Structure sense: A precursor to competency in undergraduate mathematics

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Introduction

Despite having passed the required level of secondary mathematics, many tertiary students struggle with first year tertiary mathematics. In recent years, university mathematics and statistics departments have been revising both curricula and pedagogy as well as providing extra support to help students through this transition (MacGillivray, 2008). In the study reported in this paper, we have analysed student work in order to track potential sources of students' difficulties. It seems that these difficulties may lie, not in new concepts, but in unsound foundations and in limited and inflexible structural understandings of mathematics established earlier in their experience of mathematics.

Much has been written about young students' number sense, which develops as students meet and engage in a variety of number-related activities ranging from early counting experiences to working with rational and irrational numbers. Howden (1989) describes number sense as "good intuition about numbers and their relationships. It develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms". Greeno (1991, p. 170) refers to number sense as involving "flexible mental computation (recognition of equivalences in order to regroup numbers in mental or calculator-free multiplication), numerical estimation, and quantitative judgment". Arcavi (1994, p. 24) describes number sense as "a 'non-algorithmic' feel for numbers, a sound understanding of their nature and the nature of the operations, a need to examine reasonableness of results, a sense of the relative effects of operating with numbers, a feel for orders of magnitude, and the freedom to reinvent ways of operating with numbers differently from the mechanical repetition of what was taught and memorized".

Arcavi (1994, 2005) suggests that there is an equivalent symbol sense associated with working with symbols in algebra. His description of this symbol sense includes: an understanding of and an aesthetic feel for the

power of symbols, an ability to manipulate and also to 'read through' symbolic expressions, the ability to use symbols to represent problem situations and the realisation that symbols can play different roles in different contexts. Barbeau (1995) notes that some students with technical proficiency lack insight, frequently having got by in the past with rote learning, others have poor skills in algebraic manipulation whilst still others "know variables only as placeholders for numbers, and therefore are totally unprepared to regard algebraic entities in different ways… the ability to conceptualize required for tertiary algebra is not present for many students" (p. 140).

For many students, the introduction to algebra in the early secondary school years occurs while their number sense is still poorly developed. Even for students with well-developed number sense, the links between number and symbolic structure may not be obvious. A common mistake of students, for example, is incorrectly to cancel the 2 in the numerator and in the denominator of a fraction such as

$$\frac{x^3+2}{2}$$

resulting in x^3. Illustrating that this is incorrect by using an equivalent number example, such as

$$\frac{6+2}{2} \neq 6$$

may not necessarily convince all students.

A key aspect of working with both numbers and symbols is a feel for the structure of numerical and symbolic expressions. Anecdotal evidence of common misconceptions and student errors observed by tertiary mathematics tutors in the written work of recent cohorts of their students indicates that students are making similar errors to those typically seen with middle years' secondary students. A sample of errors that had been recorded by the tutors includes:

$$\int \left(x^2\right)^3 dx = \int x^5 dx$$

$$\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$$

$$\sqrt{a^2+b^2} = a+b$$

$$\frac{\cancel{x^2}+2}{\cancel{x^2}+3} = \frac{2}{3}$$

Increasing access to computer algebra systems prompted Pierce and Stacey to consider the mathematical thinking needed for efficient and intelligent use of such technology. This led to their definition of algebraic insight in terms of two broad aspects: algebraic expectation and ability to link representations (Pierce & Stacey, 2001). It is the first of these categories, algebraic expectation, with a sub-category 'identification of structure' (see Table 1) which is of relevance for this paper. Pierce and Stacey suggest that "identifying objects, strategic groupings, and simple factors are all ways in which students can

demonstrate an identification of structure in an expression" (p. 422). Students with a well-developed sense of algebraic structure will recognise at a glance that $2x + 1$ is a common factor $(2x + 1)^2 - 3(2x + 1)$ or notice that the bracketed objects in $(2x + 1)^2 - 3x(2x + 5)$ differ. They will be able to look at the expression $\sin^2 x + 2\sin x\cos x + \cos^2 x$ and recognise that it can be strategically grouped in two different ways, allowing it to be expressed as the perfect square $(\sin x + \cos x)^2$ or as $\left(\sin^2 x + \cos^2 x\right) + 2\sin x\cos x = 1 + \sin 2x$. Competence at the senior secondary level with the three categories: recognition of conventions and basic properties; identification of structure; and identification of key features is critical to the success of tertiary students.

Table 1. Sub-categories of algebraic expectation from Algebraic insight framework (adapted from Pierce and Stacey, 2001).

Aspects	Elements	Common instances
Algebraic expectation	Recognition of conventions and basic properties	Know meaning of symbols Know order of operations Know properties of operations
	Identification of structure	Identify objects Identify strategic groups of components Recognise simple factors
	Identification of key features	Identify form Identify dominant term Link form to solution type

Several researchers (see Booth, 1988; Greeno, 1991; Kieran, 1992; Lins, 1990) suggest that many of the fundamental difficulties experienced by beginning algebra students are due to their failure to identify equivalent forms of an algebraic expression. Linchevski and Livneh (1999) refer to the importance of students developing structure sense: "this means that they will be able to use equivalent structures of an expression flexibly and creatively. Instruction should promote the search for decomposition and recomposition of expressions and guarantee that the mental gymnastics needed in manipulating expressions makes sense" (p. 191). Sfard and Linchevski (1994) note students' lack of awareness that "strings of symbols" might be interpreted in many different ways, depending on the context. An algebraic expression may be regarded, for example, as a set of symbols to be manipulated (a process perspective), or as a structural object (object perspective). Gray and Tall (1994) use the term "procept" to describe the duality of meaning associated with a symbolic representation that has both concept meaning and process meaning: "an elementary procept is the amalgam of three components: a process that produces a mathematical object, and a symbol that represents either the process or the object" (p. 121). Van Stiphout, Drijvers and Gravemeijer (2013) assert that "a limited feel for the structure of expressions, and for (sub)expressions as objects is an obstacle for reaching a higher level of conceptual understanding" (p. 76).

Hoch and Dreyfus (2006) define a similar set of abilities with respect to structure sense for high school algebra: recognising a familiar structure in its simplest form; dealing with a compound term as a single entity and through an appropriate substitution recognise a familiar structure in a more complex form; and choosing appropriate manipulations to make best use of a structure. They note that some students who can easily solve $3x^2 - 2x = 1$ do not recognise similar quadratic structure in the equation $3\cos^2 x - 2\cos x = 1$. Hoch (2003) reports on a similar lack of "structure sense" when students fail to observe related structural units in an expression, for example, $\left(x^2 - 4x\right)^2 - x^2 + 4x = 6$: "perhaps they think that brackets must always be opened as a first step to solving an equation; or they don't notice that the term $x^2 - 4x$ appears twice, once inside brackets, and once multiplied by –1; or they are just not aware that it is possible to operate on a term as a single unit, substituting one symbol in place of the term to obtain a simpler equation" (p. 1). Students with well-developed structure sense will notice that the equation can be expressed as $\left(x^2 - 4x\right)^2 - \left(x^2 - 4x\right) - 6 = 0$, a quadratic equation in which the left side can be factorised readily.

Hoch and Dreyfus (2006) contrast structure sense with manipulation skills: solving an equation or factorising an expression when given explicit instructions; and substituting correctly in a given formula. In a study with high-achieving mathematics students, Hoch and Dreyfus found that the majority of these students did not use a high level of structure sense and also displayed poor manipulation skills when solving exercises requiring the use of algebraic techniques. Students who applied structure sense made fewer mistakes than those who did not, which Hoch and Dreyfus suggest may indicate that "using structure sense leads to shorter, more efficient solutions, and thus leaves less room for calculation errors" (p. 311).

Van Stiphout, Drijvers and Gravemeijer (2013) consider structure sense as part of symbol sense rather than being a separate ability: "the part of symbol sense that involves seeing structures and patterns in algebraic expressions and equations, which is needed while carrying out algebraic manipulations such as simplifying expressions and solving equations" (p. 64).

Structure sense in the written solutions of undergraduate mathematics students

In this article we analyse the written solutions of some first year undergraduate mathematics students from Victorian universities as they answered tutorial exercise questions relating to complex numbers and differentiation. These students had studied at least Mathematics Methods or its equivalent at secondary school. Complex numbers was a new topic for some students. The focus in these examples is on the extent to which students display an awareness of structure in complex number and algebraic expressions. We assert that

structure sense includes an awareness of structure in numeric as well as in algebraic expressions. We emphasise that the lack of structure awareness reported here was not observed with all students—there were obviously many students who had a sound grasp of numeric and algebraic structure. However our observations suggest that there are also many students for whom this is not the case.

The following examples and figures relate to introductory tutorial worksheets on complex numbers—the topic of the students' lectures (note: the example numbers have been changed for the clarity of this report). The following tutorial worksheet problem (Example 1) requires students to express a complex number division in Cartesian form.

Example 1

Express the following number in Cartesian form $a + ib$ with a and b real.

$$\frac{3-4i}{1-i}$$

Student A, whose solution is shown in Figure 1, multiplied the fraction by

$$\frac{1+i}{1+i}$$

but when asked about this, she did not know why. She knew she needed to make the denominator of the fraction a real number, but had learned this only as a procedure and did not recognise $(1 - i)(1 + i)$ as a difference of squares. When further questioned, she also did not recognise $(x + y)(x - y)$, a difference of squares. The tutor commented later that some students recognised a difference of squares only if it was expressed as $(a+b)(a-b) = a^2 - b^2$, a memorised rule. This suggests a lack of flexibility in the students' thinking. Hoch and Dreyfus (2009) noted that naming a structure such as a difference of squares is an important part of learning it—the name is part of the definition and helps students to use it. A further indication of the student's lack of attention to structure is the expression of the answer as

$$\frac{7-i}{2}$$

(Figure 1) rather than in the required Cartesian form

$$\frac{7}{2} - \frac{1}{2}i$$

Rather than demonstrating flexible thinking and responding to the new mathematical context of complex numbers, this student is attached to the 'common denominator' form to 'simplify' fractions.

$$\frac{3-4i}{1-i} \times \frac{1+i}{1+i} = \frac{3-4i+3i-4i^2}{1+i-i-i^2}$$

$$= \frac{7-i}{2}$$

Figure 1. Student A's solution for Example 1: Expressing a complex number in Cartesian form.

The next tutorial worksheet problem (Example 2) requires students to simplify a complex number expression.

Example 2

Simplify the following complex number

$$\left(\frac{2-2i}{\sqrt{3}+i}\right)^{12}$$

Student B, whose solution is shown in Figure 2, did not notice the common factor of 2 in the numerator, which would have simplified the working. Nor did she recognise that $\sqrt{8}$ could be expressed as $2\sqrt{2}$ or demonstrate a flexible command of symbolic notation that would allow her to replace $\sqrt{2}$ with

$$2^{\frac{1}{2}}$$

Even when the student was given some prompts, she did not link

$$\left(2^{\frac{1}{2}}\right)^{12} = 2^6$$

We could expect a student to identify the structure related to the familiar index law $\left(2^a\right)^b = 2^{ab}$ (typically taught in Years 8 or 9) but this student (and others) apparently did not recognise this syntax when it was not presented with the symbols that they had met before.

The following problem (Example 3) requires students to find the modulus of a complex number without multiplying into Cartesian form

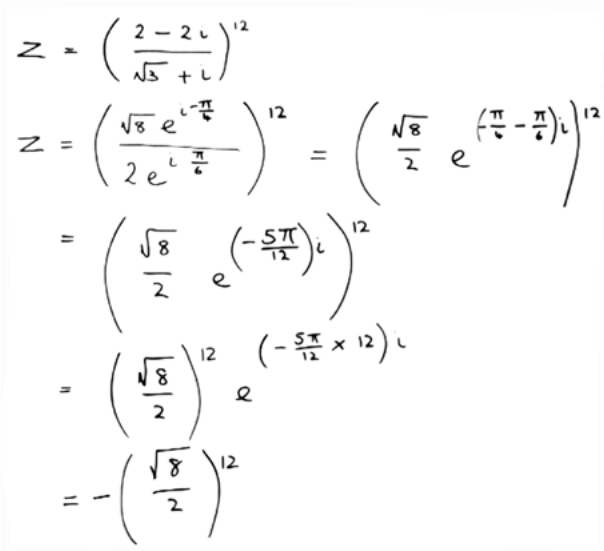

Figure 2. Student B's solution for Example 2: simplifying a complex number expression.

Example 3

Find the modulus of the following complex number without multiplying into Cartesian form.

$$\frac{-7i(2-5i)(2+3i)}{(6+4i)(5+2i)}$$

Several features of student C's solution (see Figure 7) are noteworthy. Given his approach to finding the modulus, the solution steps are all correct, although it would be expected that a tertiary mathematics student might recognise that

$$\frac{\sqrt{13}}{\sqrt{52}} = \sqrt{\frac{13}{52}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

and hence be able to simplify his answer

$$\frac{7\sqrt{13}}{\sqrt{52}} \text{ to } \frac{7}{2}$$

Again flexible thinking is required to understand that $\frac{\sqrt{W}}{\sqrt{W}}$ and $\sqrt{\frac{W}{W}}$ are equivalent syntax templates.

Of greater significance, though, is the fact that this student did not consider that a more elegant solution might be expected of a first year university student. Even though students had been plotting complex numbers on an Argand diagram, student C has not recognised the underlying structure of the complex numbers expression: that the moduli of $2 - 5i$ and $5 + 2i$ are the same. Nor has he noticed the common factor of 2 in $6 + 4i$ and then that the moduli of $3 + 2i$ and $3 + 2i$ are the same. Although the context of this question is complex numbers, the ability to scan the expression and note its structural features resembles the algebraic insight associated with successful students of algebra. The issue is less about students providing a mathematically valid answer than it is about their failure to unravel the subtleties of the question. This requires them to go beyond the syntax template □ – □i and view the expressions as complex numbers, interpreting the moduli in the geometrical sense (Bardini, Pierce & Vincent, 2015). It is the context of the question that signals the efficient approach to be taken. A solution more appropriate to students at this level is shown below:

$$\frac{7(\sqrt{4+25})(\sqrt{4+9})}{(\sqrt{36+16})(\sqrt{4+25})}$$

$$= \frac{7\cancel{(\sqrt{29})}(\sqrt{13})}{(\sqrt{52})\cancel{(\sqrt{29})}}$$

$$= \frac{7\sqrt{13}}{\sqrt{52}}$$

Figure 3. Student C's solution for Example 3: finding the modulus of the complex number expression.

$$\left|\frac{-7i(2-5i)(2+3i)}{(6+4i)(5+2i)}\right|$$

$$= \frac{|-5i||2-5i||2+3i|}{|6+4i||5+2i|}$$

$$= \frac{7|2-5i||2+3i|}{2|3+2i||5+2i|}$$

$$= \frac{7}{2}$$

The next tutorial problem (Example 4) relates to an introductory tutorial worksheet on implicit differentiation—the lecture topic at the time.

Example 4

Find the gradient and equation of the tangent to each of the following curves, at the point: (4, 2):

i) $x^3 + y^3 - 9xy = 0$

ii) $x^4 - x^2y^2 + 3y^5 = 4$

Student D, whose solution to part (i) is shown in Figure 4, has completed the implicit differentiation correctly and has obtained the correct gradient for the tangent to the curve $x^3 + y^3 - 9xy = 0$ at the given point. Although

there are no mistakes in the student's working, the common factor 3 in both the numerator and denominator seems to have escaped the student's notice. Although in this case taking out and cancelling the common factor would not have shortened the working and the numbers are relatively small, there would be no particular advantage in factorising and cancelling, but once again the student's working may be an example of lack of flexible thinking.

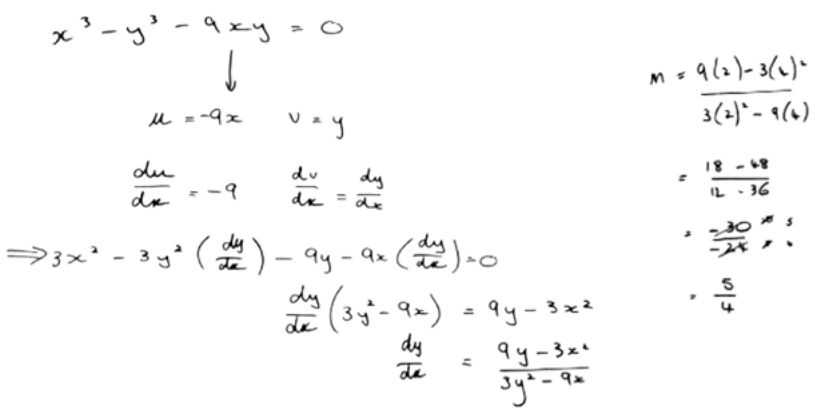

Figure 4. Student D's solution for Example 4: finding the gradient of the tangent at a given point.

The failure to notice common factors in numerical calculations is typical of many students and often leads to working with unnecessarily large numbers, impeding checking by estimation and leading to consequent arithmetic mistakes or failure to simplify. This is exemplified by the working shown in Figure 5 for part (ii) where another student, student E, substitutes in the expression for $\frac{dy}{dx}$ to find the equation for the tangent to the curve $x^4 - x^2y^2 + 3y^5 = 4$ at the point (4, 2), leaving the equation in the form

$$y = -\frac{224}{176}x + \frac{896}{176} + 2$$

It would be expected that with an easily identifiable common factor of 8, a student at this level would simplify the gradient to

$$-\frac{14}{11}$$

before finding the equation for the tangent. The student appears not to have considered the numeric structure of the expression. These examples were recorded in a technology free tutorial. It may be that the student normally worked with a calculator and was in the habit of 'outsourcing' such simplifications.

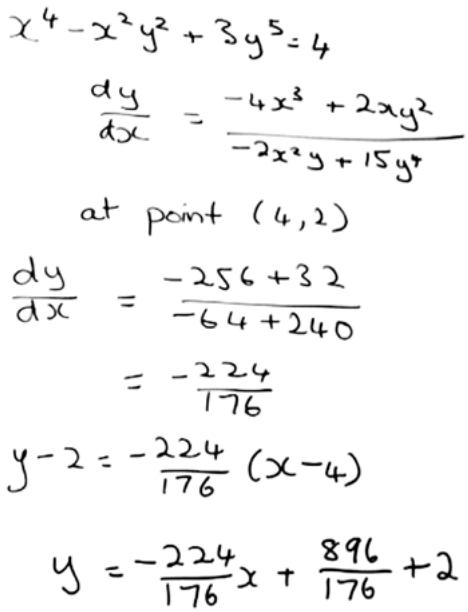

Figure 5. Student E's solution for Example 4: finding the equation of the tangent at a given point.

Conclusions and implications for teaching

Flexible mathematical thinking requires number sense and algebraic insight. The solutions presented above are illustrative of weaknesses in student work that we observed and that tutors described. In seeking to explain the lack of number sense and algebraic insight displayed one could argue that this is merely a result of the students focusing on new mathematical knowledge and skills and that in their tutorials they see no reason for taking care with the presentation of their solutions: obtaining an answer is what matters. Perhaps in assessed work (assignments and examinations) they would take greater care with the appropriate use of symbols and display a greater awareness of number and algebraic structure. However, anecdotal evidence from the tutors suggests that this is not necessarily the case. There are students who fail to notice that numerical fractions may be simplified and students who do not recognise equivalent structure in the terms of a complex number or an algebraic expression. If students see that their answer does not match that on the tutorial solution sheet, some will recognise that their answer is actually correct with respect to the new topic albeit that they did not demonstrate structural awareness that should have been established several years earlier. However, it is likely that there will be other students who compare their answers with those on the tutorial solution sheet and think that there is something about the new topic that they do not understand.

Our observations of the first year university mathematics students suggest that greater attention needs to be given to the development of a sense of numeric structure as well as algebraic structure throughout school mathematics. Connections should be made between different contexts where the same algebraic structures arise. The factors of a difference of two squares, for example, should be recognised in an algebraic expression but also when they occur in rationalising the denominator of a surd expression. Flexible mathematical thinking that looks for alternative syntax, alternative solution paths and alternative representations should be encouraged from the earliest years. Analysis of first year university students' work, of which there are just a few examples in this paper, suggest that many errors and unwieldy solution strategies are a result of unsound foundations from middle years of schooling (Years 5–9) and rigid thinking impeding students' ability to apply, what should be, familiar syntax patterns in new mathematical contexts.

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