A New Look at the Representations for Mathematical Concepts: Expanding on Lesh's Model of Representations of Mathematical Concepts

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ABSTRACT

This article investigates the potential of utilizing technology to apply Lesh's representations for teaching mathematical concepts. Lesh's model includes written symbols, verbal symbols, real life situations, pictorial representations, and concrete representations. The author suggests that, with the wealth of technological applications, teachers have an opportunity to expand that model to include "moving pictures" using technology, thus linking concrete representations to their still-picture counterparts.

The Introduction

The mathematics that students need to know today is different from the knowledge needed by their parents. In our global society and a world of ever changing technology, our current students will become adults facing new demands for mathematical proficiency (Casner-Lotto & Barrington, 2006). Children in the United States are not developing acceptable levels of proficiency in mathematics (The Business Roundtable, 2005; Gonzales, 2007; National Mathematics Advisory Panel, 2008; The Nation's Report Card, 2011). Concern about achievement in mathematics of US students has grown, and research has shown that improvement in the teaching and learning of mathematics is needed (Ball, Hill, & Bass, 2005; Fleishman, et al, 2010; Gonzales, 2007). Results from the Trends in International Math and Science Study (TIMSS) Report (Gonzales, 2007), served to reinforce that concern regarding how mathematics is taught in the US. The scores from the 2009 Program for International Student Assessment (PISA) show 15-year-old students in the US scoring below average in mathematics. Out of 34 countries, the US ranked 25th in math (Fleischman et al., 2009). Only 27% of US students scored at or above proficiency level 4, which is the level at which students can complete higher order tasks such as solving problems that involve spatial reasoning in unfamiliar context and carrying out sequential processes (The Nation's Report Card, 2011). These results have brought about a need for reform in how students are taught and how teachers are prepared to teach them. There needs to be an emphasis on representational fluency in our classrooms in order for our students to gain mathematical understanding on a much deeper level.

The Common Core State Standards of Mathematics (2010) stipulate that students need opportunities to solve problems and have multiple occasions to communicate mathematical ideas with others, and that teachers should focus on student understanding rather than on "right" answers. Researchers have recently examined instruction more closely by investigating the choice and use of various academic tasks (Johnson, 2015). Mathematics has many types and levels of representation which build upon one another as mathematical ideas become more abstract (Lesh, 2003). Physical representations serve as tools to mathematical thought and communication. They help illuminate ideas in ways that support reasoning and build comprehension. Mathematics requires representations. It is because of the abstract nature of mathematics that people use representations to access to mathematical ideas.

In my previous research observing the use of representations for mathematics instruction, much of my attention focused on Lesh's model of mathematical translations of representations (Lesh, 1987; Lesh, 2003). It investigates elementary teachers' use of pictures, words, symbols, concrete materials, and real world contexts to help their students make sense of mathematical ideas. During the course of this investigation, however, evidence surfaced that an additional model, technology, should be considered as an additional form of representation. Three sources of data were collected: video-recorded lessons, interviews, and a focus group. Original analyses indicated that although concrete representations were accessible to all three teachers, they were least used among the available representations. Verbal expression was most prominent, followed closely by abstract written symbols. Technology, however, which was not one of the mathematical representations reported, appeared regularly throughout lesson observations, interviews and the focus group. An implication that surfaced in this study is that technology is becoming more visible in classrooms as one form of mathematical representation. This article suggests an expansion of Lesh's model to include technology (or *moveable pictures* through the use of technology) as an additional means of representation.

The Big Picture

To gain better insight into this topic, I used constructivism as a theoretical foundation and Lesh's model of translations of representations (2003) to guide this study. Ideally, students move within and among five forms of mathematical representation in order to construct meaning of mathematical concepts (Cramer, 2003). Representational fluency is the ability to use several different representations and translate among these models with relative ease. This ability is foundational in students' mathematical proficiency (Fennell and Rowan, 2001; Goldin and Shteingold, 2001).

Mathematical thinking can be "represented" in many ways. It can be represented through drawings and pictures, written or oral words, through manipulatives and, all of these, alongside the abstract (numbers) (Lesh et al., 2003; Goldin, 2003; Kamii, Kirkland, & Lewis, 2001). In order for the reader to have clarity, I have defined each of the forms of representation as I use them within the study and the context of this article.

Manipulatives, also referred to as *concrete* representations, are objects designed to allow students to learn a particular mathematical concept by manipulating them (Reys et al, 2007; Van de Walle, 2005). The use of manipulatives allows students to learn difficult concepts in developmentally appropriate, hands-on, experiential ways (Reys et al, 2007). Some examples of manipulatives are: base ten blocks (which can be used for computational strategies with whole numbers and decimals); Geoboards (which can be used to explore two dimensional geometric shapes, area and perimeter, angles, etc.); pattern blocks (which are used in creating tessellations and exploring patterns in our world around us); fraction pieces (which can be used for computational strategies with fractions); and attribute blocks (which are effective in categorizing and organizing by characteristics).

Pictorial representations, also referred to as pictures, refers to anything hand-sketched or computer generated that represents concrete objects. It could be a photograph, a hand-drawn picture, tallies, graph, or chart. These may include any two-dimensional representations (Ainsworth, 1999; Tabachneck-Schijf & Simon, 1998).

Real-life representation refers to events and objects happening in the real world that allow students to make mathematical connections. Examples may include using money in a grocery store, measuring ingredients when cooking a recipe, or measuring wooden beams when building a garage, etc. (Lesh, 1987).

Symbolic representation, also referred to within this study as symbols or *abstract*, refers to the actual letters, digits, and/or symbols used to represent numbers, formulas, or any other numerical, algebraic, or geometric concepts (Ainsworth, 1999; Tabachneck-Schijf & Simon, 1998).

Technological representation, also referred to as technology, for the purpose of this article refers to the use of any technology (iPad, computer program, website, app, etc.) that produces moveable replicas of concrete or pictorial representation. Examples might include a website with moveable base ten blocks that can express place value of whole numbers or decimals, apps that have moveable fraction pieces that can be broken apart and grouped with whole number to show a mixed numeral, or a parabola whose shape changes when the user adjusts the variables. (Options for technological representation will be discussed further at the end of the article.) Technology which does *NOT* fall into this category of representation includes things such as apps that simply keep score, tablets that allow students to copy words or equations and erase them, websites with rote drill and practice of math facts or fact families, etc.

Since the early 1900s, representation using manipulatives has come to be considered essential in teaching mathematics at the early childhood/elementary school level, prompting the National Council for Teachers of Mathematics (NCTM) in the last several decades to recommend the use of manipulatives in teaching mathematical concepts (NCTM 2000, 2006). Research from both theory and classroom studies has shown that the use of manipulatives for instruction in mathematics can positively affect student learning (Cass et al., 2003; Kelly, 2006; Munger, 2007; Olkun & Toluk, 2004). The *Principles and Standards for School Mathematics* (NCTM, 2000) highlight the role of representation in mathematics asserting that students should create representations and use them to form and communicate mathematical ideas. Learners should be able to choose, apply, and translate among mathematical representations to solve problems (Lesh, 2003; Johnson, 2015). Carbonneau, et al (2012) found that the use of manipulatives improved retention of mathematical ideas. Incorporating math manipulatives into mathematics lessons in meaningful ways can help students grasp mathematical concepts, making graphic pictorial representations more meaningful and making teaching most effective (Johnson, 2015). Pictorial representation refers to anything that represents concrete objects. It could be a photograph, a hand-drawn picture, tallies, graph, or chart. These may include any two-dimensional representations.

The Original Study

As shown in the diagram below of Lesh's 1987 model, students move within and among five forms of mathematical representation in order to construct meaning of mathematical concepts (Lesh, et al., 2003). In addition to the aforementioned manipulatives and pictorial representations, he refers to events and objects happening in the real world that allow students to make mathematical connections known as "real-life situations". Examples may include using money in

a grocery store, measuring ingredients when cooking a recipe, or measuring wooden beams when building a garage, etc. Finally symbols, both verbal and written, refers to the actual letters, digits, and/or symbols used to represent numbers, formulas, or any other numerical, algebraic, or geometric concepts (Lesh, 2003).

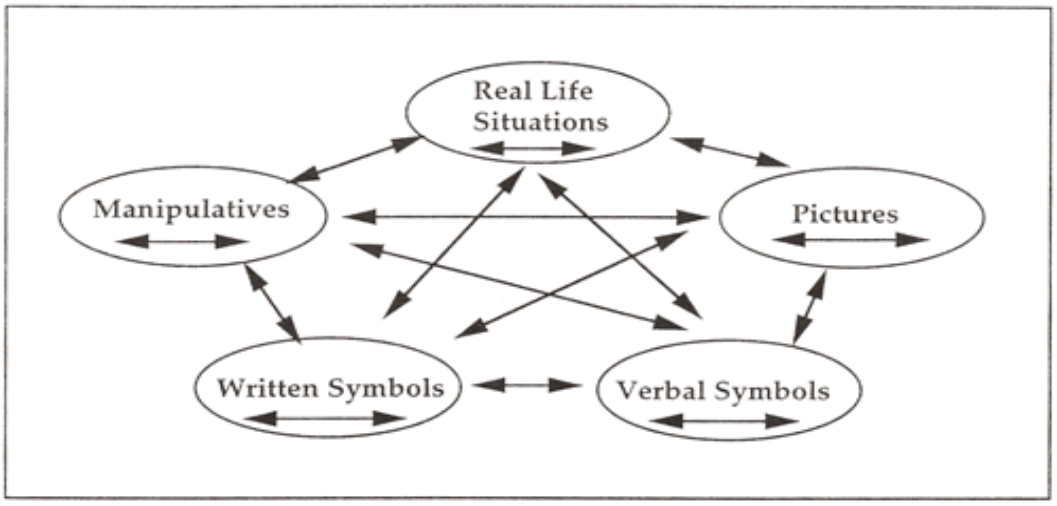

Figure 1 Lesh Translation Model

Conceptual understanding is essential to mathematics proficiency (Burns, 2005). Abstract mathematical concepts often create challenges for students in constructing mathematical understanding (Devlin, 2000; Kamii et al., 2001), therefore hands-on manipulatives and graphic pictorial representations of mathematical concepts are helpful. Representational fluency is the ability to use several different representations and translate among these models with relative ease. This ability is foundational in students' mathematical proficiency (Fennell and Rowan 2001; Goldin and Shteingold 2001; Lamon 2001). Hands-on experiences allow students to understand how numerical symbols and abstract equations operate at a concrete level (Devlin, 2002; Maccini & Gagnon, 2000). Students can benefit significantly from instruction that includes multiple models that approach a concept at different cognitive levels (Lesh & Doerr, 2003; Lesh et al., 2003; Lesh & Fennewald, 2010).

In my research looking at how teachers used representation for their mathematics instruction, I used a mixed methods case study design (Dyson & Genishi, 2005; Yin, 2003). As part of my documentation, I collected six weeks of video recorded lessons (two one-week "scoops" for each participant) (Borko, et al, 2005; Borko et al, 2007). Participants were asked to make video recordings of their instruction in mathematics, creating a snapshot of what average classroom instruction was like on a day to day basis. I asked participants to record their mathematics instruction for five consecutive days in each of their sequences. I wanted to get a true snapshot of what instruction looked like on an average day. These video-taped lessons were conducted in their natural setting of the participants' regular classroom with their own students. Participants were asked not to plan "special" lessons, since they knew they were being recorded. I wanted them to record five days of instruction that looked most like what they typically did. I then viewed the video recorded lessons, transcribing and coding the events at three minute intervals, and analyzing the lesson based on the five types of representations within the Lesh model: manipulatives, pictures, real-life situations, written symbols, and verbal symbols.

As an additional part of my documentation, I conducted teacher interviews and a culminating focus group based upon teachers experiences throughout the study. The interviews included discussions on topics such as how they felt about math, how proficient they felt they were in mathematics, how mathematics was taught to them as they came through school, and how often they perceived themselves using manipulatives and other representations in their instruction with their students. In the focus group, I looked for things that I found possibly relevant in the interviews mentioned by one or more participants that I wanted to hear further discussion from and interaction between the group members.

Related themes that emerged from the interviews and focus group were technology, behavior, and time. Teachers reported that often times they use technology instead of manipulatives due to time constraints or behavioral issues. At the time of this particular study, I did not expand on the technology piece as it was not the focus of the study. It was only after going into classrooms in different schools throughout the state over the past few years and seeing a huge growth in the use of and accessibility to technology that I revisited the data from my previous work. It is these components of my research, the interviews, the lesson sample coding, and the focus group discussion, upon which I base my assertions in this article.

A variety of representations were apparent in the videos, such as explanations, equations, and the use of various measurement tools. One thing that began to jump out at me was the use of technology within the daily instruction, such as the use of iPads by the students or pictures used by the teacher on the Promethean board. Because technology was not part of the model upon which my codes were based, I coded which item was being used that fit within the coding process (i.e. pictures, written symbols). I have since gone back and looked at the items through a new lens, focusing on how technology played a part in mathematics instruction. Table 1 shows examples of different events using a variety of representations and how they were coded. Although the codes are limited to the five components of Lesh's model, I have highlighted in bold italics the technology that was used.

Table 1 Representation Codes Sample

Event	**Code 1**
Student is working on the ***Promethean board doing a multiplication problem***.	WRI
Student is answering a question (What is 48+2)	VER
Tommy is answering the question, "How many times does three go into twelve?"	VER
Using the ***Promethean board, the teacher models how to measure to the half inch and quarter inch using a picture of a ruler.***	PIC
Using the ***ruler on the Promethean board*** teacher poses the question how long an object is (to the quarter) and points to the ruler.	PIC
Students are using a ruler to measure a paper clip.	MAN
Students are talking in partners about how they can tell the difference between an inch and a half-inch when measuring with a ruler.	VER
Teacher is ***demonstrating on the Promethean board (picture of a ruler)*** as a student describes how to read the ruler	PIC
Students were explaining their reasoning on how to determine the differences in measuring with the ruler (inch, half inch, quarter-inch).	VER
Teacher is ***showing the ruler on the Promethean board*** and demonstrating the whole, half, and quarter inch.	PIC
Students are working in assigned partners measuring items in the practice set.	MAN
Teacher says, "Yesterday we talked about measurement. What tool do we use when we are measuring?" Students respond.	VER
Students are working in partners on p 789 Am I Ready?" questions to review (multiplying) and writing their responses.	WRI

Key: MAN-Manipulatives, RL-Real-life, PIC-Pictures, WRI-Written, VER-Verbal

Based on the original analysis of the videotaped observations, manipulatives were used 12.4% of instructional time; pictorial representations were used 24.1% of instructional time; written expression was used 18.6% of instructional time; and verbal communication was used 45% of instructional time. However, upon revisiting the data, the use of technology (computer, Promethium Board, and iPad) was coded 18% of the ***total*** time. This 18% encompasses portions of pictorial, written, and verbal representation, affecting their numbers, but does not impact the coding for the use of manipulatives or real-life situations.

The "Ah Ha"

Given that technology was used in almost one fifth of the time, it raises the notion that technology is yet another way that mathematics may be represented. In recent years, the development and availability of technology has made technological representation of both concrete manipulatives and pictures a more viable option for teachers. In my interviews with teachers, issues such as time management and distribution of materials, quantity of specific manipulatives, behavior management with actual concrete manipulatives versus technological representation of manipulatives, and availability of professional development using manipulatives versus technology have furthered teachers' interests in using available technology in lieu of concrete manipulatives and paper/pencil representations.

Reference to the use of technology as a form of mathematical representation does *not* include technological use for drill and practice, recording of mathematical equations, computation games, etc., but *rather* the use of pictorial representations in moveable form. For the purposes of this article, technology is referred to as a means of "bridging" the use of manipulatives to the use of static pictures. Some examples of these *moving pictures* may be as simple as dragging counters, grouping and sliding of virtual base ten blocks, or dividing wholes into fractional parts and rearranging the pieces. They may be more complex moving models such as reflection, rotation, and translation of various shapes or a changing parabola based on the changing of its variables. Technology affords us the ability merge manipulative concrete objects with the pictorial representations currently found in textbooks and on worksheets, closing the gap in understanding the transference from one representation to another, possibly more abstract, one.

The National Library of Virtual Manipulatives has a variety of moveable pictures that are separated into each of the mathematics strands and grade levels. The users are able to easily locate the skill, and activities become more challenging as the grade levels become higher. In Numbers and Operations, students have the ability to create bar charts showing quantities and percentages, use base blocks to illustrate whole number computation in a variety of bases and explore decimals, manipulate fraction bars and circles to illustrate a fraction and identify the parts of a whole, count money and make change, investigate probability using spinners, and use number lines to illustrate numerical computation of whole numbers or fractions. For algebraic concepts, learners visualize multiplying and factoring with algebra tiles, seek multiple solutions by manipulating triominoes (triangular numeral dominoes), use a graphing tool for exploring functions, create and discover patterns working with three-dimensional blocks, solve simple linear equations employing a balance beam representation, explore functions by entering values into a function machine and observing the output.

In Geometry, students can sort attribute blocks by color and shape, build congruent and similar triangles by combining sides and angles, use a Geoboard to illustrate area and perimeter and investigate three-dimensional shapes, use geometric shapes to make patterns, use pentomino and tangram combinations to solve problems, dynamically interact with and see the results of reflections, rotations, and translations transformations, and change variables and observe patterns from a graphing simulation. For measurement, students work with analog and digital clocks to determine elapsed time, solve problems requiring students to fill and pour various amounts of liquids from a container, and use money to count and make change. Data analysis and probability can be explored as students create bar graphs showing quantities or percentages by labeling and changing values. Students can investigate percentages and fractions using pie charts, and they examine numbers and probabilities with spinners.

In addition to the National Library of Virtual Manipulatives and similar sites, there is a plethora of apps available at no cost to the user or the school district. A geoboard app allows students to build two dimensional shapes, investigate angles, and explore area and perimeter using virtual rubber bands and pegs similar to those on an actual geoboard. Number Frames is a free app that allows students to use counters and build arrays up to 12x12, compute numbers, and explore area. There are two levels of Number Pieces apps that display moveable pictorial representations of Base Ten blocks in combination with number frames and number lines. The Number Rack app looks similar to an abacus and allows students to add and subtract along one row or several rows. It has a "shade" to cover beads to model arithmetic expressions. With Number Line, students can skip count forward and backward for addition and subtraction. Intervals can be set by ones, twos, fives, etc. and spacing can be enlarged to accommodate the younger students.

If technology is used in this way, then another form of mathematical representation can be utilized, linking manipulatives to pictures in an operational manner and ensuring more operative connections for conceptual understanding.

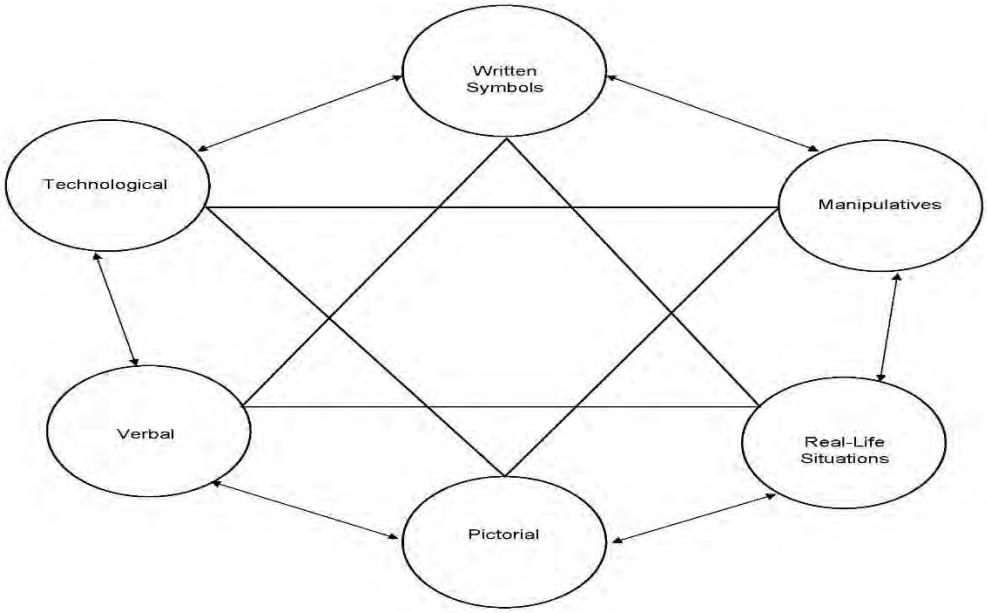

Figure 2 Johnson Mathematical Representation Model

The Conclusion

A broad idea emerged from the original study. Lesh et al.'s model of mathematical representations included manipulatives, pictures, real-life situations, written expression, and verbal expression. I found through my observations of instruction and through discussions with the participants that the findings from this study provided an opportunity to modify this model for future studies to include technology in order to reflect what I felt was observed in today's elementary classroom. Technology appeared in many forms during this study: moveable and/or manipulated pictures, written explanation, charts and graphs, displays of real-life situations, educational videos, and the use of verbal explanations of the use of symbolic representations. Since the teachers in this study used, and justified the use of, technology in ways that were different from other forms of representation, future research on representations should directly include technology as a distinct representation.

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