Eliciting algebraic reasoning with hanging mobiles


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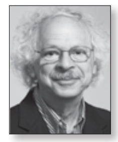
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How algebraic reasoning can be fostered within the important big idea of equivalence is demonstrated using hanging mobiles. A concrete-representational-abstract approach is used, without any formal algebraic symbolism, to elicit algebraic reasoning and higher-order thinking.

The importance of laying a foundation for algebraic reasoning at a young age is increasingly being emphasised. In this article, we report on an activity that elicits in a natural way algebraic strategies that in a later stage of learning algebra are crucial for solving equations. The activity brings about the students' spontaneous use of symbolic notations. It also makes students' reasoning visible both to themselves and to their teacher and helps build students' conceptual understanding and foster productive classroom discussion. This activity involves working with a hanging mobile.

In this article we describe the work of students aged 11–12, using a hanging mobile as shown in Figure 1. This mobile can be considered as a balance model representing an equation with unknowns. The chains on both sides of the mobile support coloured bags. The different bags each have a particular weight to make the balance 'workable', but the weight aspect is not explicitly mentioned to the students. Neither are the students told that the activity is about solving equations with unknowns. For them, this is a puzzle. They have to figure out what you can do with the bags in the hanging mobile while keeping the mobile in balance and then use these strategies to find relationships among the bags. Figuring out these puzzles does not require formal algebra, as students can physically add, remove and exchange bags and can see the result of their actions.

In this way, they develop strategies such as restructuring (e.g. moving bags with respect to each other but keeping them on one side), isolation (e.g. removing the same bags from both sides of the hanging mobile), and substitution (replacing bags with different coloured bags), strategies which are very important for solving equations. In fact, the informal strategies that come up now form a pre-stage of the most prominent strategies that are used when solving formal equations. Promoting such informal reasoning in the field of algebra is in line with the proficiency strands of the *Australian Curriculum: Mathematics* (ACARA, 2015; see also Hurrell, 2012).

The large-size physical hanging mobile that we used in this activity was inspired by mobile puzzles. The idea of these puzzles is old (Kroner, 1997) but their curricular use is newer (Goldenberg, Mark, Kang, Fries, Carter, & Cordner, 2015; see http://solveme.edc.org for an online version). Like the mobile puzzles on paper, the physical model can trigger algebraic reasoning in students. Both do this without requiring formal techniques or notations, but the advantage of the physical hanging mobile is that the students physically interact with the mobile and have an embodied experience in keeping the mobile in balance. Another positive aspect of the physical mobile is that the students can watch each other's actions, which can support classroom discussion.

Figure 1. Physical mobile.

A physical model of equivalence

A balance models the mathematical idea of equivalence, a crucial concept for understanding equations (Faulkner, Walkowiak, Cain, & Lee, 2016; Greenes & Findell, 1999; Hurrell, 2012). The mobile thus represents an equation that students can handle physically, using their sense and intuition to build the logical system of algebra. Part of this system is the meaning of the equal sign. Students often interpret the equal sign as a "to do"sign—as it is on a calculator—instead of an "is equal to"sign (Carraher & Schliemann, 2007). The hanging mobile presents this latter meaning without any notational hurdle and builds on the balancing experiences young children already have in everyday life (e.g., seesaws). The mobile gives students bodily experiences of the concept of equivalence, which can help them anchor this concept, in line with the theory of embodied cognition (cf. Núñez, Edwards, & Matos, 1999).

Lala's work with the hanging mobile

We will focus first on one child, showing how the physical hanging mobile can give relevant experience on reasoning about equivalence. Lala, 12 years old, is in a sixth-grade classroom in the Netherlands. She had no prior instruction on solving equations; early algebra is not part of the Dutch primary school mathematics curriculum. Lala was presented an empty mobile and some black, grey, and white bags that could be hung on it. Two black bags were equivalent to three grey bags and were also equivalent to six white bags, but Lala was not told this information. She was interviewed by Mara, the first author of this article.

Because we wanted to know Lala's reasoning, it was important to hear what wording she chose when working with the mobile. Therefore, we introduced the task without using terms like hanging mobile, balance, equal, and equivalence. Only when Lala spontaneously used these more formal words did Mara use them as well. Mara (M) began the activity with the following question:

M: Can you find out what happens to this thing when you add or remove bags?

L: [Hangs a white, grey, and black bag on each side (Figure 2a)].

M: Well done! You have two white ones and also a grey one remaining. Could you use these [bags] as well?

L: [Hangs the grey on the left, the two whites on the right; watches the mobile (Figure 2b)].

M: What are you thinking now? […]

L: Well, I think that these ones [Lala points to the three bags (white, grey, black) on the left side (Figure 2b), looks back and forth to the same bags on the right side] keep each other in balance. And this one [points to the remaining grey bag on the left] is equally heavy as these two [points to the remaining two white bags on the right].

M: OK, how can you find out if this is correct?

L: I don't know. Oh wait, I think I know [takes away all bags except for one grey bag on the left side and two white bags on the right (Figure 2c)].

Without explicitly being asked, Lala tried to bring the mobile into balance. From the start, she seemed to

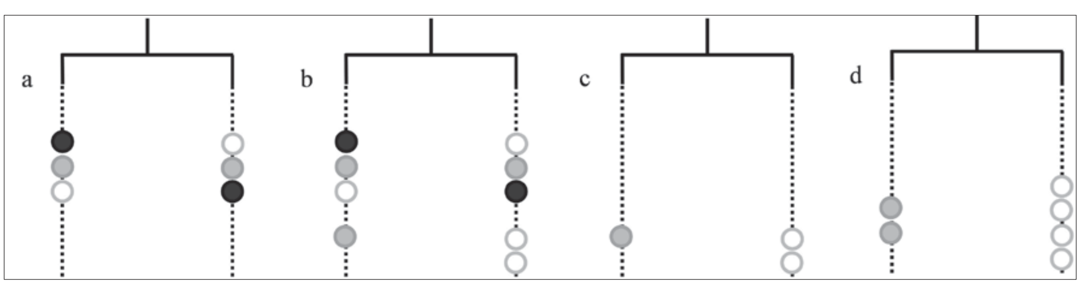

Figure 2. Schematic representation of the mobiles Lala created.

expect equivalence to apply to this situation. Moreover, she could express one unknown in terms of another unknown (one grey is "equally heavy" as two whites; in formal algebra this would be expressed as $y = 2x$). She had some difficulties choosing the right words, but she made herself understood by using gestures. Eventually she could test her assumption that one grey "is as heavy as" two whites, by applying the algebraic isolation strategy. By removing equal numbers of equal unknowns on each side (one white, grey, and black bag), only one type of unknown remained on each side of the mobile.

Mara asked if Lala could add more bags to the mobile, making sure that it remained "like this" (horizontal gesture with her arm). Lala replied that she would add "the double", and hung two grey bags on the left and four white on the right (Figure 2d):

L: One grey equals two white ones. And then if you add another grey one, you have to double the white ones as well.

Lala showed she could generate equivalent sets of bags by keeping the ratio the same.

To stimulate Lala to think about which words and notations to use—a first step towards a formal notation—we asked Lala to draw or write down her findings (Figure 3). This stimulation is important, since research has shown that even for Grade 6 and 7 students who are already able to solve equations, formal symbolising can still be a major obstacle (Van Amerom, 2003). Lala used the equal sign correctly to express the relationships between the bags (Figure 3), showing that she understood that it was not just a sign of 'here comes the answer'.

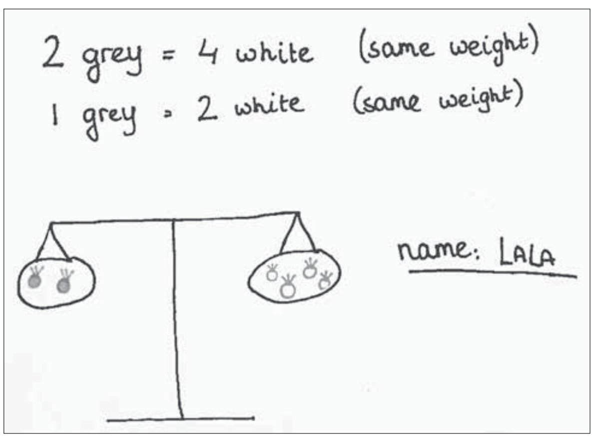

Figure 3. Lala's notes (translated from Dutch) explaining the mobile shown in Figure 2d.

From the physical mobile to drawings on a worksheet

Shifting from operating a physical mobile to solving puzzles on paper (Figure 4) helps move the experiments into one's head. We showed Lala two mobiles that we said were balanced (mobiles A and B in Figure 4). From that, she had to discover relationships between the coloured blocks, and use these relationships to complete the empty mobiles, using only one colour on each side and making them balance. This challenge involves more than the transition from the physical mobile to paper; one must combine the values expressed in two mobiles to come to an answer. In other words, students have to develop algebraic strategies such as isolating unknowns and substituting them to solve the problems. For students who have not yet encountered problems like these and have not developed a standard method for solving these problems, this requires higher-order thinking.

Mara clarifies the worksheet without giving Lala any scaffolding for solving the puzzle.

M: Here you see two examples [points to mobiles A and B, Figure 4], you can use to solve the problem. To make one that hangs like this [points to problem 1, Figure 4.1, making a horizontal gesture with the arm], how many greys do you have to put on this side [left] and how many blacks on that side [right]? Just try, you can do anything you want, including crossing things out.

L: I think… I don't know! Oh, I think that two of these blacks equal one grey [points to the four black blocks in mobile A, then to the two grey blocks].

M: Why do you think so?

L: Because eh… This is already correct: one white and another white [points to the two white blocks balancing each other in mobile A]. But here are two [points to the two greys in mobile A], and there are twice as many [points in mobile A to the four black], so I thought, if I divide this by two, then two blacks equal one grey (see Figure 4.1 for her drawing).

This pictorial notation is not algebraic, but Lala's description and reasoning are; she can simplify the equation by taking away equal unknowns on both sides of the equal sign to isolate the grey unknown. She also used this isolation principle when working with the physical mobile. Subsequently, she demonstrated that she could simplify the remaining equation (2 grey = 4 black), by performing the same operation on both sides of the equal sign (dividing by two). The experience of puzzling this out for oneself is powerful: it shows students not only that they can figure out things for themselves, but gives them a much more intuitive and logic-based understanding than an explained set of procedures is likely to give them.

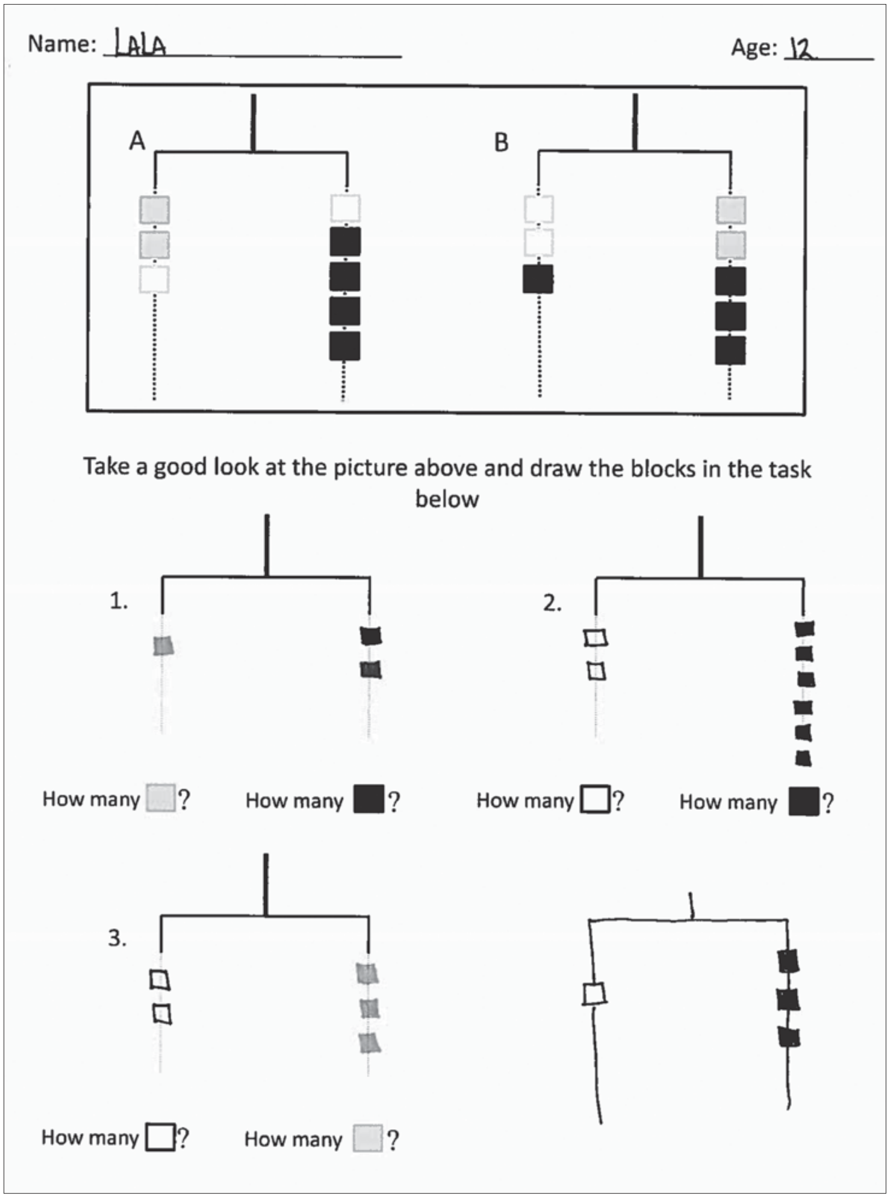

Figure 4. Lala's worksheet (questions translated from Dutch).

Lala then continued with the second task.

L: [Points first to her solution of problem 1 (Figure 4), going back and forth between the grey block on the left side and two blacks on the right]. You already know for sure that two of these [black] equal one grey. [Then points to mobile B (Figure 4)]. So two whites have to be equal to three greys. If you have one white… That would be how many blacks…? [draws one white block as a start (Figure 4.2)]. Two blacks equalled one grey [points to her solution of the first assignment]. And… Wait, I can draw two whites, that makes it easier [then draws the second white block (Figure 4.2)].

Lala drew two white blocks in Figure 4.2 probably because the given mobile B (Figure 4) also has two white blocks on the left side.

L: Two whites would imply three greys here, for example [points to where the black blocks have to be drawn]. So… yes, six! Six blacks [draws six black blocks]! So one such thing [white block] equals three blacks.

M: Why does one such thing equal three blacks?

L: Because two of these… [points to the drawn white blocks in Figure 4.2]. Oh, yes! If you divide this by two [points again to the white blocks] then you also have to divide this by two [points to the six blacks; starts drawing a new mobile (Figure 4, bottom right)].

Lala clearly expresses a proportional relationship she has seen, and she uses the algebraic principle of substitution to come to the solution, an element of algebraic reasoning we had not seen her use earlier. She used her previously found solution of two blacks equalling one grey to turn the three greys into six blacks.

In the classroom

Figure 5. Two boys working with the physical mobile in the classroom.

Lala's work is quite representative of what happened with a similar activity in a classroom. A fifth-grade class tackled the transition from working with one physical mobile (Figure 5), to combining the information of two mobiles (Figure 6) to determine what should make a third mobile balance.

Figure 6. Tessa combines information of two mobiles to explain the composition of the third.

The teacher set up the three mobiles (Figure 7) and asked the class whether, based on the first two mobiles (A and B), they could have known that the bags on third (C) had to be hung in this way, for the mobile to be straight. Although the context (and model) of the mobile plays a crucial role in eliciting students' algebraic reasoning, the contribution of the teacher is equally important. The students are challenged and put at ease. Students can see that the third mobile (C) is hanging straight, so finding the result is not an issue; the students can concentrate on figuring out how the relationships in the first two mobiles (A and B) can lead to the third mobile.

Tessa (Figure 6) explained: "Yes [you could have known that C had to look like this], because two blacks equal one grey, and two greys equal one white, so we have to double the greys to get one white, and then we have to double the other side too." The teacher then challenged the class: "Can you show this on the mobiles?" This prompted Berkay (Figure 8) to substitute one grey bag with two black bags. The physical mobile made Berkay's strategy visible to the teacher and the other students, allowing the teacher to assess progress and also to give useful feedback if needed (see, e.g., Hattie & Timperley, 2007). The teacher helped the students to structure their reasoning and provided them with the adequate language to describe this process, by explaining that Berkay substituted the grey bag with two black ones.

Subsequently, students were asked to write down their thinking when solving similar tasks on paper. Figure 9 shows the work of five students that wrote down the ratio between green and red bags (students 1 and 2) and the ratio between red and blue bags (students 3,

Figure 8. Berkay substitutes the grey bag with two black bags.

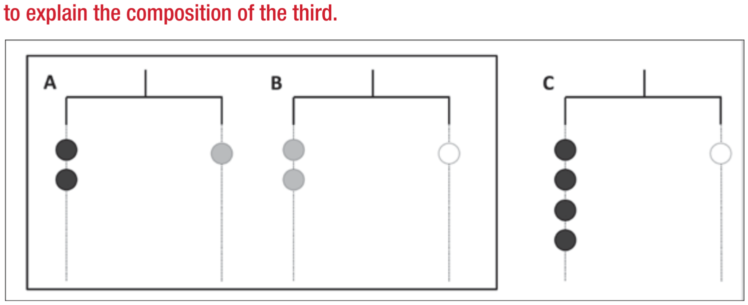

Figure 7. Schematic representations of the physical mobiles shown in the classroom.

4 and 5). While some (students 3 and 5) made themselves clear by drawing the bags, others (students 1 and 2) used words or even letters (student 4) to represent the unknowns. Moreover, students 1 to 4 spontaneously grouped the unknowns when notating them, while student 5 expanded them by drawing two blue bags instead of a blue bag with the number 2 in front (cf. student 3). Lastly, four of them used the = sign to express the relationships between the bags. This is noteworthy, since one of the common pitfalls in algebra is to see the equal sign as a signal to perform an operation, while understanding of the equal sign as a relational symbol of equivalence is important for understanding and solving of algebraic equations (Knuth, Stephens, McNeil, & Alibali, 2006). As such, asking students to write down their thinking can be a first step towards the use of formal notations.

Conclusion

The physical experience with the hanging mobile gives students the opportunity to develop embodied knowledge of balance and equivalence, and prompts them to come up with strategies such as isolation and substitution that we normally teach in algebra in secondary school. These activities show that students at the end of primary school, without having been taught formal algebra, are already able to reason about equations with unknowns and even about systems of equations with multiple unknowns.

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Figure 9. Different ways of notating ratios between the coloured bags by students 1 to 5.