Conical Pendulum Part 3: Further Analysis with Calculated Results of the Period, Forces, Apex Angle, Pendulum Speed and Rotational Angular Momentum

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Abstract
The conical pendulum provides a rich source of theoretical and computational analysis and the present work presents a seamless continuation of the previous publication. The tension force F_T and centripetal force F_C are explored further in linearization analyses and the appropriate slopes are explained. A similar analysis is applied to the period as a visual confirmation of the expected linearity and the slope is explained. The apex angle is considered as a function of both the conical pendulum period and angular speed. An analysis is presented of the conditions required for the constancy of the length of the adjacent side of the triangle defining the conical pendulum, which gives rise to an apparently counter-intuitive result and this is explained in detail. The speed of the rotating conical pendulum mass is calculated as a function of the radius and also as a function of the apex angle. The rotational speed is then used in order to calculate the rotational angular momentum. In order to maintain continuity of calculations from the previous work, the length range of the conical pendulum was maintained to be 0.435 m $\leq L \leq$ 2.130 m, the local acceleration due to gravity g = 9.789 ms^{-2} and with a mass m = 0.1111 kg. This required the same limits for the string tension of approximately $mg \leq F_T \leq$ 12 N, the calculations therefore covered an apex angular range from zero (string hanging vertically down) up to 85°.
Keywords: Conical pendulum, theoretical analysis, tension force, centripetal force, period, angular frequency, rotational speed, angular momentum, high-precision, computational analysis.

INTRODUCTION

The conical pendulum has been extensively studied both experimentally and theoretically, with varying degrees of precision, accuracy, mathematical rigor, approximation and correctness. As an example, for the situation where the orbital radius R is considered to be much smaller than the length L of the conical pendulum string ($R << L$), the orbital rotational period T was equated to the oscillation period of a standard simple pendulum (Richards, 1956). This approximation is surprisingly good, for example with L = 1.000 m, R = 0.030 m and g = 9.789 ms^{-2} (the value used throughout this paper), the simple pendulum equation gives an oscillation period of 2.008 s (to 4 significant figures). The correct equation (derived in Dean & Mathew, 2017 and also Dean, 2017) gives a conical pendulum period of 2.00777 s ≈ 2.008 s (again to 4 significant figures). An electric motor-driven conical pendulum has been used to determine a value for the

acceleration due to gravity and was claimed to be repeatedly capable of yielding results with a quoted precision of approximately 1% (Klostergaard, 1976).

When coupled to a computer-controlled data-acquisition system, (Moses and Adolphi, 1998), experimental results can be obtained by students, which are very reproducible and have a short acquisition time, thereby allowing time for further student exploration. An experiment has been reported (Dupré and Janssen, 1999), which is suitable for undergraduate students to perform and has a stated accuracy of approximately 0.1%, using a rotating rod and an appropriate data measurement system. The conical pendulum has been used as an explanatory example of the dynamics of circular motion (Czudková and Musilová, 2000), in order to highlight and answer specific test questions relating to the mathematical and conceptual understanding of circular motion. The fundamental physics laws of conservation of angular momentum and mechanical energy have been very successfully demonstrated by using a variable length conical pendulum and an experimentally unsophisticated apparatus, with a simple timing system (Bambill, Benoto and Garda, 2004). The experiment was stated to produce results that were good enough to illustrate the conservation laws and the experimental setup was a simple one to use.

An imaginative and successful conical pendulum experiment has been reported, based on the circular motion of a tethered model aero-plane (Mazza, Metcalf, Cinson and Lynch, 2007). The experiment utilized a digital video camcorder and video pointer as well as the usual mechanical apparatus. The overall assessment of the success of this imaginative experiment, suggested that the results obtained were satisfactory, with an accuracy stated to be close to 2%. An electric motor-driven, small length (0.199 m) conical pendulum was investigated in detail with specific reference to obtaining a value for the acceleration due to gravity. The average published value of g was 9.97 ms^{-2} which is in reasonable agreement with the accepted value.

Classical mechanics is applied to the detailed analysis of the conical pendulum (this being one example among several that were considered). Dimensional analysis is used to facilitate the derivation of elliptical precession and a suitable set of equations is derived (Deakin, 2012). A complex three-dimensional analysis of the Foucault pendulum, with reference to the conical pendulum, performed in a rotating frame of reference (Barenboim and Oteo, 2013) provides a series of trajectory equations for the motional path. The theory of dynamic interactions of some circular systems is investigated in detail (Lacunza, 2015), leading to the motion trajectory for the mass of a conical pendulum, by using D' Alembert's principle for classical mechanics.

Regarding the present paper, in order to maintain complete continuity with previous work, this extension of the conical pendulum analysis is based on the same figure (shown below) that was used in the first two papers that form part of this multi-part series, (Dean & Mathew, 2017 and Dean, 2017). It is noted that the conical pendulum mass (which is considered to be a point-mass) is maintained at m = 0.1111 kg, all theoretical calculations were performed using the local value of the acceleration due to gravity g = 9.789 ms^{-2} (Ali, M.Y.et al., 2014). The conical pendulum string was considered to be essentially of negligible mass (compared to the conical pendulum mass) and also physically inextensible.

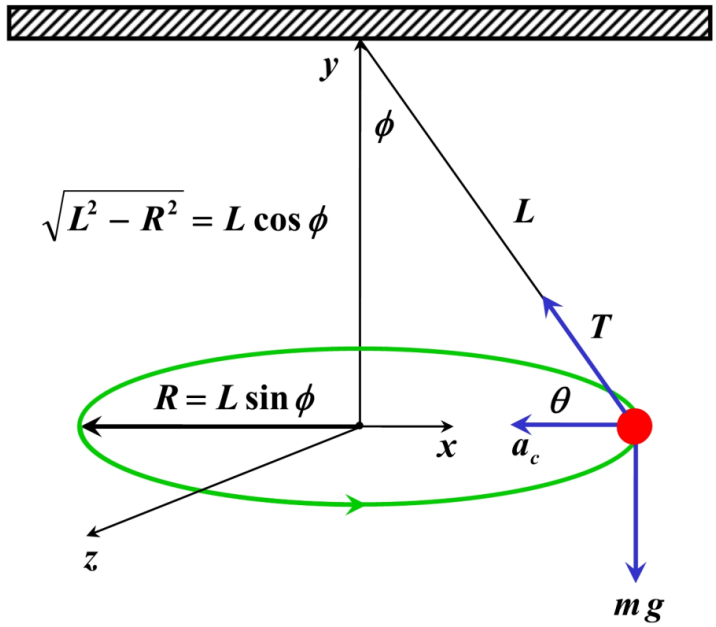

Figure 1. *Conical Pendulum Schematic*

To identify the lines that have been calculated and then plotted on the Excel charts, the reader is referred to the Appendices that contain the chart legend, (except for Chart 3).

THEORETICAL RESULTS AND ANALYSIS

The analysis that follows in this section continues directly from the immediately preceding publication relating to the conical pendulum (Dean, 2017). Inspection of the equation below enables a reasonably good approximation to be obtained in the limit of ($R << L$), such that the conical pendulum orbital period can be easily calculated from the oscillating period equation of a simple pendulum. It is noted that this approximation is surprisingly good (up to 4 significant figures), when ($R \leq 0.03L$) i.e. for when R is close to 3% of the conical pendulum length L.

$$T = 2\pi\sqrt{\frac{L\cos\phi}{g}} = 2\pi\sqrt{\frac{\sqrt{L^2 - R^2}}{g}} = 2\pi\sqrt{\frac{L}{g}}\sqrt{1 - \frac{R^2}{L^2}} \approx 2\pi\sqrt{\frac{L}{g}} \qquad (1)$$

The tension force F_T has been shown (Dean, 2017) to be given by the equation below, which can be readily plotted as a straight-line Chart, such that the slope has a value of (mg) providing an experimental value for the local acceleration due to gravity.

$$F_T = \frac{m\,g}{\cos\phi} = m\,L\,\omega^2 \qquad (2)$$

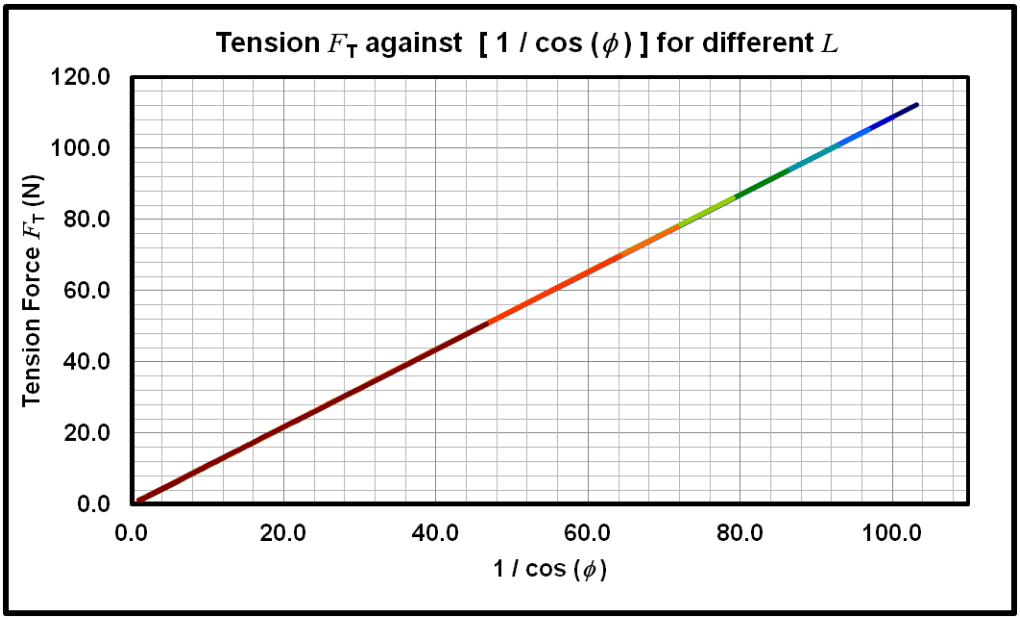

Chart 1. *F_T against 1/cos (ϕ)*

The centripetal force F_C given by the equation below, can also be plotted as a straight line as shown in Chart 2, where the slope again has the value of (*mg*), thus providing an alternative experimental value for the gravitational acceleration.

$$F_C = m\,g\,\tan\phi = m\,R\,\omega^2 \tag{3}$$

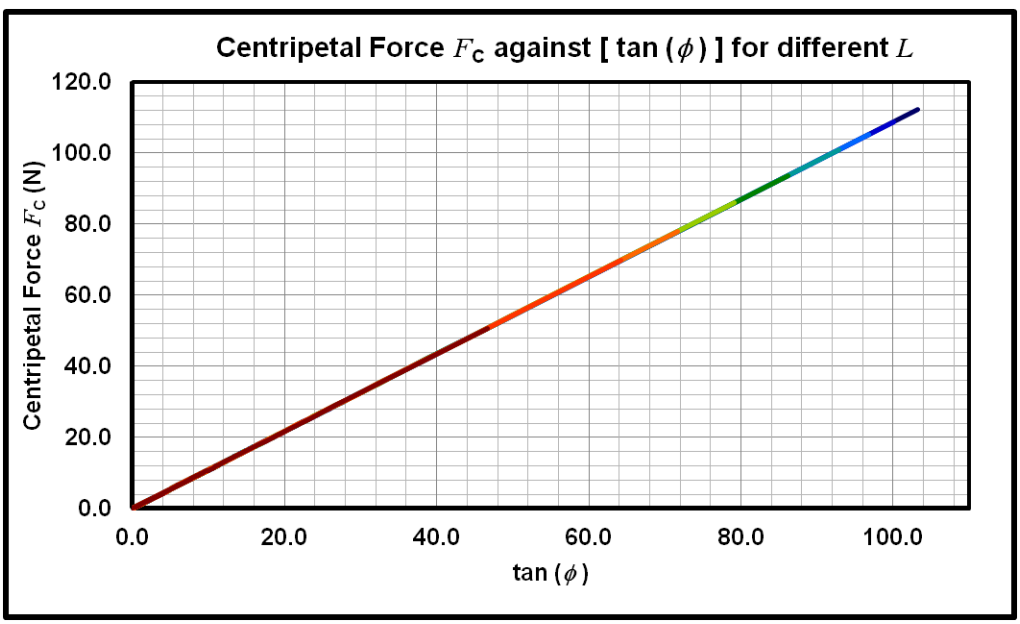

Chart 2. *F_C against tan (ϕ)*

Since the tension force F_T and the centripetal force F_C are seen to be trigonometric functions of the apex angle (although different functions), it was considered as being visually useful to calculate and plot the respective first derivatives ($dF_T/d\phi$ and $dF_C/d\phi$) of both forces:

$$F_T = \frac{m\,g}{\cos\phi} \quad \Rightarrow \quad \frac{dF_T}{d\phi} = \frac{m\,g\,\sin\phi}{\cos^2\phi} = \frac{F_C}{\cos\phi}$$

$$F_C = m\,g\,\tan\phi \quad \Rightarrow \quad \frac{dF_C}{d\phi} = \frac{m\,g}{\cos^2\phi} = \frac{F_T}{\cos\phi} \tag{4}$$

Close inspection of the above two derivative functions shows that for very small apex angles that are asymptotically approaching zero degrees ($\phi \to 0$), the denominators of both derivative equations are approximately one. Therefore, (by considering the numerators of both derivative equations), the calculated first derivative of the tension force ($dF_T/d\phi$) asymptotically goes to zero, whilst the calculated first derivative of the centripetal force ($dF_C/d\phi$) asymptotically goes to the numerical value of (mg), as shown in Chart 3. It is also observed that both first derivative functions progressively appear to merge together when $\phi \geq 80°$ and this is especially noticeable beyond 85°. As an example, when $\phi = 85°$ the first derivative of the tension force ($dF_T/d\phi$) has a value (stated to 4 significant figures and in the appropriate *SI* units) of 142.6, whereas the first derivative of the centripetal force ($dF_C/d\phi$) has a value of 143.2, which differ by only 0.4%.

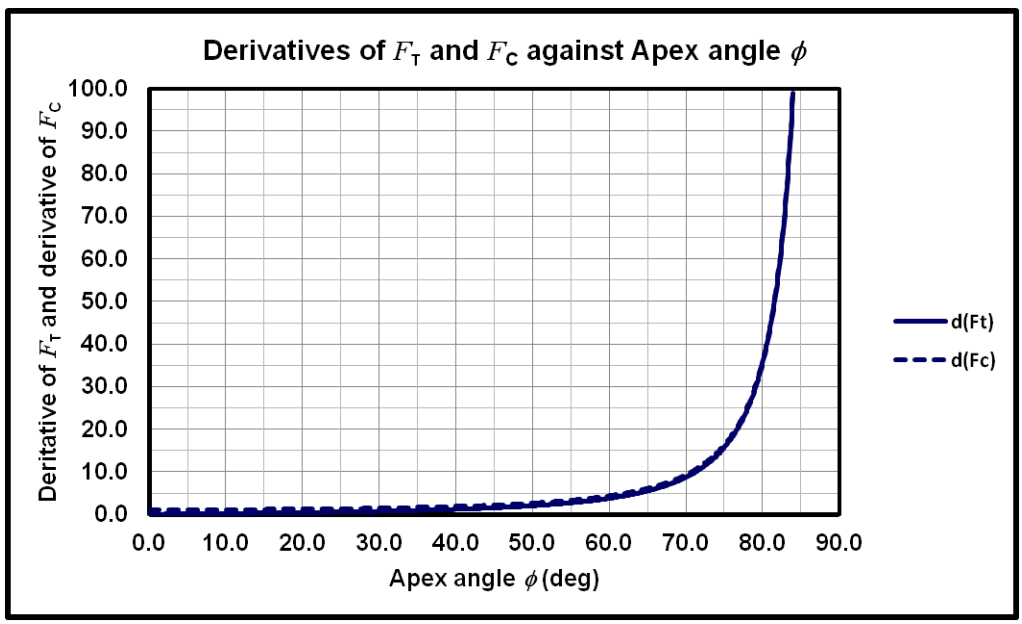

Chart 3. *Derivatives of F_T and F_C against Apex angle ϕ*

The conical pendulum period has been studied in detail and an equation derived (Dean, 2017) that shows the square of the period T^2 to be directly proportional to the square of the period for a simple pendulum $[T_{SP}]^2$ multiplied by the cosine of the apex angle.

$$T = 2\pi\sqrt{\frac{L\cos\phi}{g}} = T_{SP}\sqrt{\cos\phi} \quad \Rightarrow \quad T^2 = [T_{SP}]^2\cos\phi \tag{5}$$

This provides a straight line having a value for the slope equal to exactly 1 (Chart 4).

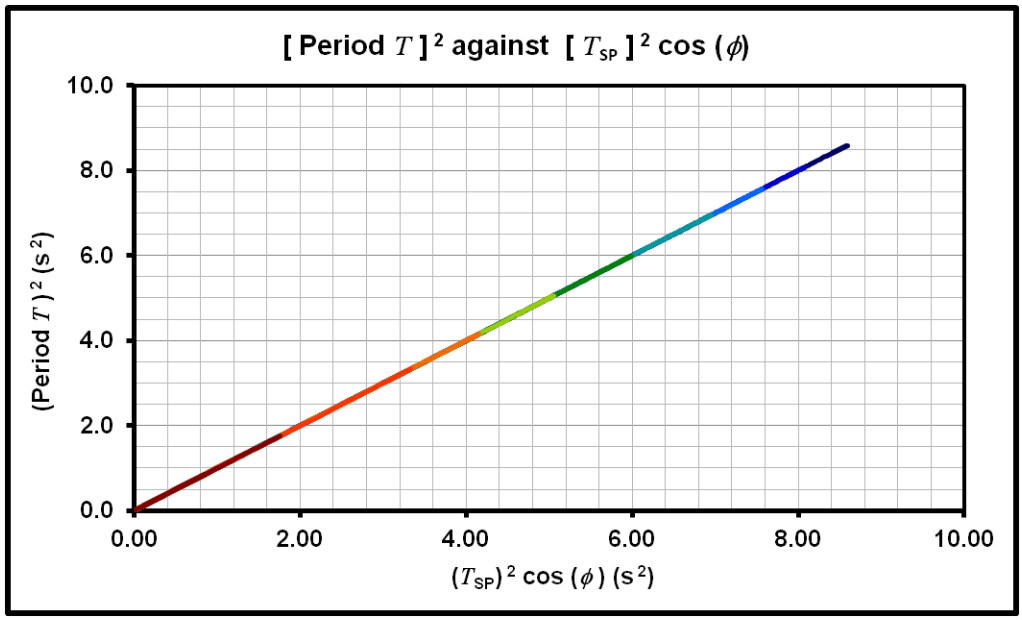

Chart 4. *T^2 against $(T_{SP})^2$ cos (ϕ)*

It is of interest to consider the dependence of the Apex angle ϕ as a function of the orbital period of the conical pendulum squared T^2, by making a simple algebraic re-arrangement:

$$T = 2\pi\sqrt{\frac{L\cos\phi}{g}} \quad \Rightarrow \quad \frac{T^2}{4\pi^2} = \frac{L\cos\phi}{g} \quad \Rightarrow \quad \phi = \cos^{-1}\left[\frac{g\,T^2}{4\pi^2\,L}\right] \tag{6}$$

This functional equation is plotted in Chart 5. The fact that all of the curves originate from a calculated (although physically unrealizable) value of $\phi = 90°$ can be explained, by considering that the minimum possible value for the conical pendulum period must coincide with the pendulum rotating at the highest possible orbital speed, which occurs in the mathematical limit of $\phi \to 90°$. It is important to note that the curves calculated for this chart are representative of the behavior of the arc-cosine function and do not take the practical physical limitations into account. The variation of the two forces (tension and centripetal) can be inferred, by considering the first two calculated charts.

As the period progressively increases, then the apex angle must correspondingly progressively decrease, which is clearly shown in Chart 5.

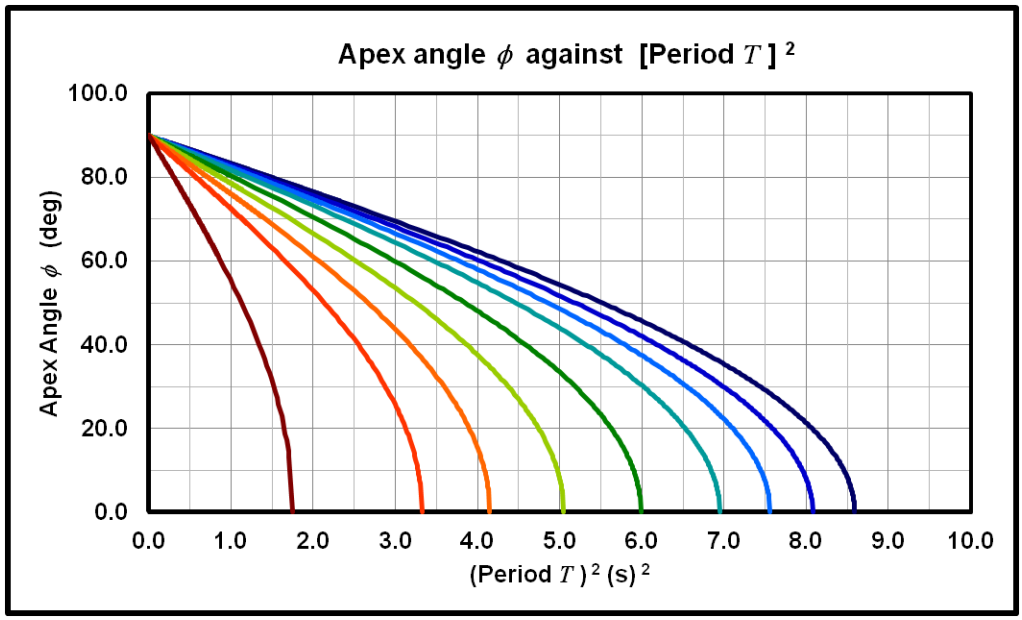

Chart 5. *Apex angle ϕ against T^2*

An alternative (entirely equivalent) representation can be used where the angular frequency squared ω^2 is taken as the horizontal axis parametric variable and plotted in Chart 6.

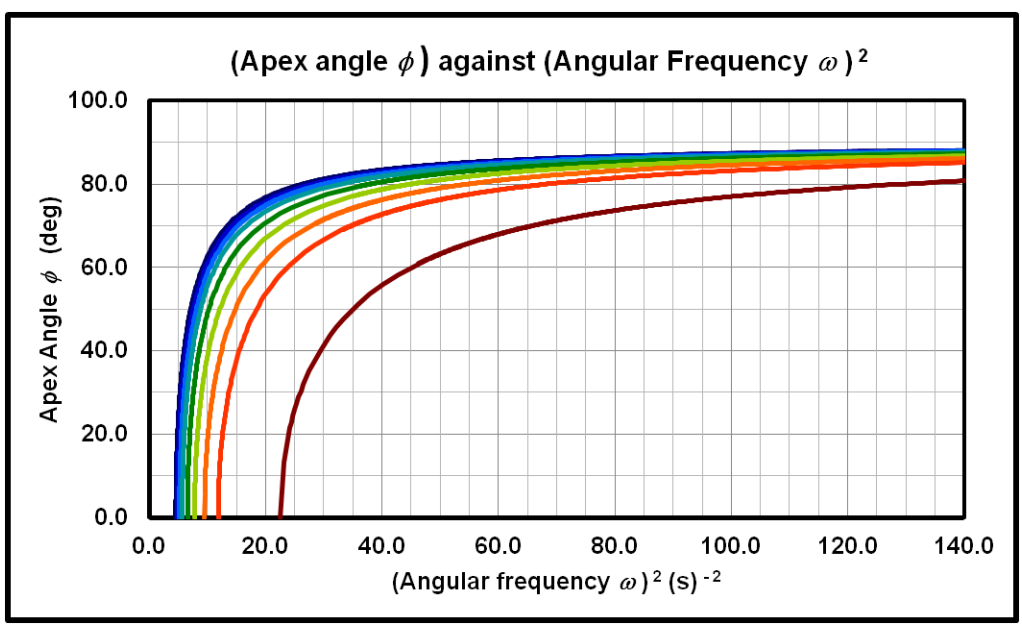

Chart 6. *Apex angle ϕ against ω^2*

The equation describing this parametric dependence is shown (7)

$$\phi = \cos^{-1}\left[\frac{g\,T^2}{4\,\pi^2\,L}\right] = \cos^{-1}\left[\frac{g}{L\,\omega^2}\right] \tag{7}$$

When the angular frequency approaches the limit of a minimum vale $\omega_{\min}$ for each of the calculated lines for the different length conical pendulums, then the apex angle correspondingly decreases to the asymptotic limit of $\phi = 0°$ (being equivalent to the situation for an oscillating simple pendulum). As the angular frequency progressively increases, the apex angle must also increase asymptotically towards the limiting value of $\phi \rightarrow 90°$ (this mathematical limit cannot be achieved physically, since it would require the angular frequency to approach infinity).

A further straightforward re-arrangement of an earlier equation enables an alternative method of obtaining an experimental value for the gravitational acceleration. The required equation is obtained in the following way:

$$T = 2\,\pi\sqrt{\frac{L\cos\phi}{g}} \quad \Rightarrow \quad L\cos\phi = \frac{g\,T^2}{4\,\pi^2} \tag{8}$$

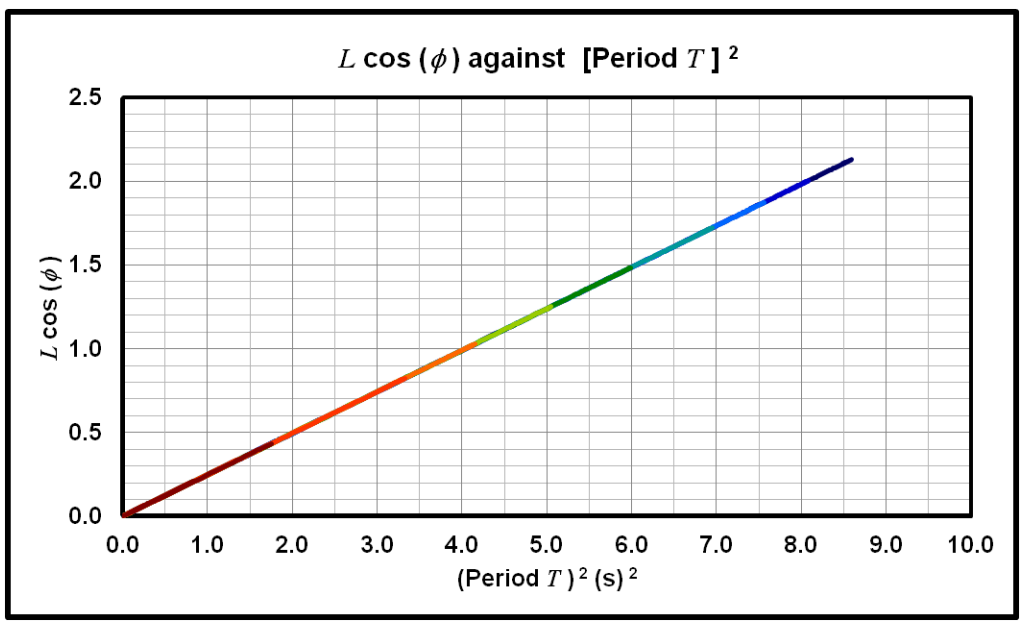

Chart 7. *L cos* (ϕ) *against* T^2

The experimental data consists of the conical pendulum period and the measured vertical distance between the conical pendulum support point and the horizontal plane of rotation. As shown in Chart 7, the calculated slope of the graphical straight line can provide an experimental value the gravitational acceleration.

In order to extend the analysis further, the orbital speed was derived as a function of the orbital radius, according to the self-explanatory analysis outlined in the equations below:

$$T = 2\pi\sqrt{\frac{L\cos\phi}{g}} = 2\pi\sqrt{\frac{\sqrt{L^2 - R^2}}{g}} \quad \Rightarrow$$

$$v = \frac{2\pi R}{T} = \sqrt{\frac{g}{\sqrt{L^2 - R^2}}}\, R = \frac{\sqrt{g}\, R}{\left(L^2 - R^2\right)^{(1/4)}} \tag{9}$$

Although the final equation (9) for the orbital speed appears to be somewhat complicated, it is in fact very readily calculated and contains only a single parametric variable. For $R = 0$ (corresponding to the situation where the conical pendulum hangs vertically downwards), the orbital speed must also be zero for all values of the conical pendulum length L. As the orbital radius progressively increases, so that $R \to L$, the denominator asymptotically tends to zero, so that the orbital speed $v \to \infty$ and this is clearly seen in Chart 8.

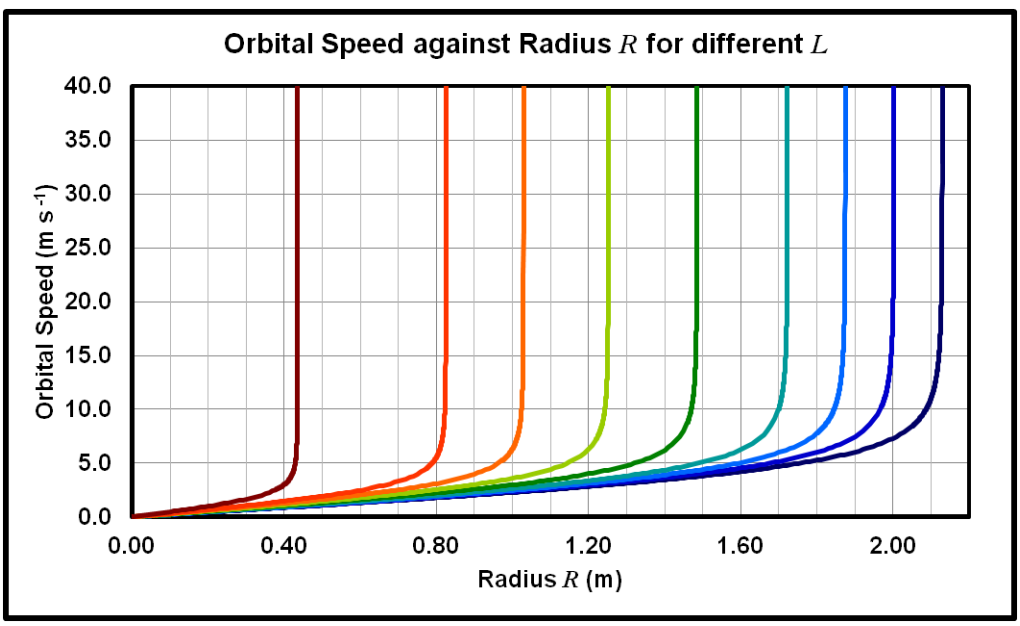

Chart 8. *Orbital speed v against Radius R*

The orbital speed can easily be re-expressed as an uncomplicated function of the apex angle, which can then be readily calculated. Starting with equation (9) for the orbital speed and then making appropriate trigonometric substitutions for the orbital radius R in the numerator and also for the function of L^2 and R^2 that appears in the denominator, yields a simple equation that can be used to calculate the functional dependence of the orbital speed v.

$$v = \frac{\sqrt{g}\ R}{\left(L^2 - R^2\right)^{(1/4)}} = \frac{\sqrt{g}\ L \sin\phi}{\sqrt{L\cos\phi}} = \sqrt{g\ L}\ \frac{\sin\phi}{\sqrt{\cos\phi}} \tag{10}$$

Consideration of the numerator of this equation indicates that when the apex angle is zero, the chart lines must all start at the origin of coordinates, so $\phi = 0°$ is equivalent to $R = 0$ for the preceding chart (Chart 8), where the conical pendulum is observed to be vertical. As the apex angle progressively increases $\phi \to 90°$ the denominator of the equation approaches zero, and so therefore the orbital speed increases and asymptotically tends to infinity (this mathematical limit being physically unattainable). Chart 9 shows this for the angular range $0° \leq \phi < 90°$.

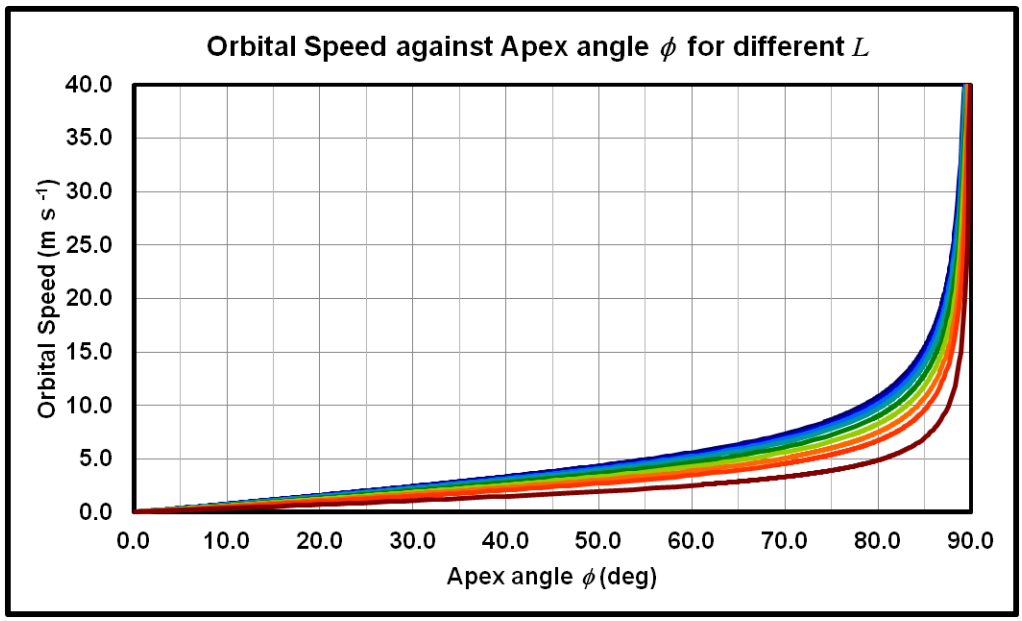

Chart 9. *Orbital speed v against Apex angle ϕ*

When the conical pendulum mass rotates in circular motion it possesses angular momentum, that will be represented in scalar terms by the lower-case letter $l = Rmv$ in the usual format. This has been calculated as a function of the orbital radius as follows in the self-explanatory analysis shown below:

$$v = \frac{\sqrt{g}\ R}{\left(L^2 - R^2\right)^{(1/4)}} \quad \Rightarrow \quad l = R\,m\,v = \frac{m\sqrt{g}\ R^2}{\left(L^2 - R^2\right)^{(1/4)}} \tag{11}$$

The dependence of the magnitude of the angular momentum as a function of the orbital radius is shown in Chart 10. The fact that l varies according to the square of the orbital radius is very clearly shown. The numerator of the equation indicates that for $R = 0$, all lines must start at the coordinate origin (just as for the speed against orbital radius). The visual appearance of the

orbital speed curves and the angular momentum curves is very similar. However, the angular momentum curves increase more rapidly and become asymptotic more far abruptly.

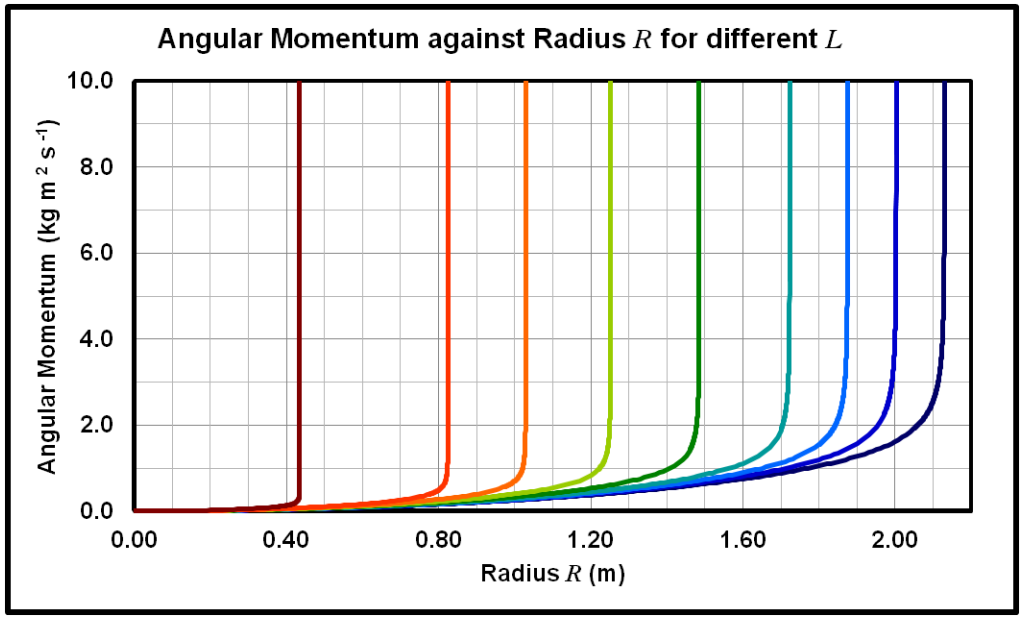

Chart 10. *Angular Momentum against Radius R*

In a similar way to the dependence of the orbital speed, the angular momentum was also considered as function of the apex angle as shown in the derivation below:

$$\left.\begin{aligned} v &= \sqrt{g\,L}\,\frac{\sin\phi}{\sqrt{\cos\phi}} \\ l &= R\,m\,v = L\sin\phi\,m\,\sqrt{g\,L}\,\frac{\sin\phi}{\sqrt{\cos\phi}} \end{aligned}\right\} \Rightarrow$$

$$l = \left(m\,\sqrt{g}\;L^{(3/2)}\right)\frac{\sin^2\phi}{\sqrt{\cos\phi}} \tag{12}$$

It can be observed from Chart 11, which illustrates the functional shape of the angular momentum curves, starting at the origin of coordinates (due to $\phi = 0°$ in the equation numerator) and progressively increasing steeply until becoming asymptotic towards infinity at high angles, which is to be expected from the square-root cosine term present in the denominator.

It can be observed from the series of Chart 11 curves, that the angular momentum steepens progressively and very rapidly becomes near-vertical for apex angles $\phi \geq 85°$. This is again a direct consequence of the root cosine term that is present in the denominator.

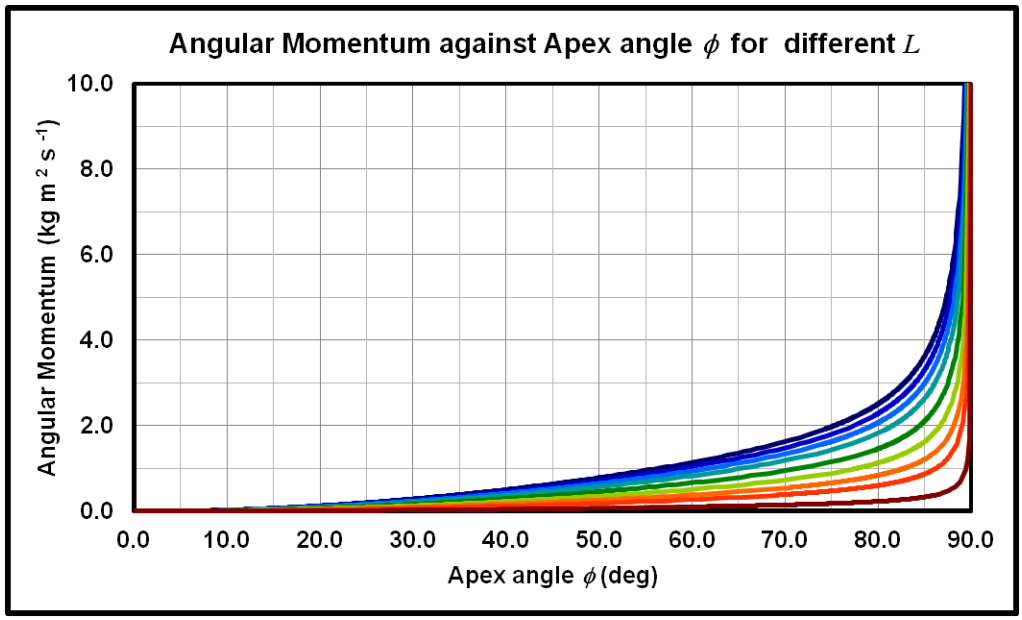

Chart 11. *Angular Momentum against Apex angle ϕ*

From the mathematical representation of the physical parameters that have been investigated, it becomes apparent that the conical pendulum period, forces, speeds and momentum equations might possibly be combined and investigated further. One interesting example analysis is presented below, which deals with the constancy of the length of the adjacent side of the schematic triangular construction used for analyzing the conical pendulum (refer to Chart 7).

ANALYSIS (*Length of adjacent side = constant*)

By considering the conical pendulum period equation T, a straightforward analysis leads to an interesting (and virtually counter-intuitive) result, which is explained in detail below:

$$T = 2\pi \sqrt{\frac{\sqrt{L^2 - R^2}}{g}} = 2\pi \sqrt{\frac{L\cos\phi}{g}} = \frac{2\pi}{\sqrt{g}} \sqrt{L\cos\phi}$$

For any particular value of the conical pendulum period T, the following can be readily obtained for two different conical pendulum lengths L_1 and L_2 that have the same period:

$$T = \frac{2\pi}{\sqrt{g}} \sqrt{L_1 \cos\phi_1} = \frac{2\pi}{\sqrt{g}} \sqrt{L_2 \cos\phi_2} \quad \Rightarrow \quad \frac{g}{4\pi^2} T^2 = L_1 \cos\phi_1 = L_2 \cos\phi_2$$

Therefore, for any particular value of the period T (and corresponding angular speed ω), a general equation can be written as follows, which applies to all pendulum lengths L_i with corresponding apex angle ϕ_i (see Figure 2 below):

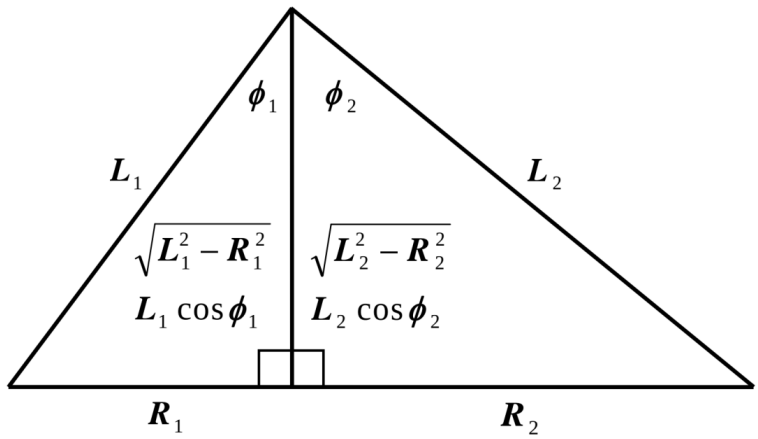

Figure 2. *Constant Length Adjacent Side*

$$L_i \cos\phi_i = \frac{g}{4\pi^2} T^2 = \frac{g}{\omega^2} = \text{constant} \quad \Rightarrow \quad \phi_i = \cos^{-1}\left(\frac{g}{L_i\,\omega^2}\right) \quad \Rightarrow \quad 0 \le \frac{g}{L_i\,\omega^2} \le 1$$

The above expression now provides an equation, which was also derived (albeit by using a slightly different analysis) in an earlier publication (Dean, 2017). Applying the two extreme mathematical limits directly gives the following:

$$\frac{g}{L_i\,\omega^2} = 0 \quad \Rightarrow \quad \phi_i = 90° \text{ physically impossible, also}: \frac{g}{L_i\,\omega^2} = 1 \quad \Rightarrow \quad \phi_i = 0° \quad \Rightarrow \quad \omega^2 = \frac{g}{L_i}$$

The second of the equation limits (which clearly corresponds to the lowest angular frequency and Apex angle ϕ = 0) shows that an appropriate graph will enable the local value of the acceleration due to gravity to be determined from the slope, where L_i are the individual lengths used for the conical pendulum.

$$\omega^2 = g\left(\frac{1}{L_i \cos\phi_i}\right) \quad \Rightarrow \quad Slope = g$$

From the analysis perspective, the lowest angular frequency used in the above, occurs when the conical pendulum has the largest possible period T and therefore behaves analogously as a simple pendulum. It is of interest to note that the equation shown below has already been used graphically in order to obtain a value of the local acceleration due to gravity (calculated from the slope of the straight line that is obtained directly from Chart 7):

$$L_i \cos\phi_i = \frac{g}{4\pi^2}\left(T^2\right) \quad \Rightarrow \quad Slope = \frac{g}{4\pi^2}$$

DISCUSSION OF RESULTS AND CONCLUSIONS

The analysis and associated charts that are presented in this paper develop from the previously published paper that discussed mathematically some of the fundamental physics involved in the rotational motion of a conical pendulum. The tension and centripetal force are both shown to be linear, as trigonometric functions of the apex angle (one being a reciprocal cosine and the other a tangent). It is shown that for both of the associated charts, the respective straight-line slopes are equal to the constant (*mg*), from which a value for the local acceleration due to gravity can be readily obtained. The square of the conical pendulum period (derived in the previous paper) was demonstrated to be linear with respect to the product of the square of the period of a simple pendulum and the cosine of the apex angle. The functional dependence of the apex angle was shown using two equivalent (but subtly different) time parameters as the horizontal axis, namely, the square of the period and the square of the angular frequency. Both charts were explained in detail, especially the point of origin of curves and any asymptotic behavior. A linear chart is included, which shows that when the distance between the point of support and the rotational plane is plotted as a function of the square of the period, the slope of the resulting straight line provides another method to determine the acceleration due to gravity. The orbital speed and also the corresponding rotational angular momentum were analyzed as functions of the orbital radius and apex angle and appropriate charts were plotted and explained in detail.

APPENDIX

Chart legend for theory lines (except Chart 3)

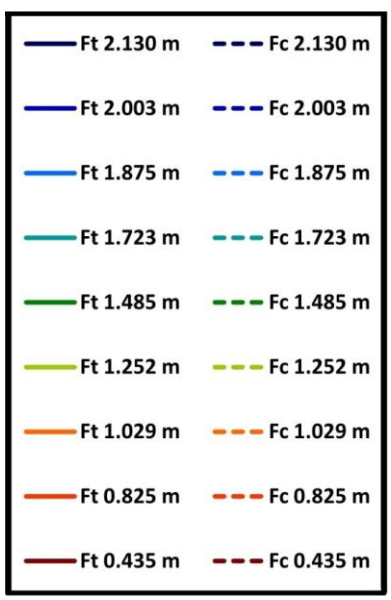

Figure 3. *Chart Legend*

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