Colour by numbers

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This is an edited extract from the keynote address given by Dr Chris Wetherell at the 26th Biennial Conference of the Australian Association of Mathematics Teachers Inc. The author investigates the surprisingly rich structure that exists within a simple arrangement of numbers: the times tables.

Introduction

When you are old enough (say, in Year 11) you might learn that a sequence is just an infinite list of numbers. That's it, just a list. Of course, it is human nature to be more fascinated by the ones which have some kind of interesting pattern—and you can easily spend hours of your life looking for them on *The Online Encyclopaedia of Integer Sequences*, http://oeis.org/. In this paper, the sequences that are deemed to be 'interesting' enough for further discussion include the counting numbers, squares, cubes, triangular numbers, tetrahedral numbers and Fibonacci numbers, among others. The search for these sequences in the times tables is motivated by well-known properties of Pascal's Triangle, a simple triangular arrangement of numbers which finds important application in the fields of algebra, probability, financial arithmetic and calculus, to name just a few.

Number patterns in the times tables

This familiar grid of numbers has rows and columns both indexed by the counting numbers, with the value in the i^{th} row and j^{th} column the product $i \times j$, as shown in Figure 1. We will ignore the shaded row and column headings in later diagrams.

×	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	8	10	12	14
3	3	6	9	12	15	18	21
4	4	8	12	16	20	24	28
5	5	10	16	20	25	30	35
6	6	12	18	24	30	36	42
7	7	14	21	28	35	42	49

Figure 1. The times tables.

We begin with a search for some of the interesting number sequences mentioned above.

Square numbers

Easy! As highlighted in Figure 2, the main diagonal contains the square numbers since when the row number and column number are both equal to n, say, then the entry is $n \times n = n^2$. Square numbers are revisited several times in the full paper.

1	2	3	4	5	6	7
2	4	6	8	10	12	14
3	6	9	12	15	18	21
4	8	12	16	20	24	28
5	10	16	20	25	30	35
6	12	18	24	30	36	42
7	14	21	28	35	42	49

Figure 2. Square numbers in the times tables.

1	2	3	4	5	6	7
2	4	6	8	10	12	14
3	6	9	12	15	18	21
4	8	12	16	20	24	28
5	10	16	20	25	30	35
6	12	18	24	30	36	42
7	14	21	28	35	42	49

Figure 3. Triangular numbers in the times tables.

Triangular numbers

Suppose you are in the $(i, j)^{\text{th}}$ cell, that is, in the i^{th} row and j^{th} column with value $i \times j$. Moving one entry to the right adds i to the value since, being in the i^{th} row, we are moving to the next number in the i-times tables. Similarly, moving one down adds j. It is possible to exploit these facts to find a sequence of cells which follow the triangular numbers starting at the 1 in the $(1,1)^{\text{th}}$ cell, as illustrated in Figure 3. Being in the first row means that moving right by two cells adds $2 \times 1 = 2$, resulting in the 3 in the $(1,3)^{\text{th}}$ cell. Now we are in the third column, so moving down by one row adds 3 to get the 6 in the $(2,3)^{\text{th}}$ cell. Now we are in the second row, so that repeating the move by two cells to the right adds $2 \times 2 = 4$ and leaves us in the fifth column. Repeating the move by one cell down adds 5 to the value. Hence, starting at 1, the overall effect of tracing this path is to add 2, then 3, then 4, then 5, … which results in the sequence of triangular numbers 1, 3, 6, 10, …, as shown. Of course, by the symmetry of the times tables the same pattern can be observed by first moving down two, then one to the right.

To formalise this argument we observe that the n^{th} combined move of "two-to-the-right-followed-by-one-down" takes us from the $(n, 2n-1)^{\text{th}}$ cell to the $(n, 2n+1)^{\text{th}}$ cell with an increase of $2n$ (twice to the right in the n^{th} row), and then to the $(n+1, 2n+1)^{\text{th}}$ cell with an increases of $2n+1$ (one row down in the $(2n+1)^{\text{th}}$ column). This results in the pattern of increases by 2 then 3, 4 then 5, … as claimed.

Alternatively, we can justify this pattern using the formula for the n^{th} triangular number. Following the description above, the value of the $(n, 2n-1)^{\text{th}}$ cell is

$$\begin{aligned} n(2n-1) &= \tfrac{1}{2}(2n)(2n-1) \\ &= \tfrac{1}{2}(2n-1)\big((2n-1)+1\big) \\ &= T_{2n-1} \end{aligned}$$

which generates the 1^{st}, 3^{rd}, 5^{th}, … triangular numbers, and the value of the $(n, 2n+1)^{\text{th}}$ cell is

$$\begin{aligned} n(2n+1) &= \tfrac{1}{2}(2n)(2n+1) \\ &= T_{2n} \end{aligned}$$

which generates the 2nd, 4th, 6th, … triangular numbers.

Consider next the zig-zag-like sequence of cells traced by repeatedly moving down then right, as shown in Figure 4. For convenience, we have added an extra row of zeros for the 'zero-times-tables'.

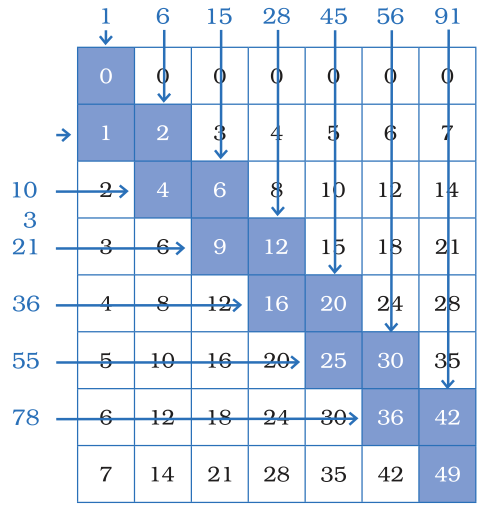

Figure 4. Triangular numbers in the sums of adjacent cells in a zigzag path.

Now calculate the sums of adjacent cells in this zig-zag path, which produces the sequence of triangular numbers alternating between vertically and horizontally adjacent cells. This can be justified using similar techniques to the pattern above. This is left as an exercise.

Cube numbers

It is a reasonably well-known fact that the sum of consecutive cubes is the square of the corresponding triangular number, that is

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(1 + 2 + 3 + \ldots + n^3\right).$$

Given the connection between T_n^2 and the $n \times n$ blocks in the top-left corner, as described in the full paper, it should be possible to find the cube numbers hidden in the Times Tables. Indeed, the induction argument used to establish the above result indicates exactly how: look for the difference between successive blocks, or in other words an L-shaped path. This is illustrated in Figure 5.

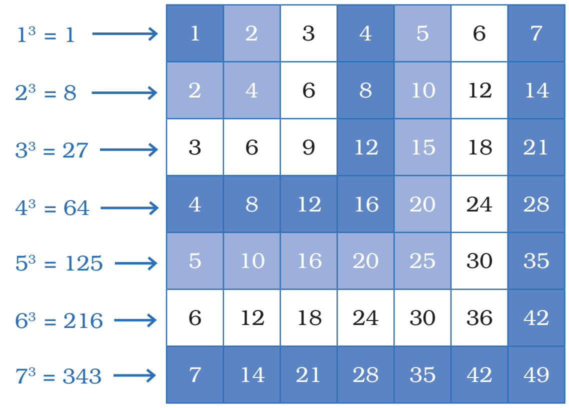

Figure 5. Cubes in the sum of L-shaped paths .

Read the full paper

The full paper develops the necessary tools to analyse the interesting sequences in more detail and also summarises the motivating properties of Pascal's Triangle, including its fractal-like structure: colouring the odds and evens black and white results in the famous Sierpinski Triangle, if zoomed out far enough. As well as continuing the search for sequences, the author demonstrates how to use Microsoft Excel to construct various colour patterns in the times tables which ultimatelylead to its own fractal-like structure. The full paper appears in *Capital Maths: Proceedings of the 26th Biennial Conference of the Australian Association of Mathematics Teachers Inc.* which is available through the AAMT website at http://www.aamt.edu.au/Library/Conference-proceedings/Capital-Maths.