na, to get 150, he did not write that. It's quite confusing, how he gets 150. That possibility is 150 from ...could be... it's reversed. It can be reversed or 15 multiplied by 10. Possible as it is.*

This is reinforced by the explanation when asked how she got the number 10.

*... Students guessed that from the existing number 12, 120, and 15 (refers to student A's work on the part of the relationship between time and many workers), the number 150 is likely to be obtained from 15 multiplied by 10.*

In the beginning, Sal attended to the students' written work according to what the students wrote without giving an argument. Next Sal used gestures while counting. It looks from the following Sal's behavior and think out loud.

*Student's thinking ... it should be ... (scrutinize and mumble, move hands doing calculations) then 15 per 12 equals x per 120.*

Sal by herself also completed the BCP at the time of interpreting students' mathematical thinking. He did it when completing the second task. Sal used her work to convince herself about the details of student's strategies and mathematical thinking in completing the BCP. Sal saw that the work or the completion of each student is different. Further, Sal used her work to see which students are already understands and which ones are not. This is evident from the following interview excerpt

between researcher (R) and subject.

- R : *Further you assure which one is right which one is wrong with?*  
 Sal : *with ... completing the BCP by myself.*  
 R : *That's means you look back to .. (Sal reply: student work) heeh... then completing the BCP, then?*  
 Sal : *I conclude which one ... students who already understand, which ones have not.*

This group uses the accuracy of observation as the basis of belief and uses evidence to infer student interpretation (Swartz, 2012). Accurate observation starts from detailing the student's strategy to completion. At the time of writing (following the work of students), the prospective teacher checks whether there are still forgotten or left behind. For example, regarding the operating errors made by student C, they only discovered after reading several times. The characteristics of the subjects in comparing work group were completed the BCP both at the beginning and end of the interpretation process; checked out the right or wrong work of the student based on his/her work; recognized student strategies those match to their strategy. Their interpretations of students' mathematical thinking characterized by : 1) pay attention to the steps used to solve problems in the form of steps to solve the word problems by looking at whether students write the given information, what is asked, detailed steps of completion, as well as withdrawal / writing conclusions, 2) give an assumptions about strategies undertaken by students both written and unwritten, 3) pay attention to the concepts understood or not understood by students in this concept of inversed ratio; and 4) pay attention to the use of variables in completion.

### ***Comparing Knowledge Model***

Hap dan Hen analysed student work by comparing student work with their own knowledge while think out loud. They did not complete the BCP to be compared with the students' written work artifacts but compared it with their knowledge. Hap compared the student's work with her knowledge to interpret the students' mathematical thinking in completing the BCP.

Hap starts by reading the questions and proceed with examining the student's work. For example, the Figure 5 is rewriting of A's written work in completing BCP.

Inversed Proportion	
Time	number of worker
15 month	20 people
$15 \cdot 3 = 12$ month	x
Number of workers added:	
$= 150 - 120$	
$= 30$ workers	

**Figure 5.** Rewriting of A's written work

Student A did not write an operation that shows an inversed proportion, A only wrote "inversed proportion". When attended to student A's work, Hap first thinks about the solution. Hap considered her knowledge of inversed proportion, i.e. comparison between variables, i.e. the time ratio is equal to

the number of people. Using multiplication will give the number of people needed. While the number of workers added is obtained by reducing the number of workers should be with many existing workers. This is apparent when Hap looks at the work of Student A on the following think out loud excerpt.

*... Students use the concept:  $Ta : Tb = na : nb$  ( $T$  denotes time,  $n$  states the number of workers,  $a$  and  $b$  state the first and second events). Later it will produce: the number of worker  $b$  (the second event) is equal to the number of worker  $a$  (first event) multiplied by  $Tb$  and divided by  $Ta$ . So the number of workers to be added is obtained by reducing the number of workers in the first time with the number of workers in the second time. That will result in the number of workers that must be added.*

Hap knew the right answer not by doing in writing but rather using his thinking (knowledge). She did it to match students' work with their thinking. In this way Hap becomes aware that the correct answer belongs to student A and student D. This is evident from the following interview excerpt.

R : *Are you scratching or not?*

Hap : *No.*

R : *why?*

Hap : *Because, from the student's work I'm direct ... from this work I was told to give a comment, I look at it from this answer. And I match with my thinking. Yes this is with ... such fondness ... and I get the correct answer between this one (A) and this (D). The point is these ones are true.*

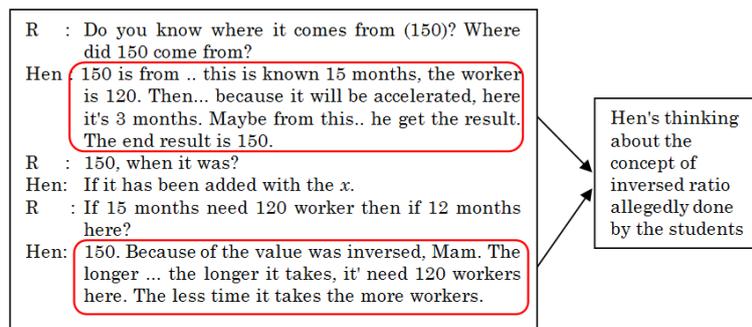
Hen also showed evidence that he was doing analysis with comparing knowledge. When attended to the students' work, Hen did not know the students' answer right or wrong yet. He analyzed by considering the formula that should be used. It was apparent when Hen looks at the student's A work on the following think aloud.

*The ratio used ... In the ratio formula there are two direct ratio and inversed proportions. The proportion used (while looking at student work) on the problems presented is a inversed proportion.*

In the interview, Hen explained her knowledge of the theory of inversed proportion. In inversed proportion, the formula used is  $A1 : B2$  equal to  $A2 : B1$ .

*Since it is presented that the ratio is reversed then the student should use a formula ... for example, using the ratio  $A1 : B2$  equals  $A2 : B1$  so. Na, this work is appropriate.*

Hen's interpretation of students' mathematical thinking A is that A's students have understood and can determine inversed ratio in BCP. The underlying reason for Hen's assumption that the student understands the inversed ratio is also apparent in the interview about how student A earns 150. According to Hen, students do a inversed ratio considering the longer the time of completion of work the fewer workers. And if work is accelerated then more workers are needed. The Figure 6 is Hen's thinking about the alleged operation A has performed.



**Figure 6.** Hen's thinking about the concept of inversed ratio allegedly done by the students

The Comparing Knowledge group also uses accurate observations of the details of student work Swartz (2012) and beliefs about concepts that are understood to ensure their understanding. They based their beliefs on their details about the students' work or strategies they attended. This belief is based on the consideration that during the process of observing, they have considered their thoughts on the concept of proportion. Therefore, they no longer check the work, the order of work, or the concept used.

The characteristics of the subjects in comparing knowledge group are: not make an written solution of the problem, only use their knowledge or thinking about the solution and concept of inversed proportion, checked out the right or wrong of the students' work based on their unwritten solution, and recognized student strategies that match with their strategy. Their interpretation of students' mathematical thinking characterized by : 1) paying attention to the operations performed or suspected of the students and the use of the property of the counting operation on the students' work, 2) paying attention to the concept used is inversed ratio explicitly or implicitly as well as students' mistake, and 3) assessing what students have or have not understood.

### ***The Use of Representation in Interpreting Students' Mathematical Thinking***

The interpretation of the characters of different answers is also different. Student A's answer is incomplete, where the students only write what is known, what is asked, and the concept that will be used, namely the inversed proportion, but do not write the operating process. With this type of answer, comparing work and comparing knowledge group makes guesses about the operations performed by students. This is related to the use of the rubric in giving an assessment (interpretation). Prospective teachers need tools to interpret students' mathematical thinking on standard or evaluation criteria. The Comparing group has a rubric that is a set of coherent criteria for student work that includes a description of the level of quality of performance on these criteria (Brookhart, 2013). Both groups use different tools / rubrics to analyze students' work, namely comparing the work of students with their own work or with their thoughts. Comparing work groups use external rubrics to analyze students' mathematical thinking, where they use their work as a tool to assess or interpret. Comparing knowledge groups use their thinking as an internal rubric, where rubrics are not expressed in external representations. Prospective teachers have obtained an idea of how the problem will be resolved, what

concepts are involved in it, and how he will provide an external or internal assessment.

Doodles about problem solving or thought of prospective teachers are basically external representations and internal representations of the rubric or the criteria they use to assess. External representations are representations that can be easily communicated to others, while internal representations are images created in mind for mathematical objects and processes (Cuoco, 2001; Hendroanto, *et al.* 2018). Comparing work groups use written solutions as a basis for analyzing or evaluating students' mathematical thinking. This is an external representation of mathematical concepts, acting as a stimulus for the senses (Janvier, Girardon, & Morand, 1993) that helps prospective teachers understand the concepts students use. The group of comparing knowledge uses their thinking as an internal representation of the rubric in solving problems. They describe mathematical ideas about inversed proportion in their thinking (Cuoco, 2001) so that they can be used to interpret students' mathematical thinking.

Representation of a problem is influenced by their past experience (Mason, 2011) where they can usually remember the same situation from their own experience. In contrast to more experienced teachers who can rely on their own teaching experience to understand students' thinking (Zhu, Yu, & Cai, 2018), prospective teachers rely solely on their knowledge and learning experience. Student B's answers immediately use reasoning without writing down what information is known and asked. Student B uses comparison reasoning worth  $12/15 \times 120 = 96$ . Students' answers end at 96 results without writing conclusions. For the work of student B, all subjects misinterpreted the students' strategies. They suspect that students only carry out operations on known information, namely  $120/15 \times 120 = 96$ . Even though they were aware of errors or irregularities that 120 divided by 15 were 8 instead of 0.8 but they were not aware of the actual operation performed by student B. Prospective teachers did not recognize the opacity of the students when they used unfamiliar algorithms. They experience challenges in interpreting student understanding or identifying key components of understanding that require attention in line with (Sleep & Boerst, 2012).

Students C and D solve the problems by writing completion steps in detail. The difference is, student C completes using a direct proportion while student D uses inversed proportion. For the work of students C and D, prospective teachers pay attention to the writing of operations in detail including student errors in using a direct proportion (student C) and the use of variable  $x$  which is used for  $t$  meanings (student D). Beyond that, attention is more to the stages of problem solving in the form of story problems such as writing down information that is known, what is asked, the process of completion, and writing conclusions. Interpretation of prospective teachers towards students' mathematical understanding tends to focus on the sequence of operations performed and the problem solving process. Prospective teachers use descriptive explanation to explain the mathematical procedure and the steps of solving problems such as delivered by Murtafiah, Sa'dijah, Candra, & As'ari (2018). They believe that if students solve problems by carrying out problem solving steps they understand the problem. The problem solving step in question refers to the problem solving phase

(Polya, 1945). The results of this study indicate that prospective teachers have similar interpretative characteristics. Interpreting of students' mathematical thinking are key teaching tasks in which teachers must generate hypotheses about how students' mathematical thinking could be developed (Fernández, Llinares, & Valls, 2012; Norton, McCloskey, & Hudson, 2011).

## CONCLUSION

The goal of this study was to gain the description of prospective teacher used model of building process in interpretation students' mathematical thinking. As suggested by literature review, the existing literature in interpreting students' mathematical thinking is relatively sparse. In trying to better understand the model prospective teacher use in interpreting students' mathematical thinking, we find that comparing model-building process is still applicable today. This study has attempted to distinguish detailed comparison models based on representations that are used as a basis for prospective teachers to provide interpretations of students' mathematical thinking.

We find that comparing model-building process in interpreting students' mathematical thinking can be distinguished as two type ie. comparing work and comparing knowledge. The distinction between these two types of comparing model is based on the prospective teacher's implicit or explicit attention to the student's work and the analysis performed. Comparing work does an analysis by considering the external representation rubric by comparing the work of students with the results of their own work in solving the same problem. Whereas Comparing knowledge analyzes by considering the internal representation rubric by comparing students' work with the knowledge they have about the problem.

Prospective teachers use the rubric as a guideline to determine their interpretation of students' mathematical thinking. The rubric is different for both groups of prospective teachers. Comparing work groups use the external representation rubric while the comparing knowledge group uses internal repetition rubrics in the analysis to obtain interpretations of students' mathematical thinking. In the comparing work, prospective teachers completed BCP in written. While in the comparing knowledge, prospective teachers did not. They used their knowledge to analyze students' strategies or students' mathematical thinking but didn't express it in writing. The characteristics of the interpretation of the two groups are relatively same, which emphasizes the concern about the operations performed or allegedly done by the students, students' mistakes, and the assessment of students' understanding. Representation is the basis for determining the model of interpretation of prospective teachers. External representations are used by the compare works group, while internal representations are used by groups of comparing knowledge. The evidence of interpretations of prospective teachers can be used to assess the progress of mathematical understanding and harmonize learning in teacher education programs in a sustainable manner by supporting and developing effective learning.

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