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Digital Fabrication and Hidden Inequalities: Connecting Procedural, Factual, and Conceptual Knowledge

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The paper uses digital fabrication as a learning environment to demonstrate the importance of conceptual understanding of one-variable inequalities. These inequalities are hidden within two-variable inequalities used to construct geometric shapes the borders of which are the graphs of basic functions from the secondary mathematics curriculum. As graphing skills in the digital era are mostly procedural, an emphasis on hidden inequalities as tools for digital fabrication of specific regions in the plane and their borders connects procedural and conceptual knowledge in mathematics. Under the umbrella of collateral learning framework, the paper discusses the notions of transitive inference and necessary and sufficient conditions. The software program used for digital fabrication is the *Graphing Calculator 4.0* produced by Pacific Tech.

Keywords: digital fabrication, algebra, inequalities, teacher education, collateral learning

INTRODUCTION

An educational paradigm of digital fabrication (Gershenfeld, 2005; Bull, Knezek, & Gibson, 2009; Walter-Herrmann & Büching, 2013) has grown out of ubiquitous movement of the 1980s to introduce a pedagogy of student-computer interaction into the schools as a way of improving teaching and learning of mathematics and science (Papert, 1980; Noss, 1987; Schwartz & Yerushalmy, 1987; diSessa, 1988; Hoyles & Noss, 1992). Conceptually, digital fabrication makes it possible "to explore how to represent a functional description of a system in a physical form and likewise to which extent a functional description of a physical system can be abstracted" (Dittert & Krannich, 2013, p. 173). As Nake (2013) noted, digital fabrication is a space where abstract algorithms that guide computational tools and specific productions of art (e.g., a computer aided

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design) meet. For example, in mathematics, fractals and strange attractors (Gleick, 1987) or even more elementary 2-D graphs, all described by abstract mathematical models, are examples of digital fabrication.

This paper continues the authors' work on bringing digital fabrication as an educational paradigm into mathematics education (Abramovich, 2011; Connell & Abramovich, 2015). It was motivated by the observation that the process of digital fabrication of different geometric shapes formed by the combinations of graphs of the basic functions from the secondary mathematics curriculum – polynomial, exponential, circular – requires deep conceptual understanding of two-variable inequalities that define those shapes. Of particular interest, two-variable inequalities alone may or may not define the expected geometric shapes in the plane desired for fabrication via the construction of a locus – a set of points determined by a specified condition formulated either through the use of an equation or inequality. The process of digital fabrication may involve the construction of a connected locus in the plane. It may also involve the construction of the locus consisting of several disjoint two-dimensional regions. In the former case, one-variable inequalities used in defining the range for the independent variable may remain hidden within the richer two-variable inequalities that define the locus. In the latter case, these inequalities have to be explicitly revealed as they play the critical role in the process of digital fabrication in the plane. In both cases, hidden inequalities are necessary for the symbolic description of the borders of the shapes. All this requires conceptual understanding of algebraic inequalities – a topic that received increased attention in secondary mathematics education research within the last decade (Boero & Bazzini, 2004; Tall, 2004; Abramovich & Ehrlich, 2007; Yerushalmy, 2009; Abramovich, 2011).

Expectations by the National Council of Teachers of Mathematics (2000) for secondary school students include the need to "understand the meaning of … equations, inequalities, and relations … using technology in all cases" (p. 296). Because mathematical methods allow for "algebra to be applied to geometry and vice versa … making visualization a tool for doing and understanding algebra" (Common Core State Standards, 2010, p. 74), the meaning of inequalities can be revealed in a geometric context of graphing. Nowadays, this context is well supported by technology-enhanced visualization based on computer graphing software. In the technological paradigm, "competence in solving [inequalities] includes looking ahead for productive manipulations and anticipating the nature … of solutions" (ibid, p. 62). That is, graphs of the functions related through inequalities when constructed with accuracy that software affords may be used for leading the way towards correct solution of the inequalities that otherwise would be difficult to solve. The use of software facilitates graphing skills that are based on the point-by-point construction of a graph from the procedurally developed (x, y)-table. Therefore, the expectation is that such skills, enhanced by a digital tool, foster conceptual understanding of algebraic inequalities including both formal (analytic) and informal (numeric) methods of finding their solutions.

In that way, the paper further contributes to the discussion of the interplay between procedural skills and conceptual understanding in the technological paradigm (Abramovich, 2015; Kadijevich, 2002; Kadijevich & Haapasalo, 2001; Peschek & Schneider, 2001). As noted by Kaput (1992), the importance of this discussion is due to the fact that "the exercise of procedural knowledge is supplanted by (rather than supplemented by) machines" (p. 549). Whereas computers, indeed, free time traditionally needed for practicing algorithmic skills, time that students, instead, could (and should) use to grow conceptually, these digital tools are conducive to developing links between the two types of knowledge. Yet digital fabrication, as its very name suggests, is not technology-free. Therefore, to do the fabrication right, one has to appreciate the notion that "the use of technology for complicated computation does not eliminate the need for mathematical thinking but rather often raises a different set of mathematical problems" (Conference Board of the Mathematical Sciences, 2001, p. 48). Revealing hidden inequalities required for sophisticated digital fabrication is one such problem.

When technology is used to facilitate inquiry into mathematics, mathematical concepts can emerge not only as a result of a computational experiment but also can be introduced as tools that structure the experiment (Abramovich, 2014). This approach has classic roots going way back to the pre-digital era. In mathematics research, Euler emphasized the importance of observations and thought processes that observations animate (see Pólya, 1954, p. 3). One can find a similar position in much earlier writings by Archimedes (1912), who called thought processes that stem from observations "a mechanical method" (p. 13), something that he was using to better understand a problem at hand before pursuing its formal solution.

John Dewey, one of the main forces behind the U.S. educational reform in the first part of the 20th century, advocated for an experiential approach to the development of knowledge by calling for a pedagogy that motivates reflective inquiry into the subject matter studied, a method that blurs the distinction between learning and doing. In turn, from this method stems what Dewey (1938) has referred to as *collateral learning* – a phenomenon that does not result from an immediate goal of the curriculum but emerges from its hidden domain. That is, a student, at any given moment, within any domain of knowledge, learns more than one particular idea. In the domain of mathematics, closely related concepts are *unintentional discovery* (Kantorovitch, 1998) when one can solve several, perhaps unrelated, problems at a time, and *hidden mathematics curriculum* (Abramovich & Brouwer, 2006) when one is expected and even encouraged to make connections among seemingly disconnected ideas and concepts, thereby motivating learning in a larger context that is originally stated. In what follows, the notion of collateral learning mapped on the related notions of unintentional discovery and hidden mathematics curriculum will serve as a conceptual framework for using digital fabrication in the context of teaching secondary mathematics that will provide cognitive support for discovering hidden inequalities and using this discovery to advance learning of algebraic inequalities through applications.

A graphing software used by the authors as a tool for digital fabrication in the context of this paper is the *Graphing Calculator 4.0* produced by Pacific Tech (Avitzur, 2011). The tool allows for graphing relations from any two-variable equations and inequalities. The notion of hidden inequalities introduced in this paper will illustrate how "mathematical planning and theoretical insights may be needed to structure efficient computation" (Conference Board of the Mathematical Sciences, 2001, p. 48). Put another way, hidden inequalities can support "the recognition of a difference between 'technological literacy' (a general set of skills and intellectual dispositions for all citizens) and 'technical competence' (in-depth knowledge that professional engineers and scientists need to know to perform their work" (Blikstein, 2013, p. 205).

DIGITAL FABRICATION USING TRANSITIVE INFERENCE

As the first example, consider the graphs of the functions $f(x)=x^2$ and $g(x)=x$ shown in Figure 1. The graphs form a parabolic segment $\Omega=\{(x,y)\,|\,y>x^2, y<x\}$ – a connected region located above the parabola and below the straight line in the (*x*, *y*)-plane. One should note that the formal mathematical definition of Ω does not include information about the variation of *x* along the *x*-axis, that is, the *x*-range of Ω. Nonetheless, this definition of Ω is complete because Ω consists of a single, connected region and seemingly missing information regarding the variation of *x* within the

parabolic segment Ω formed by the graphs of $f(x)$ and $g(x)$ is hidden within the (simultaneous) inequalities

$$y > x^2, y < x \qquad (1)$$

used to digitally fabricate Ω (Figure 2). To reveal hidden inequalities that define the x-range within the parabolic segment Ω note that inequalities (1) imply the inequality $x^2 < x$, which, through the process of cancelling out x as the common factor of its both sides, taking into account the sign of x, is equivalent to the pair of consistent inequalities $x > 0, x < 1$ or inconsistent inequalities $x < 0, x > 1$, whence

$$0 < x < 1 \ . \qquad (2)$$

It is in this sense that one can say that inequalities (2) are hidden within inequalities (1).

Inequalities (2), however, are not needed for the construction of Ω in a sense that they do not have to be explicitly communicated to a graphing facility capable of graphing relations from two-variable inequalities. Therefore, if one graphs simultaneously inequalities (1) and (2), the act of adding (2) to (1) is of no consequence. That is, if one is tasked with the transition from visual to symbolic, that is, with describing Ω in terms of x and y, inequalities (2) is extraneous information.

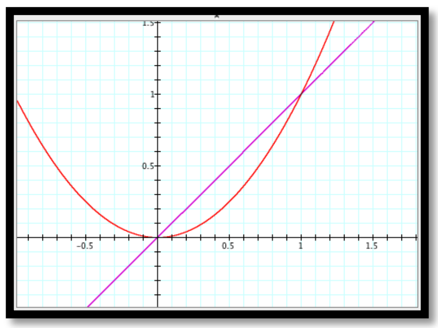

Figure 1. The parabolic segment formed by two graphs.

Put another way, the graphing tool appears to be capable of making a transitive inference (Nunes, 1992) by recognizing that if $a > b$ and $b > c$, then $a > c$. That is, the inference $a > c$ is hidden within the assumption formed by the inequalities $a > b$ and $b > c$. However, if the inequality $b > c$ is replaced by $b < c$, the relationship between a and c becomes undetermined (which is significantly different from being merely hidden) and, therefore, additional information is needed to make an inference. For example, from the inequalities 5 > -3 and -3 > -5 one can make the inference 5 > -5; however, the inequalities 5 > -3 and -3 < 6 may not be used to relate 5 and 6 using a transitive reference.

In that way, the use of the *Graphing Calculator* does "require the user [of the tool] to think conceptually before a procedure is used" (Kadijevich, 2002, p. 72). Therefore, the tool facilitates the development of links between procedural skills and conceptual understanding. The pedagogy of linking two types of knowledge, procedural and conceptual, is an important topic in mathematics education, especially in the digital era.

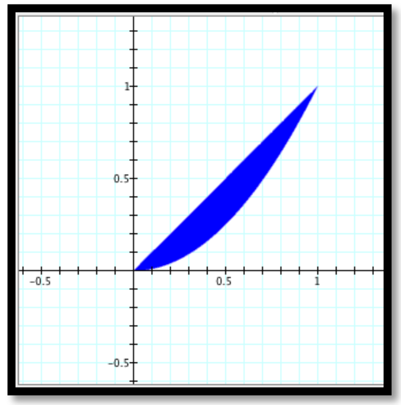

Figure 2. Digitally fabricated parabolic segment Ω.

DIGITAL FABRICATION OF BORDERS LACKS TRANSITIVE INFERENCE

Why are we talking about hidden inequalities in the first place? Do they have any use in digital fabrication? To answer these questions, consider digital fabrication of the borders (straight and curvilinear) of the parabolic segment Ω (Figure 2). The borders are not defined by the graphs of $f(x)$ and $g(x)$ as they represent only specific parts of the graphs. Therefore, unlike the case of Ω which is located between the graphs, the borders of Ω are defined independently by the corresponding parts of these graphs. Whereas inequalities can be used to construct lines (loci) of non-zero measure to ensure their visual thickness, the description of specific parts of the graphs would require an independent set of inequalities that explicitly describe the corresponding x-range. In other words, the graphing tool is unable to make a transitive inference about an x-range from the inequalities that define the lines of non-zero measure to which the borders belong.

For example, the straight border of Ω can become ε-thick by defining it as a set of points within which $|y-x|<\varepsilon$, where ε is a sufficiently small positive number. That is, the two-variable inequalities

$$-\varepsilon < y - x < \varepsilon \qquad (3)$$

define the set of points (x, y) where the difference between y and x belongs to the bandwidth $[-\varepsilon,\varepsilon]$. Likewise, the curvilinear border of Ω can be defined as a set of points (x, y) within which $|y-x^2|<\varepsilon$. That is, the inequalities

$$-\varepsilon < y - x^2 < \varepsilon \qquad (4)$$

define a set of points (x, y) where the difference between y and x^2 belongs to the bandwidth $[-\varepsilon,\varepsilon]$. One can see that inequalities (3) and (4) are mutually independent and define an ε-thick "cannula" through which each of the two graphs passes.

Graphing simultaneously inequalities (3) and (4) can mark the points (0, 0) and (1, 1), which can be interpreted as hidden endpoints where the borders intersect. Alternatively, ε-thick points common to the two borders can be fabricated by graphing the inequalities $|x-1|^2+|y-1|^2<0.05$ and $x^2+y^2<0.05$ (Figure 3). That is, for the fabrication of the borders of the parabolic segment Ω one has to introduce hidden inequality (2) and graph concurrently (3)-(2) and (4)-(2) as shown in Figure 4.

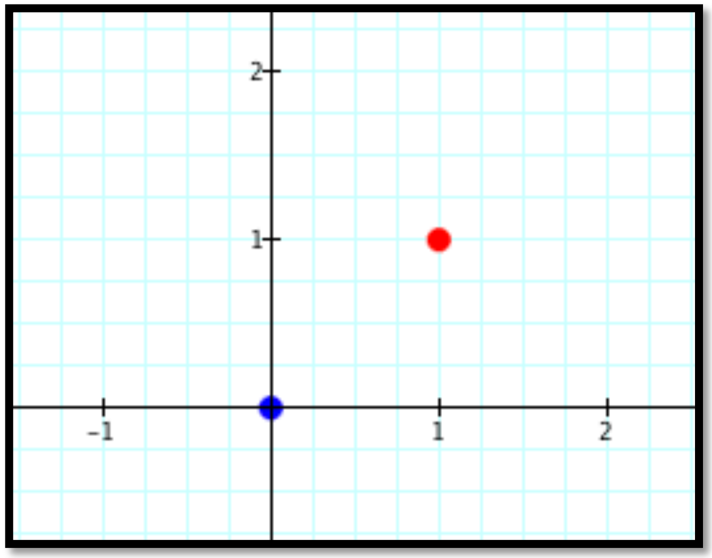

Figure 3. Graphing the endpoints of the borders of Ω .

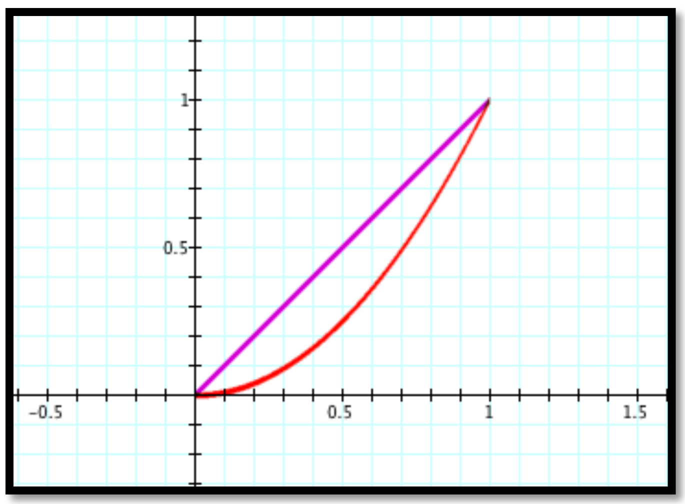

Figure 4. Using hidden inequalities (2) in constructing the borders of Ω .

A NEED FOR HUMAN-COMPUTER INTERACTION

Another example demonstrating the interplay between hidden inequalities and the digital fabrication of two-dimensional locus and their borders deals with the graphs of the functions $y = x^3$ and $y = x$. Consider the set of points in the (x, y)-plane defined as follows:

$$\Omega_1 = \{(x, y) \mid y > x^3, y < x\} .$$

Once again, the definition of Ω_1 does not include explicit information about variation of the variable x along the x-axis. However, this time, graphing simultaneously the inequalities

$$y > x^3, y < x \qquad (5)$$

fabricates two regions: one, bounded, in the first quadrant, and another, unbounded, in the third quadrant (Figure 5).

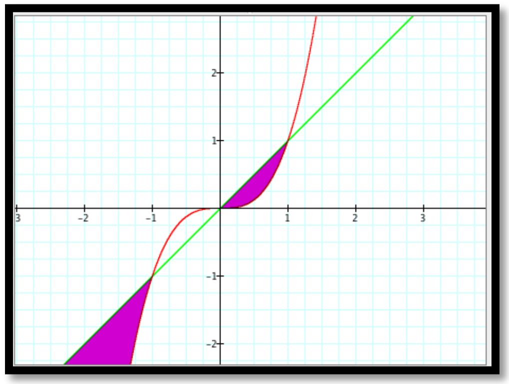

Figure 5. Digital fabrication using inequalities (5) only.

The easiest way to distinguish between the two regions is to note that in the first and the third quadrants we have, respectively, $x > 0$ and $x < 0$. So, if one of the last two inequalities has been added to inequalities (5), the graphing would yield one of the regions only. Is the inequality $x > 0$ (or $x < 0$) hidden within inequalities (5)? It follows from (5) that $x^3 < x$ or $x(x-1)(x+1) < 0$ whence, due to the sign-chart method (Dobbs & Peterson, 1991)

$$x < -1 \text{ or } 0 < x < 1. \qquad (6)$$

That is, neither the inequality $x > 0$ nor the inequality $x < 0$ is hidden within inequalities (5). At the same time, when adding either $x < 1$ or $x < -1$ to, respectively, the system of inequalities $y > x^3, y < x, x > 0$ or $y > x^3, y < x, x < -1$, the corresponding addition is of no consequence for the outcome of digital fabrication. However, just as in the previous example with parabola, hidden inequalities (6) are critical for the construction of the borders of Ω_1. Indeed, graphing the systems $-\varepsilon < y - x < \varepsilon, 0 < x < 1$ and $-\varepsilon < x^3 - y < \varepsilon, 0 < x < 1$ (for a sufficiently small value of ε) digitally fabricates these borders. That is, once again, hidden inequalities are needed for the construction of the borders of two-dimensional locus.

Making an appropriate decision regarding these conditions requires human judgment, grounded in the needs of the desired outcome. Once it is provided with appropriate instructions, the computer produces the desired outcome easily. Determining what the desired outcome is, however, is beyond the ability of the computer and requires human interaction as informed by deep conceptual awareness of the language of mathematics.

NECESSARY AND SUFFICIENT CONDITIONS AS COLLATERAL LEARNING

When exploring pedagogy of collateral learning, Dewey (1938) noted, "the greatest of all pedagogical fallacies is the notion that a person learns only the particular thing he is studying at the time" (p. 49). The topic of hidden inequalities provides a collateral opportunity for secondary school students and their future teachers alike to learn about necessary and sufficient conditions – one of the main reasoning instruments in mathematics. It is said that condition A is sufficient for condition B if condition A implies condition B (i.e., $A \Rightarrow B$). Also, it is said that condition A is necessary for condition B if B implies A (i.e., $B \Rightarrow A$). That is, condition A is necessary and sufficient for condition B if B implies A and vice versa (i.e., $A \Leftrightarrow B$).

For example, in order for the inequality $x < -1$ hold true (condition A) it is necessary that $x < 0$ (condition B). One can see that $A \Rightarrow B$. However, condition B is not sufficient for condition A as a negative number may be still be greater than -1. At the same time, the inequality $x + 1 < 0$ (new condition B) is necessary and sufficient for x to be smaller than -1. Indeed, the inequality $x + 1 < 0$ implies $x < -1$ (condition A). That is, because $B \Rightarrow A$, condition B is necessary for condition A. By the same token, the inequality $x < -1$ implies $x + 1 < 0$. That is, because $A \Rightarrow B$, condition B is sufficient for A. Therefore, the inequality $x + 1 < 0$ is necessary and sufficient for the inequality $x < -1$.

CAN HIDDEN INEQUALITIES BE NECESSARY AND SUFFICIENT FOR DIGITAL FABRICATION?

Inequalities (6) are hidden within inequalities (5). But the critical question remains: Which one is necessary, which one is sufficient, and which one is necessary and sufficient for the digital fabrication of one of the shaded regions shown in Figure 6? The inequality $x < -1$ is necessary for the fabrication of the far-left region shown in Figure 6. At the same time, $x < -1$ is also sufficient for this fabrication. That is, the inequality $x + 1 < 0$ is necessary and sufficient for the construction of the non-bounded part of Ω_1. At the same time, the inequalities $0 < x < 1$ are sufficient for the fabrication of the bounded part of Ω_1. Yet they are not necessary for that fabrication because, unlike $0 < x < 1$, the inequality $x > 0$ is both necessary and sufficient for the fabrication when added to inequalities (5). One can note that only the inequality $x < 1$ was hidden in inequalities (5) and thus it was not necessary to be added to (5) for the fabrication of the bounded part of Ω_1. It appears that hidden inequalities are either unnecessary for digital fabrication of regions formed by the graphs of functions or they are neither necessary nor sufficient for the digital fabrication.

DIGITAL FABRICATION WITH EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Consider the system of inequalities

$$y > 2^{|x|}, y < \log(100|x|) \qquad (7)$$

where the notation log stands for the base-ten logarithm. Whereas adding to (not hidden) inequalities (7) either $x > 0$ or $x < 0$ results in one of the two connected loci, for the construction of the borders hidden inequalities are needed. The locus of system (7) is shown in Figure 6. Similar to Figure 5, the locus comprises two disjoint regions and if one needs to digitally fabricate one of the regions, one has to use hidden inequalities that define the regions separately. This time, analytic solution of the inequality $2^{|x|} < \log(100|x|)$ cannot be found, only parts of the hidden inequalities can be found as both sides turn into two when $x = 1$ because of the coefficient 100, which is a special case. Nonetheless, the missing boundaries can be found graphically by graphing the equation $2^{|x|} = \log(100|x|)$. We have $x \approx \pm 0.122673$. Graphing the systems of inequalities $|y - 2^{|x|}| < \varepsilon, 0.122673 < |x| < 1$ and $|y - \log(100|x|)| < \varepsilon, 0.122673 < |x| < 1$ for a sufficiently small ε yields the digital fabrication (Figure 7, $\varepsilon = 0.01$) of the border of locus defined by inequalities (7) within which the inequalities $0.122673 < |x| < 1$ are hidden and can be revealed through cursor pointing.

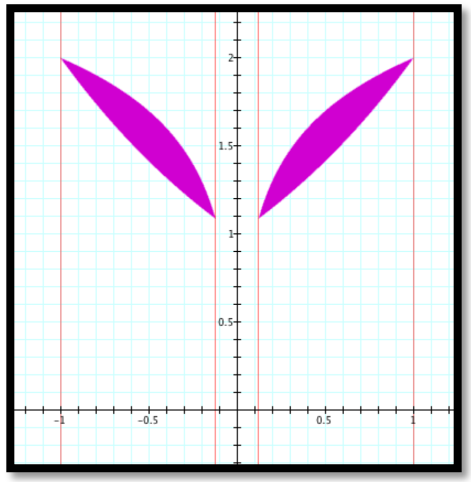

Figure 6. Revealing hidden inequalities by solving an equation graphically.

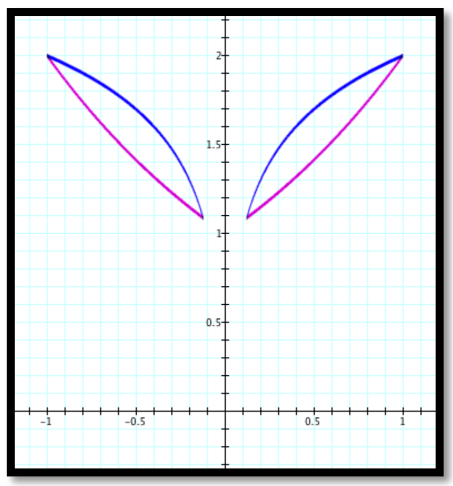

Figure 7. Fabricating borders.

DIGITAL FABRICATION WITH CIRCULAR FUNCTIONS

Digital fabrication with circular functions differs from the cases considered above in part because of the periodicity of the functions. Indeed, if the functions $y = A\sin x$ and $y = kx$ are used in the formation of a two-dimensional locus, then the number of connected regions that comprise the locus depends on the values of the amplitude A and the slope k. Consider, for example, the functions $y = \sin x$ and $y = kx$. The locus of the relation $kx = \sin x$ constructed in the (x, k) plane (Figure 8) can be used to determine positive values of k that produce a specified number of connected loci defined by the system of inequalities

$$y < \sin x, y > kx, k > 0, k < 1. \qquad (8)$$

Note that when $k \geq 1$, the line $y = kx$ has the origin as the only point in common with the graph of the function $y = \sin x$.

In this context, one can be asked to digitally fabricate the borders of the region defined by inequalities (8) under the condition that in the first quadrant the straight line $y = kx$ is tangent to the graph of the function $y = \sin x$, $0 < k < 1$, as shown in Figure 9. The case of tangency shown in Figure 9 can be defined as follows: in the (x, k)-plane, the line $k = k_0, k_0 > 0$, has exactly two points in common with the locus shown in Figure 8, one of which is the point of tangency. This time, setting $f(x) = \frac{\sin x}{x}$, the point of tangency has to be defined analytically as one of the solutions of the equation $f'(x) = 0$. Differentiation yields $x \cos x = \sin x$ whence, taking into account the sketch of Figure 8, $x \approx 7.72525$. In order to find the x-coordinate of the first point where the line $k = k_0$ crosses the locus as shown in Figure 8, one has to solve the equation $\frac{\sin x}{x} = \cos(7.72525)$ whence $x \approx 2.77706$. Now, by graphing the borders of the locus defined by inequalities (8) where $k = \cos(7.72525)$, through adding the inequalities $0 < x < 2.77706$ (where right-hand side inequality was hidden) one gets the connected region shown in Figure 10.

The curvilinear and the straight borders are defined, respectively, by the inequalities $|y - \sin x| < \varepsilon, x > 0, x < 2.77706$ and $|y - \cos(7.72525) \cdot x| < \varepsilon, x > 0, x < 2.77706$ for sufficiently small value of ε (in Figure 10 we have $\varepsilon = 0.01$). This is another way of revealing hidden inequalities in the case of a circular function. It shows that hidden inequalities emerge from the process of differentiation followed by equation solving by taking into account specific relations between the functions involved. Other examples of using circular functions in digital fabrication may be considered.

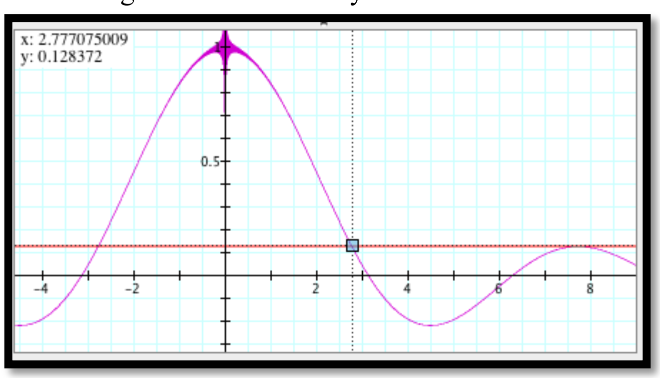

Figure 8. Locus as support system in locating hidden inequalities.

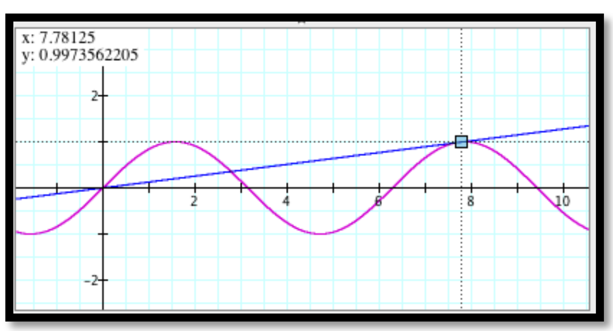

Figure 9. The line $y = kx$ is both secant and tangent to the graph of $y = \sin x$.

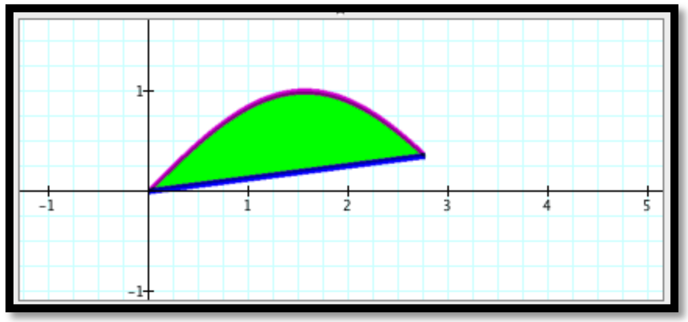

Figure 10. Completing digital fabrication using a hidden inequality.

CONCLUSION

The paper introduced the notion of hidden inequalities that one needs to consider in the context of digital fabrication of different two-dimensional loci when using computer-graphing software. Such tool-oriented perspective on an important concept of secondary school mathematics curriculum can be used to focus on some of the conceptual requirements necessary to solve practical problems in engineering and science. This not only provides an important link to related STEM areas but it also brings out the need for a greater degree of both content knowledge and the appreciation of metacognitive monitoring.

As the examples in the paper served to illustrate, digital fabrication, utilizing the basic functions of traditional school curriculum in an applied problem-solving context, requires much deeper conceptual understanding of mathematics than what one might initially expect. Recognizing the didactic value of hidden inequalities is an example of how procedural knowledge – construction of graphs – can be connected to conceptual knowledge of algebra through digital fabrication. For example, when the locus needed for digital fabrication consists of several disjoint regions in the plane, one-variable inequalities used in defining the range for the independent variable must be explicitly set out.

When students experience mathematics in the context of solving problems with an engineering/science focus, as modeled in the paper, they must "bring two complimentary abilities to bear on problems involving quantitative relationships: the ability to decontextualize … and the ability to contextualize" (Common Core State Standards, 2010, p. 6). Even if the specific example might differ, this observation serves to underscore the importance of incorporating elements of engineering/science education into the teacher education curriculum. Indeed, one can be reminded in the collateral learning format that circular (alternatively, trigonometric) functions "proved to be admirably suited for the study of sound, electricity, radio, and a host of other oscillatory phenomena" (Kline, 1985, p. 417). Teacher candidates cannot be expected to master the collateral learning pedagogy they neither understand nor have experienced. In order to enable meaningful introduction of schoolchildren to ideas that develop core STEM abilities (Bull, Knezek, & Gibson, 2009), practicing teachers must have experienced those ideas at some level personally.

Digital fabrication, being a social movement of turning ideas into things (Blikstein, 2013), is also a natural problem space to develop such experiences. As shown in the paper, when depending on a graphing context, hidden inequalities may or may not affect the outcome of digital fabrication. With this in mind, the paper demonstrated different methods of revealing hidden inequalities. These included purely analytical methods, like in the case of quadratic and cubic functions, and a combination of analytical and numerical methods, like in the case of logarithmic, exponential, and trigonometric functions.

Through these methods, using problems of digital fabrication as context, teacher candidates can experience problems that are rich in both mathematical and engineering sophistication. Furthermore, these problem situations can then be used by a mathematics education professor as a mechanism to develop links between procedural skills of using technology in constructing graphs of functions, the factual graphs knowledge related to identifying the loci necessary for the fabrications to be created, and the conceptual knowledge required for the fabrication of the specific parts of these graphs and their intersections. Developing such links requires robust understanding of algebraic inequalities and an ongoing mindful interaction between the human and the computer.

The very notion that "almost anything" (Gershenfeld, 2005, p. ix) can be fabricated in the digital context by using mathematical concepts as tools in computing applications could be a true cognitive motivation for the learners of secondary mathematics. While it appears that inequalities are the primary mathematical tools of digital fabrication at the secondary level, the methods of defining the boundaries of shapes to be fabricated, as has been demonstrated in the paper, may be different in a sense that each new method reveals a new hidden mathematical concept and/or structure. Therefore, further research on using digital fabrication as a learning environment conducive to revealing other elements of hidden mathematics curriculum is worth pursuing. Through this kind of research mathematics educators can create new collateral learning opportunities for unintentional discoveries by "mathematically proficient students" (Common Core State Standards, 2010).

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