Find the dimensions:
Students solving a tiling problem

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This paper discusses how problem-solving skills can be fostered with a geometric tiling problem.

Students learn mathematics by solving problems. Mathematics textbooks are full of problems, and mathematics teachers use these problems to test students' understanding of mathematical concepts. Problem-solving enables students to explore, develop and apply mathematical concepts in solving problems. Teachers have the responsibility of promoting problem solving experiences that create an environment that fosters student thinking and reasoning (AMTE, 2017). Problem-solving processes vary with year level. For example, the problem-solving processes that kindergarten students use may look different from the ones a Year 3 student may use. Hwang and Riccomini (2016) state that:

> Requirements for reasoning, explaining, and generalising mathematical concepts increase as students advance through the educational system; hence, improving overall mathematical proficiency is critical. Mathematical proficiency requires students to interpret quantities and their corresponding relationships during problem-solving tasks as well as generalising to different contexts (p 169).

Esmonde (2013) notes "by allowing students to mathematise problems in multiple ways, we might find that different students in the classroom develop different kinds of mathematical skills, perhaps fostering some interesting discussions when comparing these multiple strategies" (p. 24). It is imperative that teachers foster an environment that encourages students to employ varied strategies to solve a mathematical problem to support the development of deeper understanding. This paper describes two pre-service teachers (PST) who were engaged in a problem-solving task to determine the dimensions of nested squares in a tiling pattern activity (Figure 1) in a geometry class for secondary PSTs. Crucial in determining the dimensions of the squares was knowing the properties of a rectangle and a square, using spatial reasoning, and demonstrating proficiency in solving algebraic equations. Although the strategies of the two participants were different, there were commonalities that led to correct solutions. The problem we gave the students was:

> Figure 1 is a collection of squares arranged to form a rectangle. If the dimensions of the squares are distinct integers, what are the dimensions of each square, $A - J$? (McRae, n.d). Record your response in Table 1. Show all appropriate working.

Table 1. Students' worksheet.

Square	A	B	C	D	E	F	G	H	I	J
Dimension										

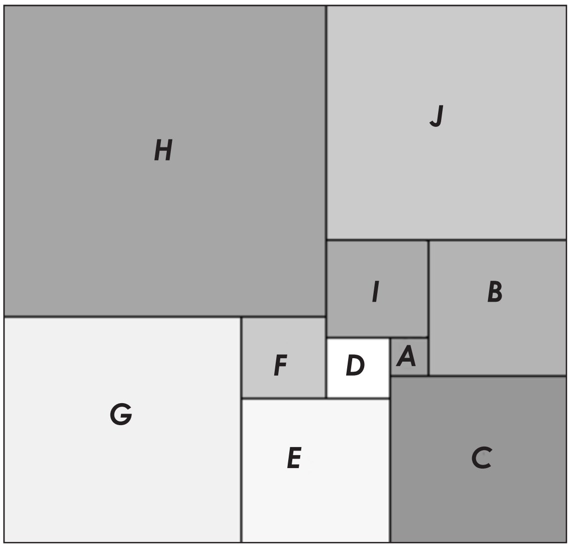


Figure 1: A collection of squares arranged to form a rectangle.

Twenty-seven PSTs participated in this activity. Jane and John (pseudonyms) were selected by the instructor based on their differing approaches to solving the problem. Both Jane and John were preparing for their final practice-teaching session at a school before graduating. To appreciate the problem-solving strategy described below, I recommend that the reader takes the time to do the activity. What are the key insights for the problem solver? What extra insights are needed for the problem solver to find all solutions (a particular solution and a general solution)? The following are important points to note when solving the problem.

- What is a distinct integer? Even though all the sides of the squares are integers, they are not all integer multiples of the side of the smallest square (for example, let the smallest be 1 and then proceed to measure the rest).
- Knowledge of algebra for solving systems of equations is needed, and the ability to know whether the result is a solution or a general solution.
- There is a need to visualise a geometrical and/or algebraic relationship that occurs vertically or horizontally among squares in the figure. For example, $F + D = E$ (the length of one side of F + length of one side of D = length of one side of E) and $F + E = G$. Some of the side lengths can be represented as differences between other side lengths or combinations of other side lengths (for example $E = C + A - D$).
- A property of a rectangle is that opposite sides are congruent.

With problem solving it is important that teachers ask students questions that will encourage thinking (Lampert, 1990) rather than giving students the answer. Samples of classroom conversations have been included to give suggestions of the type of questions that could be asked.

The case of Jane

Before starting the investigation, the instructor started the conversation with questions.

Instructor: What are distinct positive integers?

Jane: I know what an integer is—but the term distinct integer, I'm not sure.

Instructor: What does the word distinct mean to you?

Jane: Distinct means different.

Instructor: So what are distinct positive integers?

Jane: Positive integers that are not the same?

Instructor: Right

Jane then started working on the problem beginning with the two smallest squares, A and D, and labelled the length of the side $A = x$ and $D = y$ to help find the lengths.

Instructor: Why did you start with the smallest square?

Jane: I thought about that for a moment and arrived that it is much easier to start small and go big.

There was not a clear justification from Jane as to why starting with the smallest square would be an easier strategy to solve the problem. At this point, Jane was not able to move forward, because she seemed unable to relate the two lengths, x and y, to other squares around the Squares A and D. It took some time for Jane to find the length of the side of Square B. She appeared to focus only on the horizontal lengths (sides) of the squares ignoring the vertical lengths (sides).

Instructor: Jane, remember those are squares—you can also consider the vertical sides as well as the horizontal sides.

Jane did not naturally see that the length I + length A = length B (Figure 1), and that was a big obstacle for her to relate the dimensions of B using x and y.

Instructor: Jane, what are you trying to do?

Jane: My goal is to express the opposite sides of the rectangle in terms of x and y, and since opposite sides of the rectangle are congruent, then the expressions of the opposite sides could be equated to solve for x and y.

After this discussion of the dimensions of Squares A, D, I, and B, Jane came up with the dimensions of the squares as illustrated in Figure 2.

I will start with the smallest square (A).

Let $A = x$, $D = y$, where x & y are side lengths of A and D.

Then, $D + A$ = Length of square I, which means

$I = A + D = x + y$ -- (i)

$B = A + I = x + x + y = 2x + y$ -- (ii)

$C = A + B = x + 2x + y = 3x + y$ -- (iii)

$J = I + B = x + y + 2x + y = 3x + 2y$ -- (iv)

$E = C + A - D = 3x + y + x - y = 4x$ -- (v)

$F = C + E - (B + I) = 3x + y + 4x - (2x + y + x + y)$

$= 3x + y + 4x - 2x - y - x - y$

$= 4x - y$ -- (vi)

$G = F + E = 4x - y + 4x = 8x - y$ -- (vii)

$H = G + F = 8x - y + 4x - y = 12x - 2y$ -- (viii)

Since the opposite sides of the rectangle can be expressed as the sum of the sides of the square inside, then we can state:

$H + G = J + B + C$ -- (ix)
$8x + 4y = 20x - 3y$
$12x = 7y$ -- (x)
Therefore $x = 7 = A$, $y = 12 = D$

$I = x + y = 19$, $B = 2x + y = 26$
$C = 3x + y = 33$, $J = 3x + 2y = 45$
$E = 4x = 28$, $F = 4x - y = 16$
$G = 8x - y = 44$, $H = 12x - 2y = 60$
From (ix), $H + G = 104$ *and*
$J + B + C = 104$
Q E D

Figure 2. Jane's solution to the problem.

Jane had a strategy to solve the problem, but on the other hand, her process had challenges. She started with the smallest square, *A*, whose length she assigned *x*, the second smallest square, *D*, with a length *y*.

Instructor: Why did you decide to choose *x* and *y* variables?

Jane: I struggled to assign the same variable "*x*" but realised that it was not possible. I initially thought that the length of A was $\frac{1}{2}$ of *B*, but later realised that it was a wrong assumption. So by assigning *x* and *y for A* and *D*, respectively, was a strategy I thought I could use to come up with dimensions of other squares. I'm not sure that the approach will work, but a trial is better than not trying. I did also realise that the sum of the sides of *A* and *D* equal to the side of *I*. I could express the lengths of *A* and *D* regarding *I*.

After making decisions for *A*, *D*, and *I*, Jane was also faced with another struggle: which square to use to determine its length regarding *x* and *y*? Jane tried *F*, the third smallest square, but realised that it could not be expressed in terms of *x* and *y*. She was very frustrated and thought the strategy might not work. She chose *F* because it was the third smallest, and wanted to create a pattern assigning *x* and *y* from smallest to biggest.

Instructor: Can you check the squares you already know that are expressed in terms of *x* and *y* to see if you can use them to express other squares the same way?

Jane: I have tried *C* and *E* with no success.

Instructor: How about Square *B*?

Jane did spend some time studying Square *B* and noted that "neither *A* or *D* or the sum of both can sum up to fit the length of Square B." This was interesting because Jane was only considering the horizontal sum of side lengths of Squares *A*, *D*, and *I* that could fit any of the given squares. Even when the instructor hinted for her to consider the sum of the vertical lengths of Squares *A*, *D*, or *I*, she still was only considering the horizontal sum. She noted, "Squares have same lengths and therefore it does not matter which side one uses."

Instructor: How about the sum of the vertical distance of Squares A and I?

Jane: Yeah, that works. I can see I missed it—I was only fixed looking at the horizontal relationship and not considering the vertical one. Not sure why I did so.

The preceding discussion enabled Jane to find that the lengths of $A + I = B$ (*ii* from Figure 2). Similar challenges that were experienced by Jane were also experienced by other students because they only considered the horizontal distance and neglected the vertical distance relationships between squares. This resulted in challenges in more complex relationships that followed. The one that posed a greater challenge for Jane was to find the lengths of Squares E and F. For Square C, Jane was unable to visualise how the side of Square C could be expressed regarding x and y. First, Jane did not see any squares that could be added together to give the length of Square C. As noted in (*v*) from Figure 2, $E = C + A - D$. It took Jane a long time to visualise this. Similar challenges were also witnessed with square F, (*vi* from Figure 2). Knowing that opposite sides of a rectangle are congruent (*ix* and *x* from Figure 2), that the sides of the rectangle can be expressed using the squares inside it, and that all sides of a square are congruent, enabled Jane to finalise the solution. Jane was fairly well aware that the squares make up a rectangle that has opposite side congruent by definition, which led to the equations being equal (*ix* and *x* from Figure 2). When Jane was asked how she found the values of x and y from $12x = 7y$. She noted that "since they must be distinct integers and yet the two sides of the equations need to be equal therefore x can be 7 whereas $y = 12$".

The case for John

John was also asked what a distinct integer was. The instructor took the time to explain what that meant by defining the term 'distinct' as different from something else. Clarifying what a distinct integer meant played a role in John's solution process. John's approach to this problem was different from Jane's. He started with the biggest square, H, and obtained an expression for the length of H in terms of the other squares. John used the label for the square to mean the side length of that square.

Instructor: Why did you start with the biggest square?

John: Not sure why but just wanted to try it out.

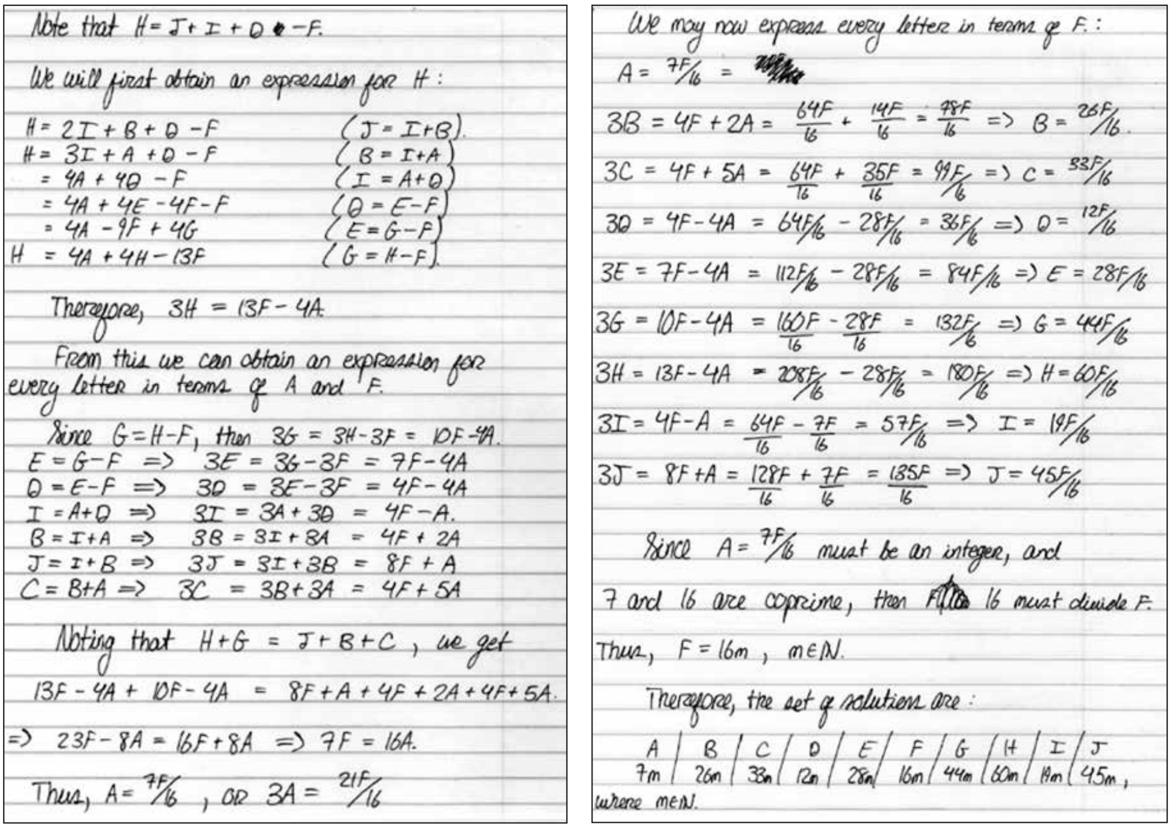
Similar to Jane, John struggled with how to express H in terms of other squares since some of the squares did not line up evenly. He wanted to have H in terms of A and F. In addition, he wanted every single square length to be in terms of A and F. Initially, John tried $H = J + I + F$, but he realised that it could not be expressed in terms of A and F. He tried other options such as $H = J + B$, $H = J + I + A$, all of which failed because John was focused on finding the sum of lengths of sides of squares to find the length of the needed part. The idea of finding the sum and taking the difference was a different way of thinking for John. The instructor gave him clues about the way to move forward.

Instructor: How about expressing H regarding the sum and difference of the squares? Check if there are any squares you can add or subtract which will give you H.

John: Ok. Let me see.

He chose $H = J + I + D - F$ because it was not possible to express H in terms of A and F. Expressing H in an obvious way, for example, $H = G + F$, could not also be expressed in terms of A and F. Note John wanted to express H in terms of A and F because 'A' is the smallest square of the terms on the right side part of the rectangle (that is, smallest of J, *B*, *C*, *A*, *I*, and *D*) that can be expressed with *A* in it (e.g., $J = D + A + B$, $I = D + A$, $D = I - A$, $C = A + B$). Similarly, on the left side there is *F*, which is the smallest of *H*, *G*, *E* and *F*. Classifying the smallest on the right and the left made it easier for him to classify the rest of the squares regarding *A* and *F*.

John found other relationships and used them to solve the problem. John used these skills to express a square regarding the other squares as he proceeded to find the dimensions of the squares (Figure 3).



Note that $H = J + I + D - F$.

We will first obtain an expression for H:

$H = 2I + B + D - F$ $(J = I + B)$
$H = 3I + A + D - F$ $(B = I + A)$
$= 4A + 4D - F$ $(I = A + D)$
$= 4A + 4E - 4F - F$ $(D = E - F)$
$= 4A - 9F + 4G$ $(E = G - F)$
$H = 4A + 4H - 13F$ $(G = H - F)$

Therefore, $3H = 13F - 4A$.

From this we can obtain an expression for every letter in terms of A and F.

Since $G = H - F$, then $3G = 3H - 3F = 10F - 4A$.
$E = G - F \Rightarrow 3E = 3G - 3F = 7F - 4A$
$D = E - F \Rightarrow 3D = 3E - 3F = 4F - 4A$
$I = A + D \Rightarrow 3I = 3A + 3D = 4F - A$.
$B = I + A \Rightarrow 3B = 3I + 3A = 4F + 2A$
$J = I + B \Rightarrow 3J = 3I + 3B = 8F + A$
$C = B + A \Rightarrow 3C = 3B + 3A = 4F + 5A$

Noting that $H + G = J + B + C$, we get

$13F - 4A + 10F - 4A = 8F + A + 4F + 2A + 4F + 5A$.

$\Rightarrow 23F - 8A = 16F + 8A \Rightarrow 7F = 16A$.

Thus, $A = 7F/16$, or $3A = 21F/16$

We may now express every letter in terms of F:

$A = 7F/16 =$

$3B = 4F + 2A = \frac{64F}{16} + \frac{14F}{16} = \frac{78F}{16} \Rightarrow B = 26F/16$.
$3C = 4F + 5A = \frac{64F}{16} + \frac{35F}{16} = 99F/16 \Rightarrow C = 33F/16$
$3D = 4F - 4A = 64F/16 - 28F/16 = 36F/16 \Rightarrow D = 12F/16$
$3E = 7F - 4A = 112F/16 - 28F/16 = 84F/16 \Rightarrow E = 28F/16$
$3G = 10F - 4A = \frac{160F}{16} - \frac{28F}{16} = 132F/16 \Rightarrow G = 44F/16$
$3H = 13F - 4A = 208F/16 - 28F/16 = 180F/16 \Rightarrow H = 60F/16$
$3I = 4F - A = \frac{64F}{16} - \frac{7F}{16} = 57F/16 \Rightarrow I = 19F/16$
$3J = 8F + A = \frac{128F}{16} + \frac{7F}{16} = \frac{135F}{16} \Rightarrow J = 45F/16$

Since $A = 7F/16$ must be an integer, and 7 and 16 are coprime, then 16 must divide F.

Thus, $F = 16m$, $m \in N$.

Therefore, the set of solution are:

A	B	C	D	E	F	G	H	I	J
7m	26m	33m	12m	28m	16m	44m	60m	19m	45m,

where $m \in N$.

Figure 3: John's solution to the problem.

Jane's and John's solutions were similar in some aspects and different in others. Jane started with the two smallest squares $A = x$ and $D = y$ and expressed the rest of squares in terms of x and y. She then solved the algebraic expressions and reached a solution, whereas John started with the largest square and expressed that square in terms of other smaller squares. John and Jane's solutions were different in terms of where they started, but very similar in terms of the thinking strategy. Jane presented one solution because her strategy was finding the integer values of x and y, whereas John gave a general solution because he gave an expression of the biggest square in terms of other squares. For example, Jane listed the dimensions of each square such as $A = 7$ and $D = 12$…whereas John had $A = 7m$, $D = 12m$ where $m \in N$ (Figure 3). Expressing all squares in the form $m \in N$ means that there was more than one solution for the squares.

Conclusion

Rectangles and squares are common quadrilaterals students encounter in geometry. They learn about their properties: squares have all sides congruent, and all angles are right angles; rectangles have opposite sides congruent and all angles are right angles. Students, in this case pre-service teachers, need to be equipped with knowledge of geometry and be able to apply that knowledge in problem solving. Spatial skills are essential in geometry because they help one to visualise related distances despite the orientation of shapes. With spatial skills, students can visualise both the horizontal lengths and the vertical lengths at the same time, without looking at these in isolation.

In this case, Jane struggled in the initial stages of the problem. Despite all the challenges the two pre-service teachers encountered, both provided an interesting solution to the problem and highlighted the need to equip teachers with the necessary tools to be good problem solvers. Engaging in the open-ended task afforded them both the opportunity to make their thinking clear, which helped the instructor to ask questions and gauge their thinking. Both students got correct answers, but by asking probing questions of Jane, the instructor could see there were gaps in her knowledge. Educators need to provide students opportunities, and scaffold understanding, to enable them to move from the initial stage of gaining mathematical concepts to independently demonstrating conceptual understanding.

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