Using linear and quadratic functions to teach number patterns in secondary school

Kok Xiao-Feng Kenan
St Margaret's Secondary School, Singapore
kenankok@gmail.com

This paper outlines an approach to definitively find the general term in a number pattern, of either a linear or quadratic form, by using the general equation of a linear or quadratic function. This approach is governed by four principles:

- identifying the position of the term (input) and the term itself (output);
- recognising that each corresponding input and output form points on a straight line or quadratic curve;
- identifying the nature of the number pattern;
- using the general equation of a straight line ($y = mx + c$) or quadratic curve ($y = ax^2 + bx + c$) to represent the general term.

Implications for teaching and learning are also offered.

Introduction

The teaching of number patterns can sometimes be a daunting endeavour as generalising a number pattern into algebraic notation can prove to be challenging for students. Research regarding number pattern generalisation around the world suggests that identifying an underlying pattern is not difficult for students but rather, formulating a rule in algebraic notation is the main challenge (English & Warren, 1995; Rivera & Becker, 2007; Stacey & MacGregor, 2001; Ursini, 1991). This was a trend that the author also observed in the Singapore secondary school classroom.

The ability to derive a functional rule between the positions of the numbers in a number pattern and the corresponding terms is crucial to algebraic thinking (Kaput, 2008; Mason, Graham & Johnston-Wilder, 2005; Radford, 2008). Therefore, it is of utmost importance that students be able to recognise the position of the terms in a number pattern and formulate a functional rule between the position of the terms and the term itself.

Research has indicated that students are more inclined to compute the next term in a number pattern using the immediate term just before it (Chua & Hoyles, 2014). This however, is inefficient as a term with a very large position number would be difficult to calculate or when the pattern is presented in a non-successive manner (Chua & Hoyles, 2014). Hence, this again highlights the importance of formulating a functional rule that connects the position of the terms and the term itself as it would enable students to determine a term given its position number and vice versa.

The purpose of this paper therefore, is to present an approach to teaching number patterns using the concepts of linear and quadratic functions, in the context of the secondary school mathematics curriculum in Singapore. In the GCE-O level Examination that Singapore secondary school students take, questions involving number patterns are usually of the linear or quadratic form (Chua & Hoyles, 2014). Therefore, this approach would provide a definitive way for students to determine the general term in a number pattern by using the general equation of a linear or quadratic function.

The approach

There are four important principles of this approach as proposed by the author:

1. Identifying the position of the term and the term itself. In this paper, it will be known as the input and output respectively.
2. Recognising that each corresponding input and output form points on a straight line or quadratic curve.
3. Identifying the nature of the number pattern. A linear number pattern has a common difference between the successive outputs while a quadratic number pattern has a no common difference between the successive outputs. Instead, the difference between the successive outputs follow a linear pattern where there is a common difference between the differences.
4. Using the general equation of a straight line ($y = mx + c$) or quadratic curve ($y = ax^2 + bx + c$) to represent the general term. To achieve this, the unknown constants, m, c, a and b in either general equation need to be found. The variables x and y would be used to represent the position of the terms and the term itself in this paper as students are naturally familiar with variables x and y in linear and quadratic functions.

Teaching linear number patterns using linear functions

The author advocates an approach that utilises the general equation of a linear function to formulate the general term of a linear number pattern. Example 1 is an illustration of a linear number pattern.

Example 1

Find the general term in the following number pattern.

1, 3, 5, 7, 9…

Principle 1

The input and output values first need to be identified. The position number of each term would increase sequentially. In this case, the position number of each term would be illustrated by the first row of numbers represented by variable x while the term itself would be represented by variable y as seen in Table 1 below.

Table 1. Input (x) and output (y) values.

x	1	2	3	4	5
y	1	3	5	7	9

Principle 2

Each of the pairs of x and y values as seen in Table 1 represent points on a straight line or quadratic curve. The coordinates of the points would therefore be (1, 1), (2, 3), (3, 5), (4, 7) and (5, 9).

Principle 3

To determine which general equation, linear or quadratic, needs to be used, the type of number pattern would first need to be determined. To accomplish that, the difference between each successive term needs to be determined first. In this case, the difference between each successive term is +2 which happens to be a common difference. This also indicates that for every one unit increase in the input, there is a corresponding increase of 2 units in the output clearly showing that the gradient of the function is a constant value of 2. Such would point to the fact that the function in question is of a linear nature.

Principle 4

Once the linearity of the function is established, the general equation of the linear function denoted by $y = mx + c$, where m represents the gradient and c, the y-intercept, can be utilised to find the general relationship between the x and y values. As $m = 2$, a pair of x and y values could be substituted into $y = 2x + c$ to determine the value of c. This would yield the relationship $y = 2x – 1$. Therefore, the general term would simply be $Tn = 2n – 1$. where x is replaced with n, which is the position number of the term, and y is replaced by Tn, the term itself.

Alternatively, the value of c can be found by finding the output for the input of 0. This can be done by finding the zero-th term (Yeo, 2010). In this case, the zero-th term is –1 since we can take 1 – 2 = –1 as the common difference between each term, 2, can be subtracted from the first term of 1. The value of c would therefore be –1.

Teaching quadratic number patterns using quadratic functions

In some cases, the number pattern does not follow a linear pattern as seen in Example 1. Instead, the number pattern could be of a quadratic nature and the general equation of a quadratic curve would therefore need to be used to find the general term. Example 2 illustrates this.

Example 2

Find the general term in the following number pattern.

1, 3, 6, 10, 15…

Principle 1

Again, the input and output values first need to be identified. The position number of each term would increase sequentially in chronological order. In this case, the position number of each term would be as illustrated by the first row of numbers represented by variable x while the term itself would be represented by variable y as seen in Table 2 below.

Table 2. Input (x) and output (y) values.

x	1	2	3	4	5
y	1	3	6	10	15

Principle 2

Each of the pairs of x and y values as seen in Table 2 represent points on a straight line or quadratic curve. The coordinates of the points would therefore be (1, 1), (2, 3), (3, 6), (4, 10) and (5, 15).

Principle 3

To determine which general equation, linear or quadratic, needs to be used, the type of number pattern would first need to be determined. To accomplish that, the difference between each successive output needs to be determined first. In this case, the difference between each successive output is not the same as in Example 1 but it increases as seen in Table 3.

Table 3. Input (x) and output (y) values showing incremental change.

x	1	2	3	4	5
y	1	3	5	7	9

+2 +3 +4 +5

+1 +1 +1

This incremental change +2, +3, +4, etc. from successive terms indicates that the incremental change follows a linear pattern as seen in Table 3. This is so because the difference of each successive incremental change is a constant of +1. Since the difference between each successive output follows a linear pattern, it can be deduced that the first derivative of the function in question is that of a linear nature. This would imply that the function itself is a quadratic one as the integral of a linear function would yield a quadratic function. Hence, we can conclude that the function in question is of a quadratic nature.

Principle 4

Once the quadratic nature of the function is established, the general equation of the quadratic function denoted by $y = ax^2 + bx + c$, where a, b and c are constants, can be utilised to find the general relationship between the x and y values. The author has outlined four methods for finding the values of a, b and c.

Method 1

Any three pairs of x and y values would be required to find the values of a, b and c. This is obtained by solving three linear equations with three unknowns, a, b and c as seen below. The coordinates of the points chosen were (1, 1), (2, 3) and (3, 6).

$$1 = a + b + c$$
$$3 = 4a + 2b + c$$
$$6 = 9a + 3b + c$$

Solving the three linear equations above, one would obtain:

$$a = 0.5$$
$$b = 0.5$$
$$c = 0$$

Hence, substituting any three pairs of x and y values would yield the relationship $y = 0.5x^2 + 0.5x + 0$. The general term would then be $Tn = 0.5n^2 + 0.5n$ where x is replaced with n, which is the position number of the term, and y is replaced by Tn, the term itself.

The problem with method 1 is that students would have to solve three linear equations in a, b and c which they would not have covered in their GCE-O level syllabus for Elementary or Additional Mathematics in Singapore. However, this can be easily overcome by teaching students how to utilise the linear equation function in their calculators which they can use for their GCE-O level examination. The values of a, b and c would be displayed in the calculator upon keying in the coefficients of a, b and c.

Method 2

Instead of solving three linear equations as seen above, a pair of simultaneous linear equations with two unknowns a and b could be formed and solved instead. This would mean that the value of c would need to be found first which can be done by finding the zero-th term. As postulated by Yeo (2010), the zero-th term can be found by looking for the output with an input of 0, which is the value of the y-intercept or c. Since the difference between the successive outputs increase in a linear pattern, i.e., +2, +3, +4, +5 etc., it can be deduced that the difference between the zero-th term and the first term is +1 based on the linearity of the pattern. Hence, the zero-th term, or the value of c is 0 as seen in Table 4.

Table 4. Finding the zero-th term.

x	0	1	2	3	4	5
y	0	1	3	6	10	15

+1 +2 +3 +4 +5

With the value of c as 0, only two pairs of points would be needed to find the values of a and b. Using the coordinates of two points as (1, 1) and (2, 3), the pair of simultaneous linear equations would be as follows:

$$1 = a + b$$
$$3 = 4a + 2b$$

Solving the pair of simultaneous linear equations, one would obtain:

$$a = 0.5$$
$$b = 0.5$$

Therefore, the relationship between the input and output would be $y = 0.5x^2 + 0.5x + 0$. The general term would then be $Tn = 0.5x^2 + 0.5x + 0$ where x is replaced with n, which is the position number of the term, and y is replaced by Tn, the term itself.

Method 2 would prove to be algebraically not as complicated as Method 1 as it involves solving a pair of simultaneous linear equations instead of three linear equations. As students have covered solving a pair of simultaneous linear equations in Secondary Two, this method would enable students to fall back on their prior knowledge.

Method 3

An alternative to finding the values of a, b and c using the general form of a quadratic function is to transform the quadratic function into a linear function using the concept of linear law. Linear law describes the relationship between two variables, X and Y and are related by the linear equation $Y = MX + C$, where M is the gradient of the straight line and C, the Y-intercept. Therefore, $y = ax^2 + bx + c$ needs to be transformed to $Y = MX + C$ as seen below.

$$y = ax^2 + bx + c$$
$$y - c = ax^2 + bx$$
$$\frac{y-c}{x} = ax + b$$

Therefore, $Y = \dfrac{y-c}{x}$, $X = x$, $M = a$ and $C = b$.

Again, the value of c can be determined by finding the zero-th term as shown in Table 4. The value of c would then be 0. This would mean that $Y = \frac{y}{x}$. Table 5 would need to be created to determine the inputs and corresponding outputs on the linear graph of Y against X.

Table 5. Input (x) and output (y) values.

$X = x$	1	2	3	4	5
$Y = \dfrac{y}{x}$	1	$\dfrac{3}{2}$	2	$\dfrac{5}{2}$	3

The gradient, M, of the linear function $Y = MX + C$, would simply be 0.5 since there is a common difference of 0.5 between each successive output per unit increase of the input. To find the value of C, a point needs to be selected and substituted into the equation $Y = MX + C$. Taking (1, 1) where $Y = 1$ and $X = 1$,

$$1 = 0.5(1) + C$$
$$C = 0.5$$

Therefore, the relationship between the input (X) and output (Y) would be $Y = 0.5X + 0.5$. This translates to $\frac{y}{x} = 0.5x + 0.5$. Manipulating the equation, one would obtain y = $0.5x^2 + 0.5x$. The general term would then be $Tn = 0.5n^2 + 0.5n$. where x is replaced with n, which is the position number of the term, and y is replaced by Tn, the term itself.

Method 3 is the most complex method as the quadratic equation would need to be transformed to a linear equation. This would involve the manipulation and transformation of $y = ax^2 + bx + c$ into $Y = MX + C$ which requires the knowledge of linear law, a topic taught only in the Additional Mathematics GCE-O level syllabus.

Method 4

A fourth method that could be used would be using the concept of the first derivative and second derivative of $y = ax^2 + bx + c$. As with Method 2, the zero-th term can be found by looking for the output with an input of 0, which is the value of the y-intercept or c. In this case, the zero-th term is 0 as seen in Table 4, implying that $c = 0$. Once that has been established, the quadratic function would be $y = ax^2 + bx$.

There is now a need to find the first and second derivative of the quadratic function $y = ax^2 + bx$. For this to be accomplished, the conceptual understanding of both derivatives with respect to the differences between the output values need to be first established. The first derivative, or the gradient function of the quadratic function, would follow the number pattern illustrated by the increasing difference between the outputs whereas the second derivative would follow the pattern represented by the constant difference between the difference of the difference of the outputs. This is shown in Table 6.

Table 6. Input (x) and output (y) values showing first and second derivative.

x	1	2	3	4	5
y	1	3	6	10	15
$\frac{dy}{dx}$		+2	+3	+4	+5
$\frac{d^2y}{dx^2}$		+1	+1	+1	

The first derivative would be $\frac{dy}{dx} = 2ax + b$ and the second derivative would be $\frac{d^2y}{dx^2} = 2a$. As seen in Table 6, the second derivative is a constant of +1 implying that $2a = 1$ and therefore $a = \frac{1}{2}$. This would yield the following quadratic function:

$$y = 0.5x^2 + bx$$

To find the value of b, the coordinates of any point can be substituted into the above equation. The point (1, 1) was chosen and hence the following was obtained:

$$1 = 0.5(1)^2 + b(1)$$
$$b = 0.5$$

One would therefore obtain $y = 0.5x^2 + 0.5x$. The general term would then be $Tn = 0.5n^2 + 0.5n$. where x is replaced with n, which is the position number of the term, and y is replaced by Tn, the term itself.

This method requires a good understanding of the first and second derivative which is only taught in the GCE-O Additional Mathematics syllabus. Thus, this method would not be accessible to students who only take Elementary Mathematics. Furthermore, this method requires the identification of the differences between the successive outputs as the first derivative and infer therefore that the difference of the differences between the outputs is the second derivative. This might prove to be a difficult endeavour for students who do not have a good conceptual understanding of derivatives.

Implications for teaching and learning

This approach is a definitive method that students can fall back on to generate the general term without having to spend time formulating a connection between the position of the term and the term itself. As research has shown, formulating the link between the position of the term and the term itself and translating that into algebraic notation poses the greatest challenge (English & Warren, 1995; Rivera & Becker, 2007; Stacey & MacGregor, 2001; Ursini, 1991). Thus, having a definitive method bodes well for students.

School administrators and department leaders may need to consider restructuring the mathematics curriculum in their respective secondary schools slightly in order to accommodate the teaching of number patterns after linear or quadratic functions are taught. This is an important consideration as the topic of number patterns is usually taught before the topics of linear and quadratic functions. A suggestion would be to teach number patterns that follow a linear pattern just after the topic of linear functions in Secondary One and teach number patterns that follow a quadratic pattern after the topic of quadratic functions in Secondary Three.

One of the keys to this approach is to link number patterns to the prior knowledge of students. In this case, their prior knowledge would be their knowledge of linear and quadratic functions. Hence, helping students make the link not only activates their prior knowledge, but also helps them bridge the gap between the abstract algebra in linear and quadratic functions and number patterns. Essentially, this could lead students to see that:

- number patterns follow a certain pre-defined pattern rooted in linear or quadratic functions; and
- points along a straight line or quadratic curve follow a certain pre-defined pattern rooted which are inherent in number patterns.

Being able to see the manifestation of one concept in another could be a powerful tool in the conceptual understanding of both number patterns and functions (linear or quadratic).

There are also some important considerations for teachers when teaching number patterns using this approach. They are outlined as follows:

Firstly, when using this approach, it is crucial for teachers to point out the input (x) and output (y) values so that students can relate the number pattern to either a linear or quadratic function. This would help students "assign meaning to the formal letters and to appreciate the variable nature of these symbols" (English & Warren, 1995, p. 6). In addition, this step is important in developing the functional relationship between the input (x) and output (y) values (Chua & Hoyles, 2014).

Secondly, it is important to present opportunities for students to generate and explore the relationship between the inputs and the outputs before consolidating and showing that the relationships generated by students all follow either a linear or quadratic pattern. This learning opportunity would prepare them to learn from their own failed or success experiences and is crucial in convincing them that the number patterns follow the general equation of a linear or quadratic function.

Conclusion

The approach, based on the four principles presented in this paper, seeks to address the problem that there is a lack of a definitive approach to finding the general term of a number pattern in the linear or quadratic form. The current approach taught in secondary schools in Singapore is difficult to utilise for finding "an expression for the nth term" (Yeo, 2012). Thus, this gave the impetus for the design of the above approach where the general term of a number pattern could be determined more concretely by using the general equation of a linear or quadratic function.

As the term 'number patterns' suggests, there is an inherent underlying pattern in the number sequence. Finding the underlying pattern is not difficult but formulating a rule in algebraic notation that connects the position of the term and the term itself is the main challenge (English & Warren, 1995; Rivera & Becker, 2007; Stacey & MacGregor, 2001; Ursini, 1991). These four principles are an attempt to mitigate this challenge by using linear and quadratic functions to find the general term in a number pattern.

Moving forward, the use of graphical representations of the linear or quadratic functions could be incorporated into the four principles in order for a better visualisation of the approach. Future research could look into how the values of a, b and c can be determined directly from the quadratic graph without having to resort to forming linear equations, using the concept of linear law, or using the first and second derivatives for a number pattern following a quadratic form.

References

Chua, B. L. & Hoyles, C. (2014). Generalisation of linear figural patterns in secondary school mathematics. *The Mathematics Educator, 15*(2), 1–30.

English, L. D. & Warren, E. A. (1995). General reasoning processes and elementary algebraic understanding: Implications for initial instruction. *Focus on Learning Problems in Mathematics, 17*(4), 1–19.

Kaput, J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds), *Algebra in the early grades* (pp. 5–17). New York: Lawrence Erlbaum Associates.

Mason, J., Graham, A. & Johnston-Wilder, S. (2005). *Developing thinking in algebra.* London: Paul Chapman Publishing.

Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalisations of patterns in different contexts. *ZDM: International Journal in Mathematics Education, 40*, 83–96.

Rivera, F. D. & Becker, J. R. (2007). Abduction-induction (generalisation) processes of elementary majors on figural patterns in algebra. *Journal of Mathematical Behavior, 26*(2), 140–155.

Stacey, K. & MacGregor, M. (1997). Building foundations for algebra. *Mathematics Teaching in the Middle School, 2*, 253–260.

Ursini, S. (1991). First steps in generalisation processes in algebra. In F. Furinghetti (Ed.), *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 316–323). Assisi, Italy: PME.

Yeo, B.W. (2010). Finding the general term for an arithmetic progression: Alternatives to the formula. *The Australian Mathematics Teacher, 66*(2), 17–21.