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### **Facilitating Pre-service Mathematics Teachers' Understanding of Directly and Inversely Proportional Relationships using Hands-on and Real-World Problems**

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# Facilitating Pre-service Mathematics Teachers' Understanding of Directly and Inversely Proportional Relationships using Hands-on and Real-World Problems

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## Abstract

The purpose of this study was to examine four pre-service middle and high school mathematics teachers' understanding of the directly and inversely proportional relationships presented through hands-on and real-world missing-value word problems. The problems included single and multiple proportions, and the PSTs were provided with plastic gears and a mini number balance system in the hands-on problems. A case study methodology with multiple cases was used in designing this study. Analysis of the semi-structured interviews revealed that for the PSTs, identifying directly proportional relationships was easier than identifying inversely proportional relationships. The PSTs recognized inversely proportional relationships in the hands-on problems but had difficulty recognizing them in the missing-value word problems. Therefore, the inclusion of the hands-on problems appeared to facilitate the PSTs' understanding of the inversely proportional relationships. When forming multiplicative relationships between quantities, they mostly reasoned about within measure spaces. Furthermore, when determining directly and inversely proportional relationships, they usually attended to the qualitative relationships and constancy of the rate of change. In the multiple proportions problems, the PSTs had difficulty finding a mathematical formula that expressed relationships between quantities compared. This result suggested possible constraints in their understanding of the multiplicative relationships when more than two quantities were present.

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## Introduction

Understanding ratios, proportions, and proportional reasoning constitutes a key area of school mathematics that is essential for students to learn but difficult for teachers to teach (Lobato & Ellis, 2010, p. 1). Proportional reasoning is also essential in understanding many situations in science and everyday life (Cramer & Post, 1993). For example, in the middle grades, students need to understand and use ratios and proportions to represent quantitative relationships (National Council of Teachers of Mathematics [NCTM], 2000), and they should be able to analyze proportional relationships and use them to solve real-world and mathematical problems (Common Core State Standards Initiative [CCSSI], 2010). Therefore, proportional reasoning plays an important role in students' mathematical development, and it is an essential concept in their elementary school mathematics and higher mathematics (Lesh, Post, & Behr, 1988).

Karplus, Pulos, and Stage (1983a) describe proportional reasoning as a term that indicates reasoning in a "system of two variables between which there exists a linear functional relationship" (p. 219), and for them proportional reasoning leads one to reach conclusions about a condition or phenomenon that can be explained by a constant ratio. Lesh et al. (1988) view proportional reasoning as a form of mathematical reasoning that entails "a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information" (p. 93). Furthermore, according to Lamon (2007), proportional reasoning refers to "detecting, expressing, analyzing, explaining, and providing evidence in support of assertions about proportional relationships" (p. 647). In this study, Lamon's (2007) definition is used to make sense of the PSTs' responses.

The Common Core State Standards for Mathematics states that being able to reason proportionally, students should "Decide whether two quantities are in a proportional relationship (7.RP.2a)" (CCSSI, 2010, p. 48). There are two types of proportional relationships between quantities: directly proportional relationships and inversely proportional relationships. Lamon (2007) states that  $y = k * x$  is the mathematical model for a directly proportional relationship. In this model, the variables  $y$  and  $x$  represent quantities that form a constant ratio, and

the amount  $k$  represents the constant of proportionality. In a directly proportional relationship, the quotient of the two covarying quantities always remains constant. Whereas, the mathematical model for an inversely proportional relationships is  $y * x = k$ , in which the product of the values of quantities always remains constant.

Despite the given importance and effort spent on teaching ratios and proportions, many studies (e.g., Clark, 2008; De Bock, Verschaffel, & Janssens, 1998; Hart, 1984; Modestou & Gagatsis, 2007) have reported students' difficulties and poor performances on proportional relationships. Furthermore, a small number of studies (e.g., Ben-Chaim, Keret, & Ilany, 2007; Cramer, Post, & Currier, 1993; Hull, 2000; Izsák & Jacobson, 2013; Simon & Blume, 1994) on pre-service teachers' (PSTs) reasoning about proportional relationships have reported that PSTs' difficulties are often similar to those of students.' One of the problems of teaching and learning proportional relationships is that traditional proportion instruction places an emphasis on rule memorization and rote computations (Izsák & Jacobson, 2013; Singh, 2000). Hence, students and teachers depend severely on using the cross-multiplication strategy and rote computations when solving proportion problems (Arıcan, 2018a; Arıcan, 2018b). Although the cross-multiplication algorithm is effective in solving proportion problems, it hinders students' and PSTs' understanding of multiplicative relationships between quantities (Farrell & Farmer, 1985; Lamon, 2007; Riley, 2010; Singh, 2000; Stemn, 2008).

A second problem with teaching and learning proportional relationships is that students and PSTs usually tend to judge nonproportional relationships to be proportional and apply proportional strategies for nonproportional situations (e.g., Arıcan, 2018b; De Bock et al., 1998; Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Modestou & Gagatsis, 2007; Riley, 2010; Singh, 2000; Van Dooren, De Bock, Janssens, & Verschaffel, 2007). In addition, the following problems have been also reported in the literature: Students and PSTs are likely to use additive strategies to solve proportion problems (Hart, 1984; Misailadou & Williams, 2003; Riley, 2010; Simon & Blume, 1994); they have difficulty creating suitable reciprocal multiplicative relationships for nonproportional problems (Izsák & Jacobson, 2013); and they have difficulty understanding ratio-as-measure and the invariance of a ratio (Simon & Blume, 1994).

In contrast to the large body of studies on students' proportional reasoning, a small number of studies aimed at investigating PSTs' and teachers' proportional reasoning (Lamon, 2007; Lobato, Orrill, Druken, & Jacobson, 2011). In particular, only a few studies have reported PSTs' and teachers' proportional reasoning regarding inverse proportions (e.g., Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Riley, 2010), and even fewer researchers have studied multiple proportions (e.g., Vergnaud, 1983, 1988, 2009). Hence, there is a need for an investigation on examining PSTs' understanding of directly and inversely proportional relationships in the mathematics education literature. PSTs must have a robust understanding of directly and inversely proportional relationships to develop their future students' comprehension of these important concepts. Therefore, this study contributes to the current literature on mathematics education by investigating PSTs' understanding of directly and inversely proportional relationships.

In the existing studies on proportions, researchers investigated teachers' proportional reasoning mostly using missing-value word problems, which usually involved a single proportional or nonproportional relationship. Similarly, instruction on proportions traditionally uses missing-value word problems in teaching. Hence, in this study, a combination of hands-on, in which plastic gears and a mini number balance system were provided, and real-world missing-value word problems were used in collecting the PSTs' responses. The use of plastic gears and the mini number balance system expected to provide hands-on experiences and generate a checking mechanism for PSTs, which would eventually help them have a robust understanding of directly and inversely proportional relationships. Furthermore, in addition to the single proportion problems, the multiple proportions problems were also included in the study. By the inclusion of the multiple proportions problems, the PSTs' proportional reasoning in complex situations was investigated. Thus, the following research questions were investigated in this study:

1. How do pre-service mathematics teachers reason about proportions in single and multiple proportions problems?
2. How does inclusion of hands-on problems contribute to the pre-service mathematics teachers' understanding of directly and inversely proportional relationships?

## Theoretical Framework

The theoretical framework of this study was developed drawing on the multiplicative conceptual field theory (e.g., Vergnaud, 1983, 1988, 2009). The multiplicative conceptual field theory was used in explaining

multiplicative structures of the problems presented in the mathematical tasks and interpreting the PSTs' responses.

### The Multiplicative Conceptual Field Theory

According to the draft Ratio and Proportional Relationships Progression (Common Core Standards Writing Team, 2011), "The study of ratios and proportional relationships extends students' work in measurement and in multiplication and division in the elementary grades" (p. 2). Moreover, elementary school allows students to develop meanings for fractions and multiplication that have important foundations on which students build an understanding of ratios, proportions, and proportional reasoning (Lobato & Ellis, 2010). Therefore, as stated by Vergnaud (1983, 1988), multiplication, division, fractions, ratios, and rational numbers are not mathematically independent of one another. Therefore, Vergnaud placed these concepts within a larger context that he called the *multiplicative conceptual field*. For Vergnaud (1988), a conceptual field is "a set of situations, the mastering of which requires mastery of several concepts of different natures" (p. 141). Two main conceptual fields are additive field and multiplicative field (Vergnaud, 1983, 1988). Additive field include a set of problems involving addition, subtraction, difference, interval, and translation; and multiplicative field include a set of problems involving multiplication, division, fraction, ratio, and similarity (Vergnaud, 1983, 1988).

One of the persistent issues in proportions is the confusion about the classification of students' strategies as either within or between (Lamon, 2007). For Lamon (2007), the confusion results from different uses of the terms *within* and *between* in the earlier research. Lamon (2007) stated that using terminologies "within or between measure spaces" or "within or between systems," which were used by Vergnaud (1983, 1988), can help us eliminate this confusion (p. 634). According to Lamon (2007), "Measure spaces usually refer to different sets of objects, different types of quantities, or different units of measure" (Lamon, p. 634). For example, if a certain car covers a distance of 60 miles in 1 hour, then driving at the same rate it can cover 120 miles in 2 hours. In this example, our measure spaces are: the distance (Measure space one or M1) and the number of hours (Measure space two or M2). Therefore, the ratio 60 miles/120 miles and 1 hour/2 hours are examples of within measure space ratios. On the other hand, the ratio 60 miles/1 hour and 120 miles/2 hours are examples of between measure space ratios. Vergnaud (1983, 1988) discussed three kinds of multiplicative structures: isomorphism of measures, product of measures, and multiple proportions other than product. In the isomorphism of measures structure, there is a simple direct proportion between two measure-spaces M1 and M2 (Vergnaud, 1988, p. 129). The product of measures structure "consists of the Cartesian composition of two measure-spaces, M1 and M2, into a third, M3" (p. 134). The problems in this structure are concerned with area, volume, Cartesian product, and work. In the multiple proportions other than product structure, a measure-space (M3) is proportional to two different independent measure-spaces (M1 and M2) (Vergnaud, 1988, p. 138). This type of proportional relationship is also called a *jointly proportional relationship*.

Vergnaud (1983) distinguished the second and third multiplicative structures as follows: In the product of measures structure, the constant of the proportionality is equal to 1, but the same is not true for the multiple proportions other than product structure. For example, the distance (D) a car traveled in some fixed time (T) at a constant speed (V) can be expressed by the equation  $V * T = D$ . In this example, the equation demonstrates a product of measures structure because the constant of the proportionality (k) is 1. On the other hand, the multiplicative relationships among the number of people (P) works in a bakery, number of minutes (M) that they need to frost some number of cupcakes (C) can be expressed by the equation  $P * M = kC$ , where  $k \neq 1$  represents the constant of proportionality, which is interpreted as "number of person-minutes per cupcake." Although the product of measures and multiple proportions other than product structures contain an inversely proportional relationship between quantities multiplied, Vergnaud (1983, 1988) did not focus on this relationship in detail. Vergnaud (1988) called the problems of the type of isomorphism of measures as simple proportion problems and called the problems of the last two structures as multiple proportions problems. As noted by Vergnaud (1988), multiple proportions problems have not been researched widely, and most teachers are unaware of students' difficulties with these problems.

## Method

### Participants and Context

A multiple-case study methodology (Yin, 1993, 2009) was used in designing this study. Because the purpose of this study was to explore the PSTs' proportional reasoning, each individual participant constituted a case. The

study was conducted in one large public university in the Southeast United States. This university was selected because it offers separate programs leading to certification for secondary grades (6-12) mathematics teachers and middle grades (4-8) mathematics teachers. The secondary grades program includes one content course with a focus on multiplicative relationships, ratios, and proportions; the middle grades program includes two such content courses. Since the focus of this study involved challenging mathematical problems, PSTs with some college level experience on direct and inverse proportions were preferred. Therefore, the PSTs who attended or were attending one of these two courses were selected. In the fall semester of 2014, I contacted two course instructors, and 10 PSTs (six from secondary grades program and four from middle grades program) agreed to participate in this study. Based on the information that they gave about their background on the direct and inverse proportions, two female secondary grades (Amy and Mary) and two female middle grades (Natalie and Emily) PSTs were recruited for the study. To maintain confidentiality, all real names were replaced with pseudonyms. Moreover, the PSTs were given 10 dollars incentive for each hour of participation.

### Data and Analysis

The data was collected through semi-structured clinical interviews (Bernard, 1994). Two video cameras were used during the interviews: One focused on the participant's written work, and the other focused on the participant and the interviewer. I conducted all of the interviews, and one graduate student operated the video cameras. Each interview lasted between 45 minutes and 90 minutes. A total of 18-hours video data were collected for this study. The steps that I followed to collect and analyze the interview data were as follows: First, I conducted semi-structured interviews for data collection. Second, the interviews were transcribed verbatim. Third, I open coded the interview transcripts line-by-line for the PSTs' reasoning strategies, and if necessary wrote memos about these strategies. Fourth, I created a code file for each task in Microsoft Excel. Then, I counted the number of occurrences of each code and entered that number in the record for the task. Fifth, I returned to the interview transcripts and recoded these to strengthen the reliability of the results. I then aggregated similar codes together to determine the connections among the codes and to identify relationships. In the last step of the data analysis, I wrote cross-task analyses of each case based on the reasoning strategies that I identified.

### Mathematical Tasks

The study involved two hands-on—Gear and Balance—and two real-world tasks—Bakery and Speed (Table 1). For the purposes of this study, I developed the Gear and Balance tasks and adapted Bakery and Speed tasks from Beckman (2011). The mathematical tasks involved either single or multiple proportional relationships and all of the problems were missing-value word problems. In the Balance and Gear tasks, the PSTs were provided with a mini number balance system and plastic gears, respectively (Figure 1). In Table 1, some of the problem examples were presented; however, during the interviews, the PSTs were provided with problems that involved varying numbers and letters.



Figure 1. Meshed plastic gears and a mini number balance system (Images retrieved from <https://www.pexels.com/photo/3d-co-ordination-gear-gears-317089/> and <http://www.eaieducation.com>)

Table 1. Descriptions of the problems

Task	Problem Examples	Problem Descriptions
Gear	<p>a) Please calculate the number of notches of Gear B, with a 2-cm radius, given that Gear A had a 3-cm radius and 12 notches.</p> <p>b) Please calculate the number of revolutions of Gear A, given that Gear B revolved 6 times.</p>	There was a directly proportional relationship between the size of a gear and its number of notches, and an inversely proportional relationship between the number of revolutions that a gear made and its size. (Single Proportion)
Bakery	<p>At a bakery, 3 people can frost a total of N cupcakes in T minutes.</p> <p>a) How many cupcakes could be frosted by 2 people in T minutes?</p> <p>b) How long would it take for 1 person to frost 2N cupcakes?</p>	The PSTs explored one inversely and two directly proportional relationships among the number of people, number of cupcakes, and number of minutes. (Multiple proportions)
Balance	W1 numbers of weights were hung on one side of the balance that has a D1 distance from the center. To balance the system on the other side, you want to hang W2 number of weights. What would be the distance, D2, in terms of D1, W1, and W2, so the system could be balanced?	The PSTs explored an inversely proportional relationship between the number of weights hung and distance (how far from the center a weight hung). (Single Proportion)
Speed	Knowing that a car covers the distance between two markers in 90 seconds driving at 60 mph then what must be its speed to cover the same distance in 60 seconds?	The PSTs investigated one inversely and two directly proportional relationships among the distance, speed, and time. (Multiple proportions)

In the direct proportion problems, there was a constant ratio relationship between two quantities. For example, in the Gear task, there was a constant ratio relationship between the size of a gear and its number of notches. The PSTs were provided two gears, which were thought to be meshed, and they were asked to calculate either the number of notches or the radius of the gears. The mathematical structure of the relationship between the radius ( $r$ ) and number of notches ( $N$ ) can be expressed with the statement  $r$  (cm) \*  $N/r$  (notches/cm) =  $N$  notches in which  $N/r$  (notches/cm) can be interpreted as  $N/r$  notches per 1 cm radius. This mathematical statement can be best explained with a *multiple batch perspective*. Assuming “ $N/R$  notches per 1 cm” as a batch, which is a unit rate, an expert can iterate this batch  $r$  times to calculate  $N$  notches.

In the inverse proportion problems, there was a constant product relationship between the two covarying quantities. For instance, in the Gear task, the product of the number of revolutions ( $R$ ) and number of notches ( $N$ ) yielded the total number of notches revolved on a gear, which was constant. Hence, the multiplicative structure of this constant relationship could be expressed with the statement  $R$  (revolutions) \*  $N$  (notches/revolution) =  $NR$  (notches). In the Balance task, because the PSTs needed to balance the system on two sides, they were required to have the same value, which was determined by multiplying the number of weights and the distance from the center of the system, on both sides. Moreover, in the Bakery task, there was an inversely proportional relationship between the number of people and number of minutes. Therefore, the contexts of the Gear, Balance, and Bakery tasks were appropriate for facilitating the PSTs’ determination of inversely proportional relationships.

In the multiple proportions problems, identifying multiplicative relationships were expected to be more difficult than single proportion problems because they involved more than two quantities. Hence, the PSTs needed to fix the value of a quantity as constant to determine the relationship between the remaining two quantities. In the Bakery task, the relationships among the number of people ( $P$ ), number of minutes ( $T$ ), and number of cupcakes ( $C$ ) can be expressed with the statement  $P$  (people) \*  $T$  (minutes) =  $C$  (cupcakes) \*  $PT/C$  (person-minutes/cupcake), where “person-minutes per cupcake” represented the constant of proportionality. Whereas, in the Speed task, the relationships among the distance ( $D$ ), time ( $T$ ), and speed ( $V$ ) can be expressed by the equation  $T$  (hours) \*  $V$  (miles/hour) =  $D$  (miles).

**Results and Discussion**

The analysis of the selected PSTs’ reasoning on the direct, inverse, and multiple proportions problems was presented for some tasks in the following pages. There were no deletions in the transcripts provided. Pauses were shown with ellipses and actions were described within square brackets.

**The Pre-service Teachers’ Reasoning on the Direct Proportion Problems**

Amy

Amy usually attended to numerical multiplicative relationships between measure spaces when determining directly proportional relationships between two covarying quantities. For example, in the Speed task, she successfully calculated the speed of a car, given that it covered 2 miles in 100 seconds, to be 72 mph using a scientific unit conversion method (Figure 2a). When asked if there was a relationship between the distance and time, she correctly determined a proportional relationship and generated the ratio table in Figure 2b.

Amy: It should be proportional I think. Yeah this should be proportional.

Int: Please tell me why it is proportional.

Amy: I mean, so you are going at a constant speed, okay so then miles and seconds [drawing a ratio table], miles is here and seconds and then we know this relationship 2 and 100 and we know 1 is 50 and 3 is 150 and 4 is 200 and so on.

Int: So, then you think it is proportional?

Amy: Yeah because all these are all these have to same like ratio this 1 over 50, and 2 over 100 is going to be 1 over 50...3 over 150 is 1 over 50, it keeps going.

The exchanges and Figure 2b confirmed that Amy’s reasoning was sufficient to see that driving at a constant speed was yielding a directly proportional relationship between the distance and time. Because Amy did not consider the referent units in her explanation of the directly proportional relationship between miles and seconds, she appeared to attend to the numerical multiplicative relationship between numbers and constancy of the quotients when justifying her determination of the proportional relationship.

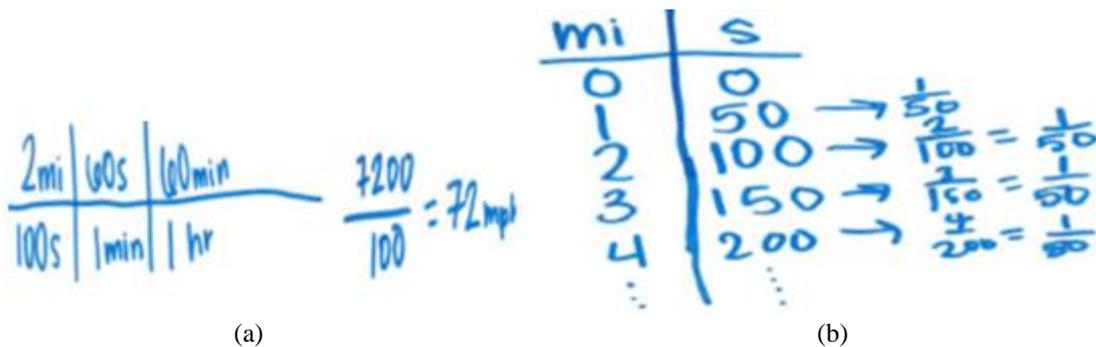


Figure 2. (a) Amy’s scientific unit conversion method; (b) Amy’s expression of the directly proportional relationship between distance and time

Mary

Although, Mary recognized the multiplicative relationships in some tasks, her main reasoning strategy for determination directly proportional relationships was that she generally attended to constancy of rate of change and qualitative relationships. For instance, in the Bakery task, when asked to talk about the relationship between the number of people and number of cupcakes, Mary used the information confirming that three people frosted 12 cupcakes in T minutes to draw a linear graph (Figure 3). She then determined the relationship as a proportional relationship.

Int: What is the relation..., what do you think the relation is?

Mary: Between cupcakes and people?

Int: Yeah.

Mary: They are proportional.

Int: What was your main idea to graph it? Like, when I ask you to identify the relationship between these two, like the number of cupcakes and people, you said I can graph it. What was the reason for graphing to identify the relationship?

Mary: So, I could show there was a linear relationship. So, that the...the ratio..., there is a constant ratio between the people and the cupcakes.

According to these exchanges, Mary's reason for drawing the graph was to show that it was linear and that there was a constant ratio relationship between the number of people and cupcakes. Her strategy of drawing a graph to determine a relationship suggested a possible coordination of a directly proportional relationship with the linearity of its graph.

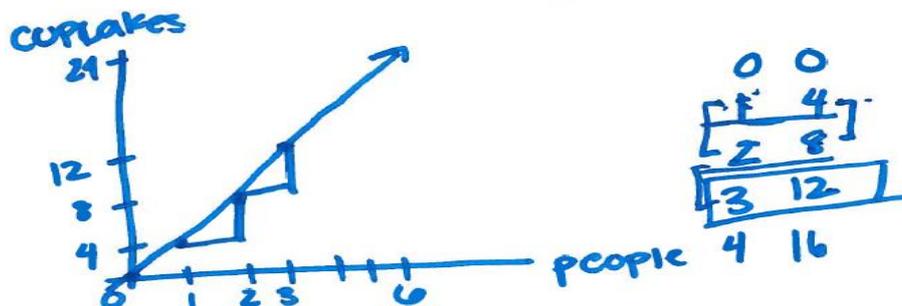


Figure 3. Mary's expression of the proportional relationship between the number of people and number of cupcakes

The exchanges above also included the first instance of Mary's mentioning the term constant ratio relationship. When asked what she meant by the constant ratio, Mary explained:

Mary: Uhh, so we have 0, 0; 1, 4; 2, 8; 3, 12, and umm so this [pointed at 12 cupcakes] is three times the amount of the one person; this [pointed at eight cupcakes] is two times the amount of whatever is made by the first person.

Int: So, the ratio is here?

Mary: I don't know. It increases by four every time. Whatever you do to the ... like if you look at the original, the 1 to 4. Whatever you do from the 1 to get to the...any amount of people that is you do to the amount of cupcakes.

These exchanges suggested that Mary was attending to a multiplicative relationship within measure spaces, "...so this [pointed at 12 cupcakes] is three times the amount of the one person, this [pointed at eight cupcakes] is two times the amount of whatever is made by the first person," and the constancy of the increments within measure spaces, "It increases by four every time." Although Mary used the term constant ratio relationship, there was not any indication that she was attending to the multiplicative relationship between measure spaces. Hence, these data suggested that she might have used the term constant ratio relationship to indicate constancy of the rate of change. Her ratio table in Figure 3 confirmed this conjecture about her reasoning because she was attending to the constancy of the increments within measure spaces. Therefore, these exchanges and Figure 3 implied that Mary's determination of the constant ratio relationship between the number of people and cupcakes was mainly based on her attention to the constancy of the rate of change.

### Natalie

Natalie usually made numerical multiplicative comparisons between measure spaces when solving the Gear problems. In the remaining tasks, she inclined towards to the multiplicative relationships within measure spaces.

For instance, in the Gear task, setting up a  $\frac{3 \text{ cm}}{10 \text{ notches}} = \frac{\frac{6}{5} \text{ cm}}{x \text{ notches}}$  proportion and cross-multiplying the values, Natalie calculated the number of notches of Gear B, with a  $\frac{6}{5}$  cm radius, to be four notches, given that Gear A had a 3-cm radius and 10 notches. When asked if she could use another strategy, Natalie said she could make a table (Figure 4) and later explained:

Natalie: So, you can do...let's do it for every, since we are working with this, let's do it for every fifth of a centimeter. So, for one notch gives you...let's do it per notch instead. So, you have notches here and then centimeters, you know for 10 notches you have 3 centimeters. And then you know for 1 notch

you have three-tenths of a centimeter. So, for 2 notches you are going to have 0.6 because for 1 notch is 0.3. If you add another notch for 0.3 centimeters, you just add 0.6. So, then you have for 3 notches, it's going to be 0.9 because you added another 0.3 and then for 4 notches, it's going to be 1.2 and so on until you get 3 centimeters for 10. And you would hopefully notice that six-fifth, you are looking for six-fifth, and since you are going to be working with decimals, I would say put that in decimal form first, which is 1.2 centimeters. And then once you hit this 1.2 you would see, oh, okay, for 1.2 centimeters, I would have 4 notches.

In her explanation, Natalie appeared to attend to the repeated addition of batches as indicated by her double counting approach (Figure 4). First, knowing that there were 10 notches for a 3 cm radius, she determined a 1-notch-to-0.3-cm radius relationship as one batch, which was the unit rate. By repeatedly adding this batch, she calculated a 1.2-cm-radius-to-4-notches relationship. Hence, to a certain degree, her reasoning in this task was attuned to the multiple batches perspective.

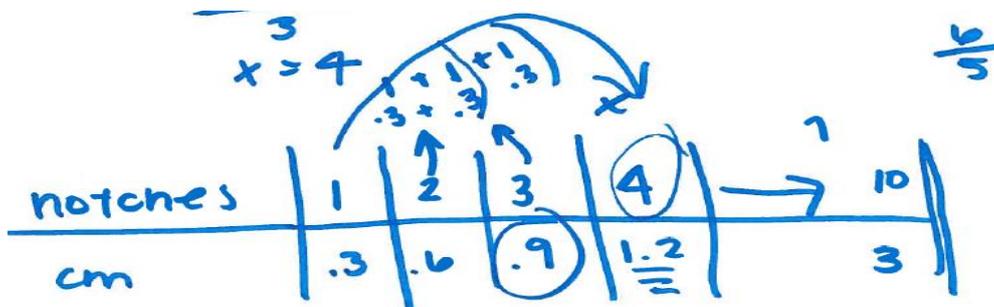


Figure 4. Natalie’s ratio table strategy

A few exchanges later, when the second interviewer asked how she was making sense of her table from the meaning of proportional relationships, Natalie explained:

Natalie: It just shows that the proportions... all of these relationships, all these ratios [pointed at 1 notch and 0.3 cm] are equivalent. The proportions are equal; they stay consistent throughout the table. So, but if this was 3 and this was 0.8, it would not be proportional. If this was anything but 0.9 for 3 notches it would not be proportional.

Int: It is because?

Natalie: Because of the unit rate we found from the original ratio we were given.

Natalie’s explanation “So, but if this was 3 and this was 0.8, it wouldn’t be proportional. If this was anything but 0.9 for 3 notches it would not be proportional” provided evidence for her understanding of a constant ratio relationship. Therefore, these data showed that Natalie’s attention to the unit rate assured the correct conclusion that the number of notches and radii were in a constant ratio.

*Emily*

Emily’s reasoning was very much similar to Natalie’s. She made numerical multiplicative comparisons between measure spaces when solving the gear problems but inclined towards to the multiplicative relationships within measure spaces in the remaining tasks. For example, in one of the gear problems, I asked Emily to calculate the size of Gear D with 21 notches, given that Gear E with a 4-cm radius had 14 notches. She explained:

Emily: Okay. I can just do the same as I did before. So, I have 4 over 14 which is two-sevenths. So, this 4 cm is two-sevenths of 14 notches, and so I have to ask myself what x amount of cm is two-sevenths of 21. So, I have 21 times two-sevenths which is 42 over 7 which is 6 cm, yeah for D.

Although Emily considered the referent units in the statement, “...4 cm is two-sevenths of 14 notches,” her reasoning suggested that she might have been attending to a numerical multiplicative relationship between the radius and number of notches. Emily’s statement, “...so I have to ask myself what x amount of cm is two-sevenths of 21,” showed that she was searching for a measurement division (e.g., Greer, 1992). Therefore, she multiplied 21 by 2/7 and found that the radius of Gear D was 6 cm. In the multiplication statement, Emily used 2/7 without its referent unit—cm/notch—and did not explain that it was showing the constant ratio relationship between the radii and number of notches. Therefore, she might have been attending to a numerical multiplicative relationship between measure spaces.

Some exchanges later, I asked Emily to calculate the number of notches on Gear B with a 6-cm radius, given that Gear A had a 3-cm radius and  $m$  notches. In her solution, Emily made multiplicative comparisons within measure spaces rather than making multiplicative comparisons between measure spaces and explained:

Emily: Well, I think since well I would say that if this is [pointing out 3 cm and  $m$  notches] unit rate so I would have basically  $1m$  like there is a certain amount of notches for 3 cm. So, since I do not know what this [pointed at the number of notches of Gear B] is and I have 6-cm, I double this [pointed 3-cm] so I would just say that  $2m$  will give me 6 cm so we are looking for.

In the previous problems of this task, Emily was given numbers that she used to identify the numerical multiplicative relationships between measure spaces. In the current problem, she identified the multiplicative relationship within measure spaces and correctly determined the number of notches to be  $2m$ . Emily's inclination towards reasoning within measure spaces might have been based on the number of notches being represented by the letter  $m$  and the fact that doubling the numbers of centimetres and notches allowed working with the same units.

### The Pre-service Teachers' Reasoning on the Inverse Proportion Problems

*Amy*

In one of the problems of the Gear task, Amy needed to calculate the number of notches of Gear K, given that it completed two-thirds of a revolution, when Gear F, with eight notches, revolved three times. Amy explained:

Amy: Let's see revolves three times and has eight notches, so that means like in total it goes through 24 notches right? Because it revolves three times with eight notches, so this is [pointed at Gear F], it goes through 24 notches in this three revolutions...and then K revolves two-thirds of a time and goes through the exact same number of notches.

Int: How do you know it goes through the same number of notches?

Amy: Because I guess like the reason why I am thinking that is because like in these three revolutions it [pointed at Gear K] goes through...goes through two-thirds of a revolution but because we know that F has eight notches we know that in total in three revolutions it is going to go through 24, so that we can say that. Okay I know, that [pointed at Gear K] is going to go through 24 notches in two-thirds of a revolution.

For Amy, Gear F was "going through" a total of 24 notches in three revolutions, which she calculated by multiplying three revolutions and eight notches. By stating, "...K revolves two-thirds of a time and goes through the exact same number of notches," Amy determined that both gears were rotating the same total number of notches. Hence, for Amy, Gear K needed to "go through" 24 notches in two-thirds of a revolution. Later, multiplying 24 notches by  $3/2$ , she correctly calculated the number of notches of Gear K to be 36 notches. Amy's explanation in this problem provided evidence for her consideration of 24 notches as constant when explaining the inverse relationship between the number notches and revolutions.

Some exchanges later, Amy worked on a new problem. In this problem, she needed to calculate the number of revolutions of Gear L, with eight notches, given that Gear M had 14 notches and revolved four times. She used the same idea—two gears go through the same number of notches—and calculated the answer to be seven revolutions. When asked if she could use a ratio table strategy to solve the same problem, Amy was able to generate one for the relationship between the number of notches and revolutions depicted in the problem (Figure 5). She recognized that the product of all rows (notches and revolutions) was equal to 56:

Amy: Okay, so well that has to be 56, I mean this is 56 here. I just know, I just kind of know that like all of these, like these two [pointed at notches and revolutions] have to multiply to give me 56 like every single time. So, I am saying what times two is 56 and that is, I do not know, 28. And then  $56/3$ , I do not [know] what that is.

Int: You can leave like that. Knowing that 56, you said 56 is the?

Amy: is the product of notches and revolutions.

In the exchange, Amy used multiplication and attended to the multiplicative relationships between the number of notches and revolutions to discuss the constant product relationship and explicitly stated that 56 was the product of notches and revolutions. Thus, Amy's idea of two gears "going through the same number of

notches,” the exchanges, and Figure 5 suggested her understanding of the inversely proportional relationship between the number of notches and number of revolutions.

notches	revolutia
56	1
28	2
$\frac{56}{3}$	3
16	$3\frac{2}{3}$
14	4

Figure 5. Amy’s ratio table expressing an inversely proportional relationship

### Mary

Similar to Amy, Mary also recognized that two meshed gears had to rotate the same number of notches in some number of revolutions. In the given gear problems, she multiplied the number of notches and revolutions of one gear and equated this value to the product of number of notches and revolutions on the second gear. On the other hand, unlike Amy, who attended to multiplicative relationships, Emily usually attended to static points and swapping pair of numbers when characterizing inverse relationships. For example, when asked to draw the graph of the relationship between the number of notches and revolutions using the 8-notches-to-3-revolutions relationship, Mary drew an inversely proportional graph (Figure 6) and determined an inverse relationship between the number of notches and revolutions:

Mary: These are inversely related.

Int: Okay, how do you know these are inversely...?

Mary: Because when there is 24 notches, there is only one revolution, but when there is 24 revolutions, there is one notch and you can find the inverse relationship between them.

Int: When you are saying inverse relationship, what do you imply with inverse?

Mary: I mean, like 24 over 1 that is the inverse of 1 over 24.

Int: Okay, because of that reason, you think that is the inverse...?

Mary: Yes and you can find that throughout the entire graph.

Int: Do you mean you will have an inverse of some...one point appears here [I pointed at the inversely proportional graph]?

Mary: Every point on the graph has an inverse somewhere else on the graph.

Mary stated that  $\frac{24 \text{ notches}}{1 \text{ revolution}}$  and  $\frac{8 \text{ notches}}{3 \text{ revolutions}}$  ratios were the inverses of  $\frac{1 \text{ notch}}{24 \text{ revolutions}}$  and  $\frac{3 \text{ notches}}{8 \text{ revolutions}}$  ratios, respectively (Figure 6). Her statement “...24 over 1 that is the inverse of 1 over 24” suggested that she was attending to 24/1 and 1/24 being reciprocal of each other when explaining the inverse relationship. She also stated that “Every point on the graph has an inverse somewhere else on the graph.” It appeared that for Mary, the points (1 notch, 24 revolutions) and (24 notches, 1 revolution) were the inverses of each other because the values of the quantities were swapped. Therefore, Mary attended to the pair of values of two quantities being swapped when determining an inverse relation between the number of notches and revolutions. While Mary’s conjecture of pair of values being swapped in an inversely proportional graph was correct, it was not sufficient to distinguish proportional relationships from nonproportional relationships.

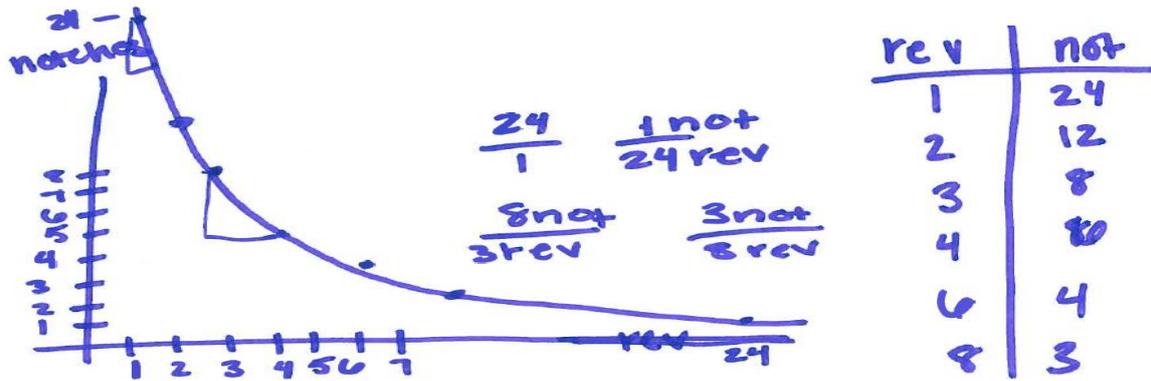


Figure 6. Mary's inversely proportional graph and ratio table

A few exchanges later, I asked Mary if she could express the relationship between the number of notches and revolutions with a ratio table. She generated the ratio table in Figure 6 and explained:

Mary: The amount of notches and the amount of revolutions...every time we multiply together, they equal 24.

In the explanation above, Mary explicitly stated the constant product relationship between the number of notches and revolutions. Because Mary also explained 24 as “the total notches travelled in both gears,” the context of the Gear task seemed to facilitate her understanding of this constant product relationship.

#### Emily

The two middle grades teachers, Natalie and Emily, both attended to inverse qualitative relationships—one quantity is increasing and the other quantity is decreasing—when determining inversely proportional relationships. Similar to Amy and Mary, they recognized constancy of the products in the Gear and Balance tasks but not in the remaining inverse proportion tasks. For instance, in the Balance task, Emily explored an inversely proportional relationship between the number of weights hung on a balance and the distance from the center of the balance. To clarify what the task was about, I explained that W1 number of weights was hung at D1 distance on one side of the balance, and W2 number of weights was hung on the second side. The first problem was about determining D2, the distance in the second side, in terms of D1, W1, and W2. By hanging different variations of weights, Emily recognized that the product of the number of weights and distance was equal on both sides. Therefore, she expressed the relationship between the number of weights and distance with  $W1 \cdot D1 = W2 \cdot D2$ . She then explained:

Emily: Well I think, well I was, I think that W1 or yeah W1 times the distance, distance one would have to equal, would has to equal W2 times the distance in order to balance [she wrote  $W1D1=W2D2$ ]. So, in order to find out what D2 is, I would just do W1 times D1 divided by W2.

Int: How do you know that [I pointed at  $W1D1$ ] one was equal to this two [I pointed at  $W2D2$ ] to balance?

Emily: Well if they are balanced, like they have to be the same on both sides in order to balance.

The exchanges provided evidence that Emily's recognition of the inversely proportional relationship between the number of weights and distance was based on her understanding of balancing, “Well if they are balanced, like they have to be the same on both sides in order to balance.” It appeared that Emily determined this inversely proportional relationship by experimenting on the balance system, but it was possible that she might have had past instruction on balancing and that might have helped her in determining the inversely proportional relationship. Hence, the context of balancing seemed to be helpful in Emily's determination of the inversely proportional relationship.

Some exchanges later, I asked Emily if she could generate a ratio table from the values of quantities that she used to balance the system on one side, given that on the other side six weights were hung on a 4-cm distance from the center. Using the balance formula, Emily multiplied 6 by 4 and got 24 and explained that she needed combinations of 24 on the other side:

Emily: Okay, so I guess the first way is try about it is that I have 4 cm and then six weights, so based on this  $W1D1=W2D2$ , I started off by thinking okay well I multiply this and that is 4 times 6 is 24 and...so, what are the like what are the...possible combinations that I can come up with. So, I can do...I can do 4 and 6, I can do...four weights on 6 cm away. So, I can try that first [she hung six weights at 4 cm], that works. And then I could do...three weights, so three weights on 8 cm.

Int: Do you want to talk about anything?

Emily: Well all these [circled pairs of weights and distances] values here like if I multiply these together they have to equal 24 for to balance.

Int: What is that 24?

Emily: Twenty four is the amount of distance and weight of the first side. Yeah, like I showed here that this is four weights on the distance of six from the center, so that is why it has to be 24.

Emily searched for “possible combinations” of 24, and by circling each pair (Figure 7), she showed that the products were all equal to 24. When asked what 24 was, she said, “Twenty four is the amount of distance and weight of the first side.” Therefore, these exchanges provided evidence for Emily’s explicit attention to the constancy of the products of number of weights and distance. These data suggested that Emily’s recognition of the constant product relationship was based on her attention to the numbers or, as I discussed in the preceding paragraph, she might have recalled this constant product relationship from a past instruction on balancing.

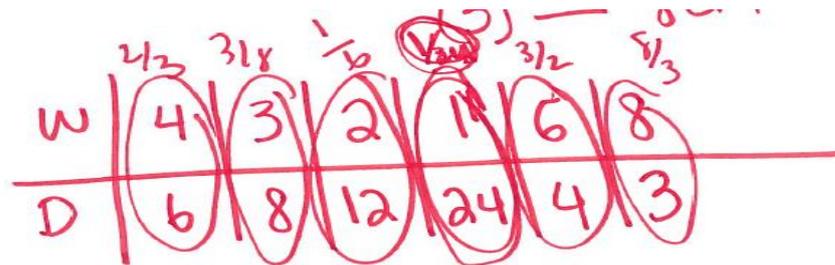


Figure 7. Emily’s ratio table for expressing the constant product relationship between the number of weights and distance

### The Pre-service Teachers’ Reasoning on the Multiple Proportions Problems

#### Mary

In the Speed task, Mary preferred reasoning within measure spaces and used a scientific unit conversion strategy, a similar example of this strategy is provided in Figure 2a. For instance, in one of the problems, she needed to calculate the speed of a car that covered a certain distance in 60 seconds, given that another car covered the same distance in 90 seconds at 60 mph. When asked what her initial impression was, Mary determined the speed to be 90 miles an hour:

Mary: It is going to be 90 miles an hour.

Int: How do you know that it is 90 mph? How did you get that quick?

Mary: I do not know. Because he is driving 60 miles an hour, so he is driving a mile a minute. So, he drove a mile and a half because he drove for 90 seconds. So, if we want to drive a mile and a half in 60 seconds...you would have to travel a mile and a half in a minute.

Int: Then how do you know your speed is 90?

Mary: Because 90 miles in an hour divided by 60 seconds is 1.5.

In the Gear and Balance tasks, Mary determined inversely proportional relationships based on numbers swapping. In this task, the numbers were also swapped, but she did not see that. Instead, Mary seemed to be focusing on the numerical relationship, without attending to the referent units, between values of the speed and time because she incorrectly stated the relationship among speed, time, and distance by saying, “Because 90 miles in an hour divided by 60 seconds is 1.5.” Later, dividing 90 seconds by 60 mph, she wrote 1.5 (Figure 8) without its units that also suggested her attention to the relationship between numbers. To further investigate this possibility, I asked her if she was getting “miles” from the division of “seconds” by “miles per hour,” and Mary stated, “I don’t know if it makes sense that I write 1.5 miles.” Her response was a reflection of her confusion about the accurateness of her distance expression. Later, when she was reasoning out her answer, she incorrectly explained that multiplying 1.5 miles by 60 seconds yielded 90 mph (Figure 8). Mary’s incorrect

expressions indicated that her difficulty with using appropriate referent units and expressing multiplicative relationship among quantities in the multiple proportions problems.

$$\frac{90 \text{ s}}{60 \text{ mph}} = 1.5$$

$$(1.5 \text{ miles})(60 \text{ s}) = 90 \text{ mph}$$

$$(1.5 \text{ miles})(50 \text{ s}) = 75 \text{ mph}$$

Figure 8. Mary's initial distance formula

### *Natalie and Emily*

In the Bakery task, I asked Natalie if she could find a formula to represent the relationships among the number of people, number of cupcakes, and time. She could not find a formula but explained that the formula that she expected to get should involve multiplication and division, not addition and subtraction. Similarly, in the Speed task, Natalie thought  $\text{Time} = \text{Speed} \times \text{Distance}$  to be the correct formula to express the relationships among the quantities but recognized that it was not the correct one. Even though she spent some time trying to generate a formula, she could not obtain one and expressed her difficulty saying, "I don't know. I'm not good with finding the equations at all." Emily also experienced similar difficulties. When I asked her to find a formula to express the relationships in the Bakery task, she generated the  $P = C/T$  formula. Emily tested this formula on one of the problems that she solved earlier and recognized that the formula was not yielding the correct answer. Thus, the need for a constant in the formula made it challenging for Emily to obtain the correct formula.

## Conclusions

This study investigated how pre-service middle and high school mathematics teachers determine directly and inversely proportional relationships presented through hand-on and real-world missing-value word problems. This is a critical topic, because existing mathematics education research documents numerous difficulties, some of which I discussed earlier in the literature review that students and teachers have with this topic. The studies in mathematics education literature generally use word problems with single proportional or nonproportional relationships to investigate how students or teachers reason about proportional relationships. However, in this study, the PSTs solved single and multiple proportions problems that were presented through hands-on and real-world mathematical tasks.

Regarding to the first research question, the PSTs' responses to the proportion problems showed their tendencies to describe relationships between quantities either as proportional relationship or as inverse relationship. The PSTs' responses to the given problems suggested that identifying directly proportional relationships was easier for them than identifying inversely proportional relationships. Their reasoning strategies when determining directly proportional relationships generally included attention to qualitative relationships—two quantities are increasing (or decreasing) together—, constancy of the rate of change, and multiplicative relationships within and between measure spaces (less frequently). However, they usually relied on the qualitative relationships—when a quantity increases, the other decreases—when determining inversely proportional relationships. Because the mathematical tasks included quantities with different units, identifying multiplicative relationships within measure spaces seemed easier for the PSTs than identifying multiplicative relationships between measure spaces. Thus, the PSTs tended to form multiplicative relationships within measure spaces. On the other hand, when they made multiplicative comparisons between measure spaces, they attended to the numerical multiplicative relationships.

In terms of the second research question, the Gear and Balance tasks, in which physical devices (i.e., plastic gears, a mini number balance system) were provided, appeared to facilitate the PSTs' understanding of inversely proportional relationships more than the Bakery and Speed tasks. Although all four PSTs recognized the constancy of the products and reasoned about the inversely proportional relationships in the Gear and Balance tasks, none of them recognized the constancy of the products in the Bakery and Speed tasks. In the Gear task, the context of the problem eased the PSTs' determinations of the constant product—the total notches moved—by coordinating the number of groups (where a group corresponded to one rotation) and the size of groups

(where the size was the number of notches). Similarly, in the Balance task, the PSTs made experiments to balance the system on both sides. When doing that they empirically determined that the product of the number of weights hung and their distance from the center was equal for both sides. Thus, there may be important differences in how students reason about multiplicative relationships among the tasks. This result encourages using hands-on activities such as the ones used in this study in teaching directly and inversely proportional relationships.

When solving multiple proportions problems (i.e., problems in the Bakery and Speed tasks), the PSTs usually fixed one quantity at a time to explain the relationship between the remaining quantities. In this study, none of the PSTs was able to obtain a complete formula to express the multiplicative relationships among the quantities in the Bakery task, and most of them had difficulty obtaining the correct formula in the Speed task. The PSTs' difficulty expressing multiplicative relationships between quantities in the multiple proportions problems suggested possible constraints in their coordination of the multiplicative relationships when more than two quantities were present.

## Notes

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## References

- Arıcan, M. (2018a). Preservice middle and high school mathematics teachers' strategies when solving proportion problems. *International Journal of Science and Mathematics Education*, 16(2), 315–335. <https://link.springer.com/article/10.1007/s10763-016-9775-1>
- Arıcan, M. (2018b). Preservice mathematics teachers' understanding of and abilities to differentiate proportional relationships from nonproportional relationships. *International Journal of Science and Mathematics Education*, 1–21. <https://link.springer.com/article/10.1007/s10763-018-9931-x>
- Beckmann, S. (2011). *Mathematics for elementary teachers* (3rd. ed.). Boston, MA: Pearson.
- Ben-Chaim, D., Keret, Y., Ilany, B. (2007). Designing and implementing authentic investigative proportional reasoning tasks: the impact on pre-service mathematics teachers' content and pedagogical knowledge and attitudes. *Journal of Mathematics Teacher Education*, 10, 333-340.
- Bernard, H. (1994). *Research methods in anthropology* (2nd ed.). Thousand Oaks, CA: Sage.
- Clark, H. J. (2008). *Investigating students' proportional reasoning strategies* (Master's thesis). Available from ProQuest Dissertations and Theses database. (UMI No. 1453188)
- Common Core State Standards Initiative (2010). *The common core state standards for mathematics*. Washington, DC: Author. <http://www.corestandards.org/Math/Practice/>
- Common Core Standards Writing Team (2011). *Progressions for the Common Core State Standards for Mathematics (draft), 6-7, ratios and proportional relationships*. <http://math.arizona.edu/~ime/progressions/>
- Cramer, K., & Post, T. (1993). Making connections: A case for proportionality. *Arithmetic Teacher*, 60(6), 342–346.
- Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 159–178). New York, NY: Macmillan. [http://www.cehd.umn.edu/ci/rationalnumberproject/93\\_4.html](http://www.cehd.umn.edu/ci/rationalnumberproject/93_4.html)
- De Bock, D., Verschaffel, L., & Janssens, D. (1998). The predominance of the linear model in secondary school students' solutions of word problems involving length and area of similar plane figures. *Educational Studies in Mathematics*, 35(1), 65–83. <https://link.springer.com/article/10.1023%2FA%3A1003151011999?LI=true>
- Farrell, M. A., & Farmer, W. A. (1985). Adolescents' performance on a sequence of proportional reasoning tasks. *Journal of Research in Science Teaching*, 22(6), 503–518. <http://onlinelibrary.wiley.com/doi/10.1002/tea.3660220605/full>

- Fisher, L. C. (1988). Strategies used by secondary mathematics teachers to solve proportion problems. *Journal for Research in Mathematics Education*, 19(2), 157–168.  
<http://www.jstor.org/stable/749409>
- Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276–295). New York, NY: MacMillan.  
<http://psycnet.apa.org/record/1992-97586-013>
- Hart, K. (1984). *Ratio: Children's strategies and errors*. Windsor, England: NFER-Nelson.
- Hull, L.S.H. (2000). *Teachers' mathematical understanding of proportionality: Links to curriculum, professional development, and support* (Unpublished doctoral dissertation). The University of Texas at Austin, Austin, TX.
- Izsák, A., & Jacobson, E. (2013, April). *Understanding teachers' inferences of proportionality between quantities that form a constant difference or constant product*. Paper presented at the National Council of Teachers of Mathematics Research Pre-session, Denver, CO.
- Karplus, R., Pulos, S., & Stage, E. (1983a). Early adolescents' proportional reasoning on 'rate' problems. *Educational Studies in Mathematics*, 14(3), 219–233.  
<https://link.springer.com/article/10.1007/BF00410539>
- Lamon, S. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol 1, pp. 629–667). Charlotte, NC: Information Age Publishing.
- Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93–118). Reston, VA: National Council of Teachers of Mathematics.
- Lim, K. (2009). Burning the candle at just one end: Using nonproportional examples helps students determine when proportional strategies apply. *Mathematics Teaching in the Middle School*, 14(8), 492–500.
- Lobato, J., & Ellis, A. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics: Grades 6-8*. Reston, VA: National Council of Teachers of Mathematics.
- Lobato, J., Orrill, C., Druken, B., & Jacobson, E. (2011). *Middle school teachers' knowledge of proportional reasoning for teaching*. Paper presented at the American Educational Research Association (AERA) conference, New Orleans, LA.
- Misailadou, C., & Williams, J. (2003). Measuring children's proportional reasoning, the "tendency" for an additive strategy and the effect of models. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 293–300). Honolulu, HI: University of Hawaii.  
<https://files.eric.ed.gov/fulltext/ED501129.pdf>
- Modestou, M., & Gagatsis, A. (2007). Students' improper proportional reasoning: A result of the epistemological obstacle of "linearity". *Educational Psychology*, 27(1), 75–92.  
<http://www.tandfonline.com/doi/abs/10.1080/01443410601061462>
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.  
<http://www.nctm.org/Standards-and-Positions/Principles-and-Standards/>
- Riley, K. R. (2010). Teachers' understanding of proportional reasoning. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1055–1061). Columbus, OH: The Ohio State University.
- Simon, M., & Blume, G. (1994). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *Journal of Mathematical Behavior*, 13(2), 183–197.  
<http://www.sciencedirect.com/science/article/pii/0732312394900221>
- Singh, P. (2000). Understanding the concepts of proportion and ratio among grade nine students in Malaysia. *International Journal of Mathematical Education in Science and Technology*, 31(4), 579–599.  
<http://www.tandfonline.com/doi/abs/10.1080/002073900412688>
- Stemn, B. S. (2008). Building middle school students' understanding of proportional reasoning through mathematical investigation. *Education 3–13*, 36(4), 383–392.  
<http://www.tandfonline.com/doi/abs/10.1080/03004270801959734>
- Van Dooren, W., De Bock, D., Janssens, D., & Verschaffel, L. (2007). Pupils' overreliance on linearity: A scholastic effect? *British Journal of Educational Psychology*, 77(2), 307–321.
- Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–174). New York, NY: Academic Press.
- Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in middle grades* (pp. 141–161). Reston, VA: National Council of Teachers of Mathematics.

Vergnaud, G. (2009). The theory of conceptual fields. *Human development*, 52(2), 83–94.

<https://search.proquest.com/openview/3b27972afbc561a32625bb489b6122f3/1?pq-origsite=gscholar&cbl=41080>

Yin, R. K. (1993). *Applications of case study research*. Newbury Park, CA: Sage.

Yin, R. K. (2009). *Case study research: Design and methods* (Vol. 5). Thousand Oaks, CA: Sage.

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