Imagining complex numbers by generating, interpreting and representing them

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Ever since their serendipitous discovery by Italian mathematicians trying to solve cubic equations in the 16th century, imaginary and complex numbers have been difficult topics to understand. They were received with suspicion back then as it was difficult to visualise them, let alone make sense of them and apply them practically. It was not until around the turn of the 19th century that a graphical representation of imaginary number was realised first by the Norwegian surveyor Caspar Wessel (1799), then by the Swiss born French mathematician John-Robert Argand (1806), the French mathematician Abbeé Adrien-Quentin Buée (1806) and the German mathematician Carl Friedrich Gauss (1831). They hit upon the idea to represent imaginary numbers as points on a plane similar to the Cartesian plane like the one devise by René Descartes in the 17th century. It was Descartes who first coined the term 'imaginary' to describe these numbers. The Swiss mathematician Leonhard Euler was the first to denote the imaginary unit as $i = \sqrt{-1}$ and Gauss who first used the term 'complex'. The plane on which imaginary (and complex) numbers are plotted is called an Argand Diagram or the Complex Plane. On it, the horizontal axis is called the real axis and the perpendicular vertical axis is called the imaginary axis. (Subsequently, it was not until around the turn of the 20th century that they were used to understand a very practical problem: electricity in the form of alternating current, AC).

Before further discussing complex numbers, it is necessary to clarify the correct sense of the word *complex*. Here the word complex is used to describe something consisting of a number of interconnecting parts. Think of a shopping complex which is made up of a number of different buildings in close proximity. The different parts of a complex number are the *real* part and the *imaginary* part and both of these are denoted by a real number. The term complex is not meant to denote a level of complexity which can often be students' first impression when encountering complex numbers.

One way to understand complex numbers is to place them in the context of the more familiar types of numbers. The first type of numbers that children learn are the positive integers, 1, 2, 3, etc., then positive fractions like $\frac{1}{2}, \frac{1}{3}, \frac{1}{4},$

etc. and decimals like 0.5, 1.6, 2.7, etc. Next, students learn about negative numbers; –1, –2, –3, etc., $-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}$, etc. and –0.5, –1.6, –2.7, etc. These are all real numbers and they serve students well throughout their primary and secondary school years and of course in daily life. The important thing is that these numbers can be represented as points on a number *line* (the real number line) with zero at the centre, positive numbers to the right of zero and negative numbers to the left. Similarly complex numbers can be represented as points: not only on the real number line but *anywhere on a plane* (the complex plane). Whereas real numbers have only one dimension, (horizontally along the real number line, being either positive, negative or zero), complex numbers can be seen as having two dimensions; a real dimension (horizontally) and an imaginary dimension (vertically) perpendicular to the real.

Just as a fraction is a number made up of two parts, so is a complex number made up of two parts. A complex number is a complex (a blend, hybrid or composite) of a real part represented by a real number and an imaginary part also represented by a real number, in the form $a + bi$, (or sometimes $x + yi$), where a is the real part and b is the imaginary part. (Mathematicians use i whereas electrical engineers use j instead of i for the imaginary symbol, since i is the symbol for current). The real part of a complex number is referenced on the horizontal real axis of the complex plane and the imaginary part is referenced on the vertical imaginary axis. A pure real number (with no imaginary part) is therefore situated on the real axis, a pure imaginary number (with no real part) is situated on the imaginary axis and a complex number with both real and imaginary parts is situated on neither axes but in one of the four quadrants. Figure 1 below shows the complex plane (or Argand diagram in honour of John-Robert Argand), with two real numbers, 3 and –5, two imaginary numbers, i and $-2i$ and four complex numbers, $3 + 4i$, $-3 + i$, $-4 - i$ and $2 - 3i$. Due to their difficulty and applications in higher mathematics, science, engineering and technology, only students going onto to study these subjects in Years 11 and 12 and beyond at tertiary institutions encounter complex numbers.

Generating complex numbers

Complex numbers can be generated quite simply by a vertical translation of a quadratic equation's curve (parabola) as it goes from having two real solutions, to one real solution and then to no real solutions. When a quadratic equation has no real solutions, it instead has two complex solutions. A quadratic equation has two real solutions when its curve crosses the x-axis twice, one real solution when the vertex of its curve touches the x-axis and no real solutions when its curve does not cross the x-axis.

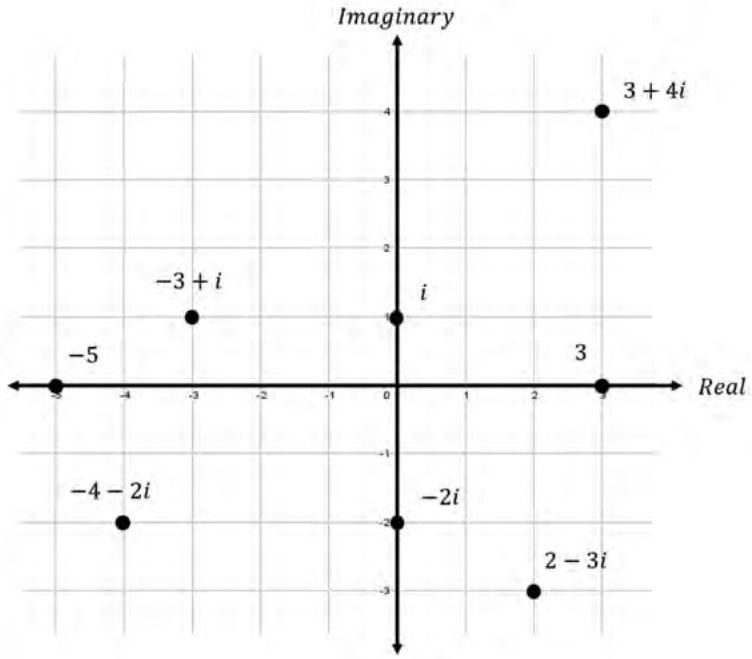


Figure 1. Real and complex numbers.

When solving for the solutions of a quadratic equation in standard form $ax^2 + bx + c$ the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is used to find its solutions. It is the value of the discriminant, the term under the square root sign, $b^2 - 4ac$, that determines whether the quadratic equation has real or complex solutions. If the discriminant is greater than zero, the quadratic equation will have *two* real solutions. If the discriminant equals zero, the quadratic equation will have one real solution. If the discriminant is less than zero, the quadratic equation will have no real solution but *two* complex solutions because the square root of a negative number has no real solutions.

To illustrate this, consider the simple quadratic equation $x^2 - 4x + 0 = 0$. It has two real solutions, $x = 4$ and $x = 0$. By increasing the constant term from zero to 3, then to 4, 5, 8 and 20, the solutions to these equations are as follows in Table 1 below:

Table 1

Quadratic equation	Factors	Solutions
$x^2 - 4x + 0 = 0$	$(x - 4)(x - 0)$	Two real solutions: $x = 4$ and $x = 0$
$x^2 - 4x + 3 = 0$	$(x - 3)(x - 1)$	Two real solutions: $x = 3$ and $x = 1$
$x^2 - 4x + 4 = 0$	$(x - 2)(x - 2) = (x - 2)^2$	One real solution: $x = 2$
$x^2 - 4x + 5 = 0$	No *real* factors	Two complex solutions: $x = 2 + i$ and $x = 2 - i$
$x^2 - 4x + 8 = 0$	No *real* factors	Two complex solutions: $x = 2 + 2i$ and $x = 2 - 2i$
$x^2 - 4x + 13 = 0$	No *real* factors	Two complex solutions: $x = 2 + 3i$ and $x = 2 - 3i$

When the quadratic equation $x^2 - 4x + 5$ is solved using the quadratic formula, the discriminant is –4. As the square root of a negative number cannot be taken, –1 is factored out of the –4 resulting in $-4 = 4 \times (-1)$. This valid algebraic step allows the simplification to continue until the result is a set of complex numbers. The symbol i is also used to represent $\sqrt{-1}$. So $i = \sqrt{-1} \Rightarrow i^2 = -1$. This is one method of generating complex numbers. The following is the solution to $x^2 - 4x + 5 = 0$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16 - 20}}{2} \\ &= \frac{4 \pm \sqrt{-4}}{2} \\ &= \frac{4}{2} \pm \frac{\sqrt{4}\sqrt{-1}}{2} \\ &= 2 \pm \frac{2i}{2} \\ &= 2 \pm i \end{aligned}$$

Similarly, when $x^2 - 4x + 8 = 0$ and $x^2 - 4x + 13 = 0$ are solved, the solutions are $x = 2 \pm 2i$ and $x = 2 \pm 3i$ respectively. These solutions are confirmed when they are substituted back into their respective quadratic equations. The symbol i follows the normal rules of algebra and $i = \sqrt{-1} \Rightarrow i^2 = -1$. For example, substituting $x = 2 \pm 3i$ into $x^2 - 4x + 13 = 0$ gives

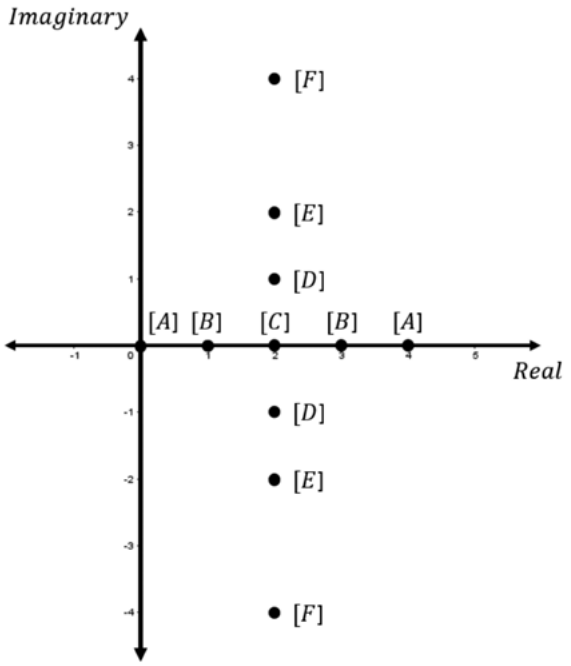
$$\begin{aligned} x^2 - 4x + 13 &= 0 \\ (2+3i)(2+3i) - 4(2+3i) + 13 &= 0 \\ 4 + 6i + 6i + 9i^2 - 8 - 12i + 13 &= 0 \\ 4 + 9i^2 - 8 + 13 &= 0 \\ 4 + 9(-1) - 8 + 13 &= 0 \\ 4 - 9 - 8 + 13 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} x^2 - 4x + 13 &= 0 \\ (2-3i)(2-3i) - 4(2-3i) + 13 &= 0 \\ 4 - 6i - 6i + 9i^2 - 8 + 12i + 13 &= 0 \\ 4 + 9i^2 - 8 + 13 &= 0 \\ 4 + 9(-1) - 8 + 13 &= 0 \\ 4 - 9 - 8 + 13 &= 0 \\ 0 &= 0 \end{aligned}$$

When solving and plotting the above six quadratic equations with their solutions, the parabola is shifted upwards as a result of increasing the constant term in the quadratic equation and the two real solutions of the quadratic equations get closer to each other in value until they equal each other at $x = 2$. At this point, the parabola's vertex is on the x-axis and occurs when $x^2 - 4x + 4 = 0$.

As the constant 4 in the quadratic equation increases in value, the parabola is shifted higher still until it no longer cuts the x-axis. As a result, the quadratic equation no longer has real solutions. Instead, it now has *two complex solutions*. Notice the position of the two solutions; they are no longer on the real axis.

The two complex solutions lie on the parabola's axis of symmetry, $x = 2$. As the constant term in the quadratic equation becomes even larger, the parabola is shifted higher still and the two complex solutions spread further and further apart. Their real parts remain the same, $x = 2$, but their imaginary parts become more distant and move equal distances away from the real axis. In summary, while the discriminant is positive but decreasing, the two real solutions of quadratic equations get closer to each other along the real or x-axis until they equal each other (one real solution) when the discriminant equals zero. At this point, the vertex of the quadratic curve is on the real or x-axis. As the discriminant becomes negative and then more negative, the single real solution 'splits' into two complex solutions, rotated by 90° and move further apart from each other vertically or the imaginary dimension. Figure 2 below shows the pairs of solutions to the above six quadratic equations (without their parabolas for clarity).



$[A]$ *Real solutions to* $y = x^2 - 4x + 0, x = 0, 4$

$[B]$ *Real solutions to* $y = x^2 - 4x + 3, x = 1, 3$

$[C]$ *Real solutions to* $y = x^2 - 4x + 4, x = 2$

$[D]$ *Complex solutions to* $y = x^2 - 4x + 5, x = 2 \pm i$

$[E]$ *Complex solutions to* $y = x^2 - 4x + 8, x = 2 \pm 2i$

$[F]$ *Complex solutions to* $y = x^2 - 4x + 20, x = 2 \pm 4i$

Figure 2. The transition from real to complex numbers.

Interpreting complex numbers

Why then is $i = \sqrt{-1}$ and how does it relate to a rotation of 90° as was seen in the above set of quadratic equations? When solving quadratic equations, the discriminant, $b^2 - 4ac$ can sometimes be negative. This becomes a problem since the square root of a negative number cannot be taken, i.e., there is no real number that when multiplied by itself will give a result of –1, because $(-1) \times (-1) = 1$ and $1 \times 1 = 1$. To overcome this limitation, mathematicians 'imagine' some number, call it i for imaginary, such that $i \times i = -1$. This simplifies to $i^2 = -1$. Finally, taking the square root of both sides then gives $i = \sqrt{-1}$.

One of the interpretations of i, ($\sqrt{-1}$), is a rotation of 90° counter clockwise (CCW) about the origin of the Argand diagram. To understand why this is the case, consider the number +1 on the real number line. If +1 is multiplied by –1 the result is –1 or a 'counter clockwise rotation' about the origin of 180°. If –1 is multiplied by –1 again, another 'jump' of 180° is performed and the result is +1 again. If another axis is added, a vertical (imaginary) axis at $x = 0$, this will represent the imaginary numbers. Since $i = \sqrt{-1}$, this also means that $i^2 = -1$. How then can +1 be turned into –1 by doing the same operation twice? By 'rotating' 90° counter clockwise about the origin twice in quick succession. When +1 is multiplied by i the result is i, i.e., one unit up the vertical axis. When i is multiplied by i again the result is i^2 or –1. When –1 is multiplied by i again the result is $-i$. Finally, when $-i$ is multiplied by i again the result is $-i \times i = -(i^2) = -(-1) = +1$; see Figure 3.

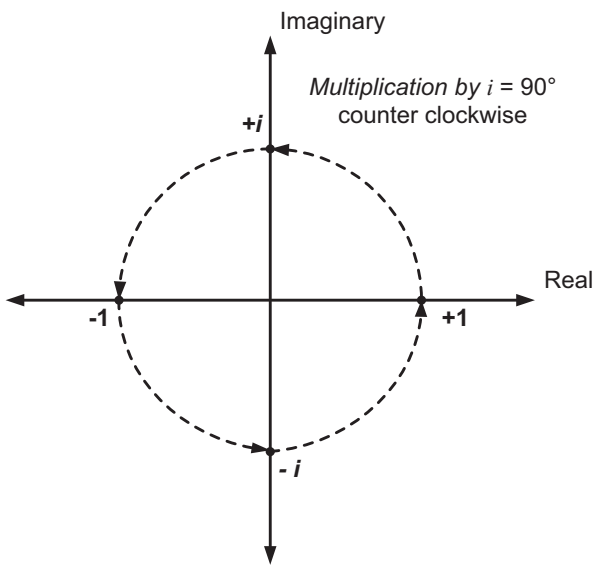


Figure 3. i as a rotational operator.

To demonstrate the idea that i is a rotation of 90° counter clockwise about the origin of the Argand diagram, consider what happens when a real number is multiplied by i. For example, the real number 3 is situated at the coordinates (3, 0), i.e., 3 units to the right and zero units up. Multiplying 3 by

$i = 3 \times i = 3i$ and its coordinates are now $(0, 3)$. The angle between the original 3, the origin and the resulting $3i$ is 90° counter clockwise. If 3 is multiplied by i twice, the result is $3 \times i \times i = 3i^2$. Recalling that $i^2 = -1$, this results in $3 \times (-1) = -3$. The two multiplications of i this time resulted in a $90° + 90° = 180°$ counter clockwise rotation about the origin, that is, the angle between 3, the origin and the resulting -3 is 180° counter clockwise. The same effect also occurs when a complex number is multiplied by i. For example, multiplying $(4 + 5i)$ by i results in $(4 + 5i) \times i = 4i + 5i^2 = 4i + 5(-1) = -5 + 4i$. Again, the angle between the original $(4 + 5i)$, the origin and the resulting $-5 + 4i$ is 90° counter clockwise. This rotation of 90° counter clockwise about the origin of a complex number comprising of a real and imaginary part occurs no matter where on the Argand diagram the complex number is, i.e., it could be in any quadrant or situated on any of the positive or negative, real or imaginary axes. So multiplying by i is a rotation of 90° counter clockwise about the origin; see Figure 4.

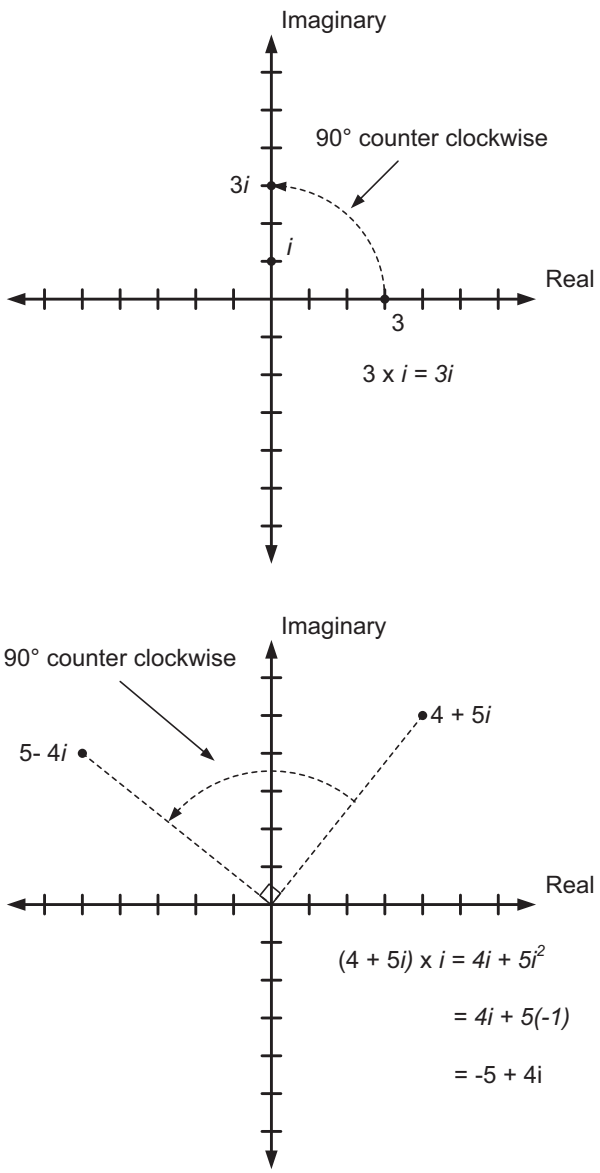


Figure 4. Multiplying by i.

Conversely, if an imaginary number such as i, coordinates (0, 1) is divided by i this would imply that $\frac{i}{i} = 1$, coordinates (1, 0). Therefore dividing an imaginary number by i takes that imaginary number from the imaginary axis and onto the real axis in exactly the opposite way that multiplying a real number by i takes that real number from the real axis onto the imaginary axis. The angle between the original i, coordinates (0, 1), the origin and the resulting 1, coordinates (1, 0) is a rotation of 90° clockwise. Dividing a complex number by i is performed by multiplying the quotient by the conjugate of i which is simply the value of the imaginary part with the opposite sign. Therefore, if a complex number such $3 + 4i$, coordinates (3, 4) is divided by i, the result is $4 - 3i$ as shown in Figure 5:

$$\begin{aligned}\frac{3+4i}{i} &= \frac{3+4i}{i}\cdot 1\\ &= \frac{3+4i}{i}\cdot\frac{-i}{-i}\\ &= \frac{-3i-4i^2}{-i^2}\\ &= \frac{-3i-4(-1)}{-(-1)}\\ &= \frac{-3i-(-4)}{1}\\ &= 4-3i\end{aligned}$$

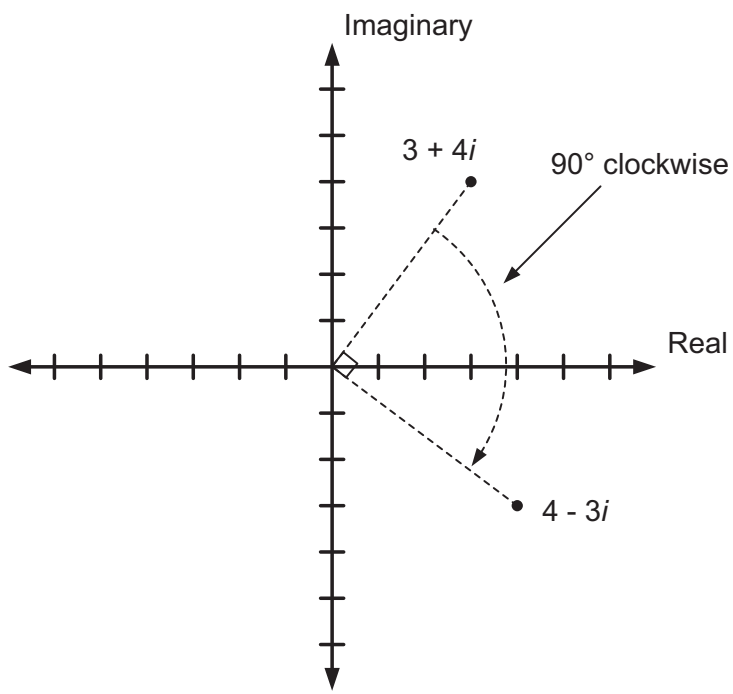


Figure 5. Dividing by i.

Again, the angle between the original $3 + 4i$, the origin and the resulting $4 - 3i$ is 90° clockwise. As in the case of multiplying by i being a rotation of 90° counter clockwise, this rotation of 90° clockwise about the origin of a complex number comprising of a real and imaginary part also occurs no matter where on the Argand diagram the complex number is. So dividing by i is a rotation of 90° *clockwise* about the origin.

$i = \sqrt{-1} = (0, 1) = 1\angle 90°$ = a rotation of 90° counter clockwise about the origin.

Representing complex numbers

Two ways of representing complex numbers are by using rectangular notation or Cartesian form ($a + bi$) and polar notation ($r\angle\theta$) or $rcis(\theta)$. Since a complex number is a point on a plane whose coordinates are referenced from the origin or intersection of the real and imaginary axes, the following diagram shows the relationship between these representations using only middle secondary school algebra, that is, Pythagoras' theorem and basic trigonometry.

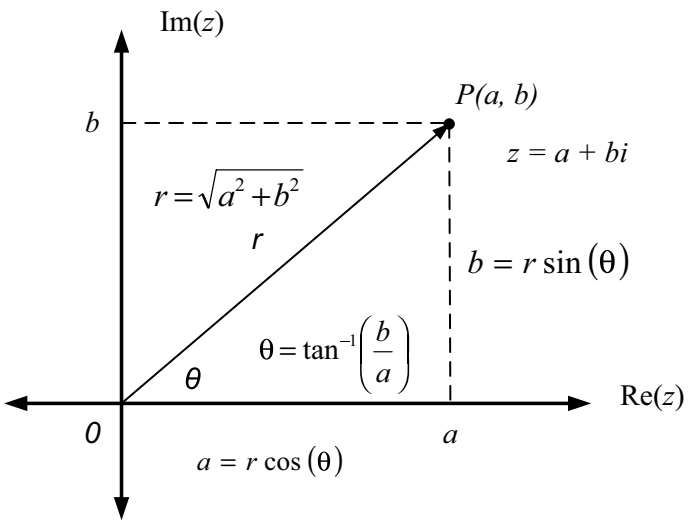


Figure 6. The relationship between rectangular and polar notation of complex numbers.

Since a complex number can be in any of the four quadrants, the inverse tangent ($\tan^{-1}$) function has a caveat. If the complex number is in the first or fourth quadrant, because a (the real part) > 0, then

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

and, if the complex number is in the second or third quadrant, because a (the real part) < 0, then

$$\theta = 180° + \tan^{-1}\left(\frac{b}{a}\right)$$

A complex number such as $3 + 4i$ in rectangular notation becomes

$$\begin{aligned} a+bi &= r\angle\theta = \sqrt{a^2+b^2}\angle\tan^{-1}\left(\frac{b}{a}\right) \\ &= \sqrt{3^2+4^2}\angle\tan^{-1}\left(\frac{4}{3}\right) \\ &= 5\angle 53.13^\circ \end{aligned}$$

Therefore $3 + 4i = 5\angle 53.13^\circ$. This is the same as the coordinates (3, 4). Conversely, a complex number such $13\angle 67.38^\circ$ as in rectangular notation becomes

$$\begin{aligned} a &= r\cos\theta \\ &= 13\cos(67.38^\circ) \\ &= 5 \end{aligned}$$

$$\begin{aligned} b &= r\sin\theta \\ &= 13\sin(67.38^\circ) \\ &= 12 \end{aligned}$$

Therefore $13\angle 67.38^\circ = 5 + 12i$. This is the same as the coordinates (5, 12).

Engineers use the $\angle$ symbol whereas mathematicians use *cis* to denote the angle of a complex number expressed in polar notation. As $a = r\cos(\theta)$ and $b = r\sin(\theta)$, with b being the imaginary part, $bi = i\sin(\theta)$, both these expressions simplify to

$$\begin{aligned} a+bi &= r\cos(\theta)+i\sin(\theta) \\ &= r(\cos\theta+i\sin(\theta)) \\ &= r\,cis(\theta) \end{aligned}$$

where *cis* is an abbreviation of $\cos(\theta) + i\sin(\theta)$. Therefore $13\angle 67.38^\circ = 13\ cis(67.38^\circ)$. The angle component can be expressed in either degrees as preferred by engineers or radians as preferred by mathematicians.

Real world applications of complex numbers

One of the things that distinguish our modern society from those of centuries ago is our use and mastery of electricity; it could be said that electricity literally runs the world. From the megawatts of electricity that is generated and distributed to power homes, businesses and industries, to the milliwatts of powers used to operate Bluetooth devices, these and much more are all designed with the use of complex numbers. Complex numbers help engineers design such systems because the electricity of AC obey the rules of complex numbers.

As well as the electricity that flows along solid metal conductors, systems that use invisible electromagnetic waves that travel through the air and the vacuum of space and designed with the help of complex numbers. These

systems include radios, televisions, microwave ovens, infrared remote controls, lasers, UV lighting, X-rays and gamma waves to help kill cancers in people. Without complex numbers our modern world would simply not exist. Those Italian mathematicians back in the 16th century could never have imagined the wonderful uses that their esoteric discoveries would one day lead to.

References

Holland, P., Hodgson, B., Liston, H., Rozen, R., Douglas, S., Nolan, J. & Phillips, G. (2013). *Maths quest 12: Specialist mathematics*. Singapore: John Wiley.

Mazur, B., (2003). *Imagining numbers (Particularly the square root of minus fifteen)*. St Ives, UK: Penguin Books.

Nahin, P. J. (1998). *An imaginary tale: The story of the square root of –1*. Princeton, NJ: Princeton University Press.

Stewart, I., (2008). *Taming the infinite: The story of mathematics from the first numbers to chaos theory*. St Ives, UK: Quercus Publishing.