Tracking change in primary teachers' understanding of mathematical reasoning through demonstration lessons

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This paper reports on the impact of a professional learning programme on participating teachers' perceptions of mathematical reasoning. A total of 26 teachers participated in this study from four schools in Victoria, Australia and one school in British Columbia, Canada. The participants observed two demonstration lessons prepared and taught by the research team, attended pre- and post-demonstration lesson group discussions and taught each lesson in their classroom. Interviews with participating teachers before beginning the program, after trialling the first demonstration lesson, and after trialling the second lesson provided data for analysis. The Primary Teachers' Perceptions of Mathematical Reasoning Framework previously established by the research team was used to track the shifts in teachers' perceptions and understanding of mathematical reasoning across the program. We theorise that intentional foci on salient aspects of reasoning demonstration lessons, highly collaborative reflections, and teacher enactment of the demonstrated lessons have the potential to develop teachers' perceptions and understanding of reasoning.

Keywords · teacher professional learning · demonstration lessons · mathematical reasoning · post-lesson discussion · inquiry community · primary school

Whilst reasoning is a mathematical proficiency espoused in many national mathematics curricula, studies have shown that often primary school teachers only incorporate some aspects of reasoning and not others (Clarke, Clarke, & Sullivan, 2012) or they show evidence of being confused or unsure how to define reasoning (Loong, Vale, Bragg, & Herbert, 2013). Research has shown that in order to help students have a firm grasp of mathematical reasoning, it is essential that teachers understand what mathematical reasoning entails (Lannin, Ellis & Elliott, 2011; Stylianides & Ball, 2008). This paper reports on the impact of a professional learning program on participating primary school teachers' perceptions and ideas about mathematical reasoning. We track the way teachers articulate their perceptions of mathematical reasoning across the duration of a professional learning program called the *Mathematical Reasoning Professional Learning Research Program (MRPLRP).* This tracking is carried out using the Primary Teachers' Perceptions of Mathematical Reasoning Framework (Herbert, Vale, Bragg, Loong &

Widjaja, 2015). We argue that teachers' verbalisation of their perceptions regarding reasoning provides us with a glimpse of their understanding and tacit knowledge of reasoning.

Mathematical reasoning involves developing and communicating arguments with the intention to convince (Brodie, 2010; Pedemonte, 2007). Two elements of reasoning that have been found to be necessary in inquiry-based activities are generalising and justifying or proving (Carpenter, Franke & Levi, 2003; Ellis, 2007; Lannin et al., 2011; Stylianides, 2010). There is a long history of teachers using inquiry to elicit students' mathematical thinking in their mathematics lessons (e.g. Fennema et al., 1996; Franke et al., 2009; Kazemi & Franke, 2004, Reid, 2002). Curricula worldwide place increasing emphasis on students' ability to reason mathematically (e.g. Department of Education UK, 2012; Ministry of Education Singapore, 2012; National Council of Teachers of Mathematics, 2014). In Australia this is seen with the introduction of the four articulated mathematical proficiencies in the Australian Curriculum: Mathematics namely: Understanding, Fluency, Problem Solving and Reasoning (Australian Curriculum Assessment and Reporting Authority, 2012). In the Australian Curriculum: Mathematics reasoning is explicitly stated as a proficiency to be developed in students and defined as being the '… capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising' (p. 5). The researchers in this project used this definition of reasoning for planning the demonstration lessons.

The increased emphasis on reasoning highlights the importance of teachers' pedagogical content knowledge of mathematical reasoning: what it entails and how to embed and encourage reasoning in their lessons (National Council of Teachers of Mathematics, 1999; Reid, 2002; Stylianides & Ball, 2008). Despite this emphasis, teacher enactments of this proficiency in classrooms have been found to be infrequent (Boero & Dapueto, 2007; Stacey, 2003). It is unclear why this is so, as there is a dearth of information about what primary teachers perceive mathematical reasoning to be, and to what extent their mathematical content knowledge (Ball, Thames, & Phelps, 2008) and pedagogical content knowledge (Shulman, 1987) enables them to plan and implement tasks that promote students' development of this proficiency (Stylianides & Ball, 2008). There is a need for more opportunities for teachers to learn about students' mathematical reasoning and to learn ways of supporting its development (Francisco & Maher, 2011).

Professional learning opportunities are required that specifically target developing knowledge and understanding of mathematical reasoning for primary teachers. The MRPLRP was established by the Mathematical Reasoning Research Group (MaRRG) at Deakin University in response to a need to assist schools in addressing this issue. In this program teachers observed two demonstration lessons taught by an expert teacher: participated actively in post lesson group discussions; and taught the lesson or a modified version of the lesson individually or in collaboration with a colleague with their students. We interviewed teachers about their reasoning perceptions at various stages in the program. This paper reports on an analysis of teachers' understanding of reasoning throughout the program implementation. The Primary Teachers' Reasoning Perception Framework previously established by the research team (Herbert, Vale, Bragg, Loong & Widjaja, 2015) was utilised to identify and map teachers' growth in their understanding of reasoning over the duration of the program.

The findings in this paper answer the following research question:

How does participation in demonstration lessons, aimed at developing children's reasoning, impact teachers' knowledge and understanding of reasoning?

The research design enabled exploration of teachers' perceptions about reasoning at different stages in the program, and the defining aspects of the professional development program that enhanced teachers' understanding of reasoning.

Literature review

The following sections review the literature on professional learning in general and mathematical reasoning in particular. A theoretical framework regarding the nature of learning from a phenomenographic perspective is also presented.

Professional learning programs

Every professional learning program aspires to impact participants through some form of professional growth (Department of Education and Early Childhood Development, 2013). Goldsmith, Doerr and Lewis (2014) broadly defined teachers' learning as 'changes in knowledge, changes in practise and changes in dispositions or beliefs that could plausibly influence knowledge or practice' (p. 7). This is consistent with Booth's (1997) view that learning is a change in the way an object of learning is perceived. Goldsmith and colleagues' review of over 100 articles on professional learning programs found that teachers' professional learning tended to occur incrementally and iteratively. In addition, research shows that effective teacher professional learning is often intensive; ongoing; connected to practice; addresses specific curriculum content; and promotes strong professional collaboration (Darling-Hammond, Wei, Andree, Richardson, & Orphanos, 2009; Jaworski, 2004, 2008).

Jaworski (2004) saw value in teachers, who are the practitioners, co-learning with university researchers in collaborative inquiry to build and sustain the development of mathematics teaching and learning. A variety of such professional learning programs in mathematics have been conducted nationally and internationally. Some of these programs have been school-initiated (e.g. Anderson, 2007) while others have been collaborative teacher-researcher based studies such as Demonstration Lessons (Clarke, 2011; Clarke et al., 2013; Grierson & Gallagher, 2009). Other teacher-researcher based programs such as Japanese Lesson Study (Hunter & Back, 2011; Lewis, 2002; Olson, White, & Sparrow, 2011; Takahashi, 2011; Widjaja, Vale, Groves & Doig, 2015), Chinese Lesson Study (Huang, Su, & Xu, 2014) or Teacher Research Groups (Yang, 2009) have long standing traditions in their country of origin.

The design of the MRPLRP reported in this paper was informed by variation theory (Marton, Runesson & Tsui, 2004) and previous work on Demonstration Lessons (Clarke et al., 2013; Grierson & Gallagher, 2009) following the model of 'pre-brief, teaching and debrief'. It draws on some elements of Japanese Lesson Study in the planning of the demonstration lesson. While both Japanese Lesson Study and Demonstration Lessons are similar in that both models of professional learning occur in the classroom context and is focused on a particular inquiry in a given lesson, there are some key differences (Bruce, Ross, Flynn, & McPherson, 2009). Demonstration lessons usually occur over a relatively short period of time focusing on the overall classroom environment and teacher actions in one lesson. Lesson Study is cyclical and requires a longer term commitment where the foci are intentional observation of students and their work. Bruce et al. (2009) found that demonstration lessons are appropriate at the entry level where teachers have limited experience or difficulty visualising a strategy and recommended the benefits of cyclical demonstration lessons as opposed to a one-off event.

Teacher noticing of instructional events in the mathematics classrooms is important and central to teachers' improvement of their practice (Mason, 2002; Schoenfeld, 2011). Choy (2013) asserted that the most productive mathematical noticing involves attending to relevant information; relating this information to prior experiences; and combining this new understanding to generate different possible responses.

Stylianides and Ball (2008) argued that besides having knowledge of the logico-linguistic structure of proof, teachers ought to have knowledge of situations for proving which includes knowledge of different kinds of proving tasks and knowledge of the relationship between proving tasks and proving activity. Aspects of this knowledge were evident in Brodie (2010) who documented and analysed the experiences and challenges faced by five secondary mathematics teachers who were undertaking post graduate coursework as they promoted mathematical reasoning in reform-oriented South African classrooms. The collaborative action research projects undertaken by these teachers, supported by an experienced researcher, resulted in the development of broader understandings of mathematical reasoning. They became more accomplished in choosing tasks and shifting interaction patterns to support and accommodate learner contributions and ideas. These teachers faced emerging challenges as they moved between old and new practices and endeavoured to stimulate mathematical reasoning among all learners. Ball and colleagues (2008), argued that teachers need to develop specialised mathematical content knowledge to enable them to be 'nimble in thinking about numbers, [pay] attention to patterns, and [be] flexible thinking about meaning…' (p. 401) as they support mathematical thinking.

Assessment of learning in teacher professional learning programs

As Goldsmith, Doerr and Lewis (2014) noted, it is important that any professional learning program assess the extent to which teachers have learnt. According to variation theory, learning is considered to have occurred when there is a change in the way an object of learning is perceived (Booth, 1997; Ramsden, 1988). The learner's existing cognitive structure or schema has been adapted, rejected or supplemented (Piaget, 1970). This is often demonstrated by a change in the learner's expression of their perceptions of the content and is normally characterised by the learner becoming aware of additional features or aspects of a concept or phenomenon not previously discerned (Bowden & Marton, 2004). The amount of attention given to aspects of the learning intention accounts for differences in how people learn (Marton, Runesson, & Tsui, 2004). In phenomenographic research, learning can be said to have occurred if the learner views the phenomenon differently from their initial perception of the phenomenon (Bowden & Green, 2005).

Phenomenography is a research methodology which has been employed in a range of educational research (see for example Kilinc & Aydin, 2013; Marton, Runesson & Tsui, 2004). It seeks to describe the set (outcome space) of qualitatively different categories of perceptions of a phenomenon (Marton, 1988) held across a diverse group of participants (Marton & Booth, 1997). This interpretive research approach describes a phenomenon as seen by the participants and aims to capture the variation in perceptions of the phenomenon under investigation (Marton, 1988), such as, mathematical reasoning. Typically interview transcripts are examined to find ways of grouping responses into categories of description according to common features and characteristics (called dimensions of variation). Categories are distinguished by the values of dimensions associated with each category (Cope, 2000). The final categories and dimensions emerge over a series of iterations each leading to a refinement of the outcome space through the interrogation of pooled, de-identified, discourse fragments on the collective level. Categories are an 'aggregate of similar perceptions' (Akerlind et al., 2005, p. 92). The dimensions structure the categories into an hierarchical outcome space such that the categories are ranked in increasing levels of perception of the phenomenon (Bowden, 1994; Marton, 1988). '[The] validity of the outcomes is related to the processes that are used at all stages'(Akerlind et al., 2005, p. 89), that is: the selection of the sample for maximum variation; the conduct of consistent, open interviews; and the iterative inductive analysis which remains focused on the meanings expressed in the set of transcripts taken as a whole. With some exceptions (e.g. Carpenter et al., 2003; Clarke et al., 2012) primary school studies tend to focus on children's reasoning rather than teachers' perceptions, understanding or practice of teaching reasoning.

The Framework of Teachers' Perceptions of Mathematical Reasoning

As an initial effort to understand the nature of teachers' perceptions of mathematical reasoning, the research team employed phenomenographic analysis (see Herbert & Pierce, 2012; Bowden & Green, 2005) of interview data to develop a framework of primary teachers' perceptions of mathematical reasoning (Herbert et al., 2015). This framework provided a general overview of the qualitatively different categories of primary teachers' perceptions of mathematical reasoning. Briefly, these categories were delineated one from another in accordance to the values of the dimensions of variation evident in each category. For example for the dimension 'Audience' the values were related to whether the reasoning was shared with others or not; for the dimension 'Purpose' the values that came to the fore was whether reasoning was related to recounting, comparing and contrasting, making choices, explaining, arguing step-by-step, articulating reasons, justifying, hypothesising, generalising, proving, evaluating or connecting ideas. The other dimensions were related to the mode of presentation of the reasoning (the values were verbal, diagrammatic/written, symbolic or gesture/action) and the type of reasoning (the values being inductive, deductive, abductive or adaptive). Hence it is the particular values of the dimensions which determine the commonalities and differences between the categories. The range in teachers' perceptions of mathematical reasoning was structured according to these dimensions to indicate a growing awareness of the complexity of mathematical reasoning, with awareness of more values of the dimensions indicating a more inclusive perception. Table 1 shows the framework with the categories of teachers' perceptions of mathematical reasoning listed from least complex or least inclusive (Category A) to the most complex or most inclusive (Category G) along with a brief description of each.

Table 1
Primary Teachers' Perceptions of Mathematical Reasoning (Herbert et al., 2015)

Category	Reasoning is perceived to be:	Descriptor
Category A	Thinking	Thinking in general and involves making choices and personal reflection
Category B	Communicating thinking	Recounting thinking to others using verbal means of presentation
Category C	Problem solving	Using thinking to solve problems
Category D	Validating thinking	Explaining, articulating reasons or justifying either verbally or diagrammatically
Category E	Forming conjectures	Hypothesising (generalising) and explaining and/or justifying.
Category F	Using logical arguments for validating conjectures	Using step-by-step logical arguments for justifying conjectures
Category G	Connecting aspects of mathematics	Bringing together different aspects of prior mathematical knowledge to solve problems and make sense of mathematics

In this paper the Framework is used to track changes in primary teachers' perceptions of mathematical reasoning after engagement in the program of professional learning described earlier A comprehensive discussion of the development of the categories can be found in Herbert et al. (2015). By using this framework to categorise the perceptions articulated by individual teachers about mathematical reasoning and comparing their perceptions across the duration of the professional learning program, growth in teacher awareness towards the reasoning proficiency may be identified thus providing a measure of whether teachers have learnt from the professional learning experience or not (Bowden & Green, 2005). It is hoped that this awareness and ability to articulate their ideas about reasoning will enable teachers to be more adept at recognising when and what type of reasoning is evident in the tasks they set for their students and how to foreground the reasoning in their class discussions.

We are mindful that teachers' awareness and subsequent articulation of the reasoning that they perceive as evident, depends on the types of reasoning that are present in the lessons observed. Hence a brief analysis of the reasoning types that are embedded in each of the demonstration lessons is presented in the description of the lessons below. The reasoning actions that are anticipated of students in each of the demonstration lessons are highlighted.

Method

The model of demonstration lesson utilised in the MRPLRP required an expert teacher to model a live teaching episode that demonstrates how reasoning can be embedded in lessons developed around number concepts. It draws on practices from Japanese Lesson Study where the lesson is structured with documentation including anticipated solutions and the launch of the problem as well as lesson observations involving teams of teachers with an extended post-lesson discussion. There is a strong emphasis on teachers teaching the lesson or a modified version of the lesson with their class after observing the demonstration lesson. We chose this model because not only did it include opportunities for collaboration and trialling by teachers, but reasoning had only recently been specified as a proficiency in the Australian curriculum and some teachers may have been at entry level with teaching practice of this proficiency (Bruce et al., 2009).

Research design and participants

When approached by the research team, several schools were interested and consented to participate in the professional learning and research project. Teachers of the consenting schools were then provided with information about the project and individually decided to participate or not. In all the schools the teachers were already members of school year level teams although not every member of the school team participated. A total of 26 teachers participated in the professional learning program of which 19 were from four public primary schools in Victoria, Australia and seven were from one public primary (locally called elementary) school in British Columbia, Canada.

Participants' background information and a brief summary of their participation is provided in Table 2 (see Results section). The mathematics teaching experience, leadership roles and the year levels taught by the participants were diverse. Teaching experience ranged from two years to more than 25 years with some holding leadership positions as teaching coaches, assistant principals, or year level coordinators.

The MRPLRP was delivered following the research design (see Figure 1) that is, two cycles of demonstration lessons each followed by a post-lesson discussion and then the teachers trialled the lesson in their classroom. The first cycle occurred at the end of 2012 and the second at the beginning of 2013. The first demonstration lessons were held less than a month after the first interviews were conducted. On the day of the demonstration lesson, a pre-lesson briefing was held immediately prior to the demonstration lesson and a post-lesson discussion was conducted immediately after the demonstration lesson for that day. Teachers were encouraged to teach the lesson in their own classrooms using the adapted material provided after each demonstration lesson. Interview 2 followed their trialling of the lesson in their classroom. There was approximately four months between demonstration lessons. It was envisaged that by observing, reflecting, and trialling the demonstration lessons, teachers would raise their awareness and notice reasoning actions and would be able to articulate them during post-lesson discussions and interviews following trialling of the lesson in their classroom.

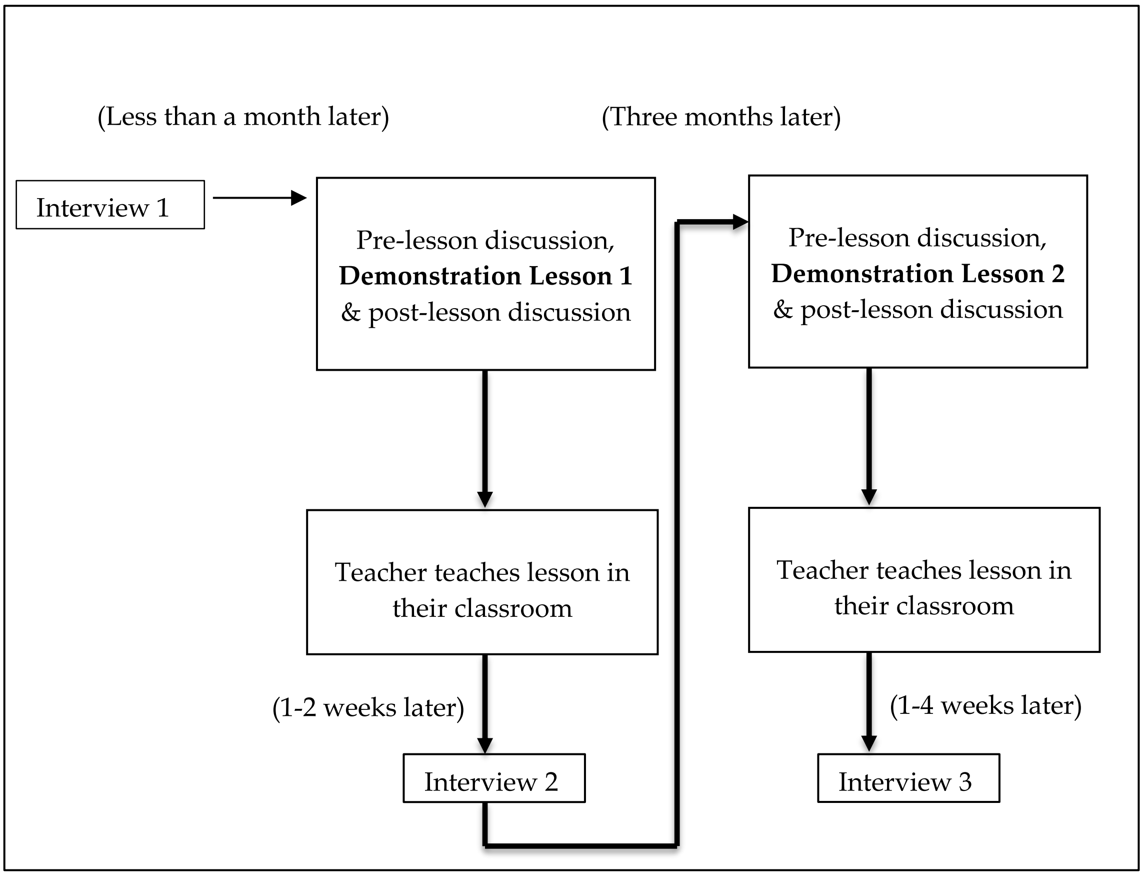

Figure 1. Research design

The interview questions were semi-structured with opportunity for follow-up questions at the discretion of the interviewer. Four members of the research team individually conducted all the interviews with available participants at their research school. Each interview lasted approximately 30 minutes. In the first interview we gathered information regarding teachers' teaching background, professional learning, approach to mathematics teaching, awareness and knowledge of reasoning in the mathematics curriculum, and reasons for participating in the reasoning project. Questions included: Would you describe the types of mathematics PL that you have participated in the past two years; The Australian Curriculum: Mathematics includes four proficiency strands – understanding, fluency, problem solving and reasoning. What does **reasoning** mean to you? In the second interview, teachers were asked to provide feedback about the demonstration lesson with questions such as: What do you think were the benefits of this lesson for the students' reasoning and learning? What do you think were the benefits of observing this lesson for your understanding of reasoning and your teaching of reasoning? What were the benefits of the post-lesson discussion? They were asked to provide examples of reasoning examples that they saw in their trialling of the lessons and if there were any changes in their understanding about reasoning and what they have learnt from the demonstration lessons. In the third interview, participants were asked to provide feedback on the demonstration lesson and post-lesson discussion, and to report on the trial of the second reasoning lesson, how their understanding of reasoning in mathematics has developed during the course of the project, what they saw as the most beneficial aspects of a focus on reasoning in mathematics lessons and what aspect(s) of this professional learning program had been most influential in developing their understanding of reasoning and their practice of teaching reasoning in their classrooms.

Due to administrative constraints, only two of the seven participating teachers in Canada were able to observe the second demonstration lesson, trial the lesson, and be interviewed. Similarly, administrative constraints did not allow some teachers from the Australian schools from participating in all three interviews. Hence out of the 26 teachers, 17 teachers were interviewed three times, seven were interviewed twice and two were interviewed only once. As it was not possible to obtain evidence of development in the two teachers who were interviewed once, evidence was only drawn from the remaining 24 teachers.

Implementation of the demonstration lessons

Demonstration lessons were planned by the research team and taught by a team member or an external expert teacher. All the participants at each school observed the same lesson, except in the Canadian school where the first demonstration lesson was taught twice so that all participants could observe one of the lessons. The first demonstration lesson titled 'What Else Belongs?' was adapted from Small (2011) (See Bragg et al. (2013) for a detailed explanation of this lesson). The aim of the lesson for Year 3 and 4 students was to form a conjecture for three numbers (e.g. 30, 12 and 18) belonging to a group, justify their conjectures, and extend the group of numbers by identifying other numbers that could belong or conversely did not belong. The lesson plan included a number of teacher prompts and questions to support a range of reasoning actions such as: *How did you convince each other that your reason works?* Analysis of children's reasoning during this lesson found that students compared, generalised, explained, verified, justified and used logical argument (see Vale, Widjaja, Herbert, Bragg & Loong, 2016).

The second demonstration lesson entitled 'Magic V' was designed using the problem from NRICH (2014). The aim of this lesson was for children to notice similarities and differences in the arrangement of the numbers 1 to 5 in a V shape and find arrangements of these numbers to make a 'Magic V'. (It is 'magic' when the sum of numbers on each arm of the V are equal). After the students explored the Magic V task, a whole class discussion was carried out using prompts such as: *What do you notice about all the Magic Vs you found?* Children were then asked to test the conjecture: *Sam said 'It is impossible to make a Magic V with an even number at the bottom with a set of numbers from 1 to 5.' Is Sam right? Explain why or why not.* See Bragg, Loong, Widjaja, Vale, & Herbert (2015) for a detailed explanation of the lesson. The lesson was planned to elicit reasoning actions such as comparing and contrasting, generalising, forming and testing conjectures and explaining properties. These were made explicit in the lesson plan objectives and illustrated by the expert teacher in her line of questioning with individual students and during whole class discussion of explanations and justifications. See Widjaja (2014) for an analysis of children's reasoning.

In each cycle of the project, the 60-minute lesson plans written by the research team were provided to teachers several days in advance. During the 20-minute pre-lesson briefing on the day, observation sheets and seating plans were provided for teachers to record what they saw and heard. Teachers were encouraged to record observations of reasoning by children as they communicated with each other as well as with the teacher but were cautioned against teaching the students during the lesson. Teachers were asked to document the variety of solutions that individual students used to solve problems, including errors.

Immediately after each demonstration lesson, a post-lesson discussion lasting 30-40 minutes was conducted to provide teachers with the opportunity to share and discuss their observations. The post -lesson discussions focused on teachers reporting evidence of children's reasoning and when and how that happened. They were also asked what teacher actions they thought elicited reasoning and what aspects of the lesson contributed to reasoning and learning and how the tasks could be altered to enhance reasoning and learning.

Following these discussions, teachers were asked to teach the demonstration lessons to their own students at an appropriate time. To support teachers to trial the lessons in their classroom, suggestions of alternate numbers were provided.

Data analysis

As indicated earlier three individual interviews were used to collect data for this study. The interview questions collected data about teachers' perceptions of mathematical reasoning, their experience and strategies for teaching reasoning, their learning from each of the stages, and elements of the MRPLRP. To respond to the main research question for this paper which is the analysis of teachers' awareness of reasoning and growth in their awareness, data were drawn from the individual interviews conducted with the teachers who participated in at least two interviews. The Primary Teachers' Perceptions of Mathematical Reasoning Framework (Herbert et al., 2015) was used to map changes in teachers' perceptions or understanding of reasoning over the course of their involvement in the program.

The first step of analysis for this study was to identify the initial category of perception of reasoning for each participant. To confirm inter-rater reliability each member of the research team independently analysed the transcripts of the third interview of the same two teachers from two different schools to classify the teachers into one of the categories of the framework. There was concurrence in the analyses of all five researchers demonstrating a high inter-rater reliability. Each of the researchers was subsequently assigned a set of first, second, and third interviews to code teachers' statements according to the categories in the framework. This coding was checked by the first author, so the categorisation of every teacher was conducted at least twice to verify inter-rater reliability. The codes and categories for each interview were recorded, and the results for individual teachers were collated and are presented in Table 2 (see Results section).

Due to space limitations it is not possible to provide evidence of the categories assigned for each teacher. However, the case of Faye below provides an example of the evidence used to categorise Faye's perceptions of reasoning for two of the interviews. In the first interview teacher Faye said,

> …but I think reasoning would have a lot to do with the understanding behind the mathematical concepts…, so not the process of the mathematical concept but more the *thinking* that goes on during a process maybe.

Faye was classified as having a Category A perception of reasoning as she mostly referred to reasoning as thinking in general and regarded it as a private and independent action. In the third interview, however, Faye said,

> …Well the kids were constantly having to explain, because they work in partners which meant they could make their *thinking out loud*, you could always hear them *justifying*, thinking about other reasons why things won't work, or the reasons why things do work…

Faye's words expressed a shift in understanding, now classified as Category D, as she referred to the importance of children justifying their thinking and doing this verbally to *others*. Faye was not classified as being in Category E because she has not mentioned terms related to 'hypothesising' or forming conjectures which was one of the key values in the 'Purpose' dimension of Category E.

By coding each teacher's expressions of reasoning and determining if there were changes across the three interviews, we hoped to identify the teachers who changed their perception and articulation of reasoning. We then analysed the interview data inductively to identify themes that described the ways our professional learning program (MRPLRP) had contributed to growth in the primary teachers' perceptions and understanding of mathematical reasoning. These themes were derived from the participants' responses to each question asked.

Results

By coding the responses of the 24 teachers over the duration of the project, we were able to identify the reasoning perception categories that teachers were perceived to predominantly hold at different stages. The category of each participant's perception of mathematical reasoning perception at each interview is presented in Table 2.

Table 2

Participant Demographics and their Reasoning Perception Category at each stage of the Project and Grouped according to Shift over Time

GROUP A - Teachers with prior knowledge of reasoning

School	Pseudonym	Teaching experience (Years)	Year Class 2012	Year Class 2013	Other year levels taught	Interview 1	Interview 2	Interview 3
B	Heather[1]	20	6/7	6/7	4-6	D	D	G
D	Winnie	10	3	4	K-6	D	D	B
A	Carol	11		5/6	2, 3, 5/6	D	D	D
D	Zoe, coach	5	4	2	1, 4, 5	D	D	D
B	Mallory	20	1	1	K - 4	D	D	-
D	Tori	4	4	4	K, 3	E	D	D
B	Nick	12	6/7, 7	6/7, 7	1-2	F	G	-
B	John	28	2/3	2/3	2 - 4	F	G	-

GROUP B - Teachers who demonstrated shift from a less complex to more complex perception of reasoning

School	Pseudonym	Teaching experience (Years)	Year Class 2012	Year Class 2013	Other year levels taught	Interview 1	Interview 2	Interview 3
A	Grace	>20	5	5/6	K-5	A	B	G
C	Sonya	10	3/4	3/4	1/2, 4/5	A	C	C
A	Faye	3	3/4	3/4	none	A	D	D
D	Vivian	2	4	4	none	A	D	D

C	Olive	>7	3/4	3/4	K-6	A	G	E
C	Rose	13	2/3	5/6	1 - 6	B	B	D
A	Darlene	4	1/2	1/2	2	B	D	D
A	Ally[2]	>25		3	3 – 6	B	D	F
B	Ida	16	K	K	3, 4, 5	B	D	-
C	Pam	4	2/3	x	1,	B	D	-
D	Ulla	11	5	4	3, 6	B	F	D
A	Elizabeth	4	K	K	1 - 2	C	D	D
D	Yianna	2	3	1	none	C	-	D

GROUP C- Teachers who did not demonstrate a shift in their perception of reasoning

School	Pseudonym	Teaching experience (Years)	Year Class 2012	Year Class 2013	Other year levels taught	Interview 1	Interview 2	Interview 3
C	Queenie	>16	3/4	3/4	Not evident	A	A	B
B	Lucy	18	3/4	3/4	1-2, 4 - 5	B	B	B
B	Keira	20	4/5	4/5	1 - 12	B	B	-

Note 1: Assistant Principal
Note 2: School's Teaching and Learning coach in 2012.

Changes in teachers' perceptions of reasoning

Analysis of changes in perception or understanding of mathematical reasoning over time for the 24 teachers as evident in Table 2 indicated that there were three main groups:

- Teachers who had prior knowledge of reasoning,
- Teachers who demonstrated a shift in perception and understanding of reasoning from less to more complex, and
- Teachers who did not demonstrate shifts in perception or understanding of reasoning.

Teachers who had prior knowledge of reasoning

Eight teachers articulated more complex understanding of reasoning at the start of program. More complex reasoning is taken to be that which includes explanation, generalisation, validation and justification which are found in Category D and above. In the examples below, Heather and John were classified as being in Category D and F respectively at the first interview. These teachers demonstrated an ability to articulate what reasoning meant to them and used terminology that indicated an understanding of the more complex reasoning actions. In-depth study of their interview transcripts revealed that these teachers held a strong teaching philosophy where they emphasised the use of reasoning on a daily basis and encouraged the reasoning proficiency by using questioning techniques and discussions to elicit reasoning responses from students.

> …Well I guess they're reasoning every day, with the types of things that they're doing. Why are they choosing the method that they are? Not simply because I told them so? What's your motivation behind trying it this way? And I do a lot of discussion with them. (Heather, Interview 1)

> …So in terms of discussion, I think if I'm working with an individual student or looking at what they're doing, I might ask them a question, rather than just saying, well that's wrong or something like that. I might say, well if this is true, then this – then this must be true, that sort of thing. And again we use it to see logical connections between things... (John, Interview 1)

The teachers in this group have taught for four to 11 years and over a range of year levels but do not seem to hold leadership positions in schools. Among this group, three teachers appear to have not shown further development and two teachers were classified in a less complex category in the third interview because evidence of awareness of reasoning was not expressed.

Teachers who demonstrated a shift in perception and understanding of reasoning. Of the 24 teachers, 13 showed a growing awareness of mathematical reasoning from a less complex perception at the start to becoming aware of more complex aspects of reasoning after the first demonstration lesson. For example, Faye (quoted above) shifted from Category A to Category D, while Olive shifted from Category A to Category G. In the first interview Olive said that reasoning is 'The thinking, mathematical reasoning – the thinking behind why things work' (Category A). However in the second interview she began to articulate more complex elements of reasoning when she said,

> …get them to analyse, how did you get that and why did you do it that way and what strategies did you use… these 2 girls [said], well we'll find our reasons first and then make numbers fit that reason, that to me, that's exactly what we were trying to get out of this. …They were [having the conversations] …but while you're working you could start thinking about your journal entry. And then it becomes a record of their learning as well. ...They interlink because if you can give your reasons you seem to have a higher level of understanding of some of the processes that you need in the maths. (Olive, Interview 2, Category G)

This group comprised almost 50% of the participants. Of the 13, three continued to show further development in their understanding after the second demonstration lesson. After the third interview three other teachers showed development of understanding while five showed no further development and one were classified in a less complex category. Further examples of teachers' shift in their perceptions of mathematical reasoning are reported in Herbert, Widjaja, Bragg, Loong & Vale (2016) and Bragg, Herbert, Loong, Vale & Widjaja (2016).

Teachers who did not demonstrate shifts in perception or understanding of reasoning. Three teachers did not show any major shift in their articulation staying at Category A or B by the end of the program. These teachers were using less complex reasoning verbs such as 'thinking' or 'communicating their thinking'. Careful analysis of their interview showed that these teachers were noticing the effects on the classroom dynamics rather than the reasoning capability of the students or the strategies used by the demonstrating teacher in drawing out the children's reasoning. They were saying things like:

> …I really liked the pacing of the lesson, I know we didn't get through as much as we wanted to, but again it was the first time and I really liked the way that they were at their – at the group, at the carpet together and then sent back and then at the carpet again…. That – it just gave them the chance to get up and walk around, it gave them the chance to talk… then when they were listening, they were really listening. (Lucy, Interview 2, shifted from B to B to B)

There were neither comments in their interview that showed that they noticed students justifying or trying to convince their peers, nor did they mention anything about the questioning techniques that stimulated the reasoning capabilities of children.

These results seem to suggest that the first demonstration lesson impacted on the teachers more profoundly than the second demonstration lesson. This was evidenced by 50% of the participants (12) shifting from using less complex reasoning verbs to more complex reasoning verbs after participating in at least one demonstration lesson cycle. It seemed that teachers who did not progress to a more complex level by the end of the project, regardless of whether they had a pre-established knowledge of understanding or not, were those who did not express deeper awareness of the strategies for eliciting reasoning used by the demonstrating teacher nor the reasoning skills of the students. It is likely that these participants either did not fully articulate their perception or the interviewer did not provide an opportunity for them to fully express their perception of reasoning in subsequent interviews. This finding will be further addressed in the Discussion section.

Factors that influenced shifts in teachers' reasoning perceptions

The interview data revealed several factors that contributed to teachers developing more complex perceptions of reasoning. In agreement with the literature (Clarke et al., 2013; Grierson & Gallagher, 2009), the participants attributed their shifts to watching the demonstration lessons, engaging in post lesson discussions and the actual teaching of the lessons in their own classrooms. Evidence of each of these factors are presented below.

a) Benefits of demonstration lesson observation

Participants identified four benefits derived from the experience of observing a demonstration lesson. The first of these benefits concerned developing their own understanding of reasoning.

An increased understanding of reasoning. Faye clearly articulates her improved understanding arising from observing the demonstration lesson:

> …I think it was really worthwhile for both the students and their learning and their understanding, but more importantly for me and my understanding. (Faye, Interview 3, shifted from A to D to D)

Ally, who is a seasoned teacher and coach, felt that by watching the questioning that was being carried out in the lesson gave her better understanding of what reasoning is and how she could model that next time she coaches teachers.

> …For my understanding of reasoning I really honed in on the questioning that you were doing with the students. (Ally Interview 2, shifted from B to D to F)

Besides gaining a personal understanding of what reasoning is, teachers noted the pedagogical strategies that helped promote it.

Opportunity to pay attention to salient or new pedagogical strategies, techniques and representations.

> …it just reminded me of a way to structure a maths lesson and the way to question. Just you consistently saying "can you give me a reason for that", and "why did you decide that", and "can you explain it", and "does this match, does that not match, why doesn't it?" So drawing further information out of the children. (Grace, Interview 3, shifted from A to B to G)

Grace was reminded of the importance of structuring the lesson in a way that makes reasoning become an instructional objective so students develop the ideas for themselves. Olive highlighted the importance of modelling versus reading about it.

> … a CRT [substitute teacher] can get a teacher's notes and would still teach the lesson differently to what the teacher had in mind. So just reading someone's notes you can interpret it differently, but watching someone do it, you've got, you've seen the conversations, you've seen how they've done it so you get so much more out of it. (Olive, Interview 2, shifted from A to G to E)

Whilst teachers recognised the strategies that stimulated reasoning they also noted the different representations, such as diagrams and gesture rather than words or text that demonstration teacher accepted as a means for students to express reasoning.

> …not just getting … stock-standard responses. How she was able to prompt them to think outside of the box and get them to come up with answers that you might not necessarily expect, especially grade 3's, to come up with regarding their reasoning and their thinking with those particular numbers or grouping of numbers. And being able to get them to, even if they couldn't explain verbally, to show on the board in some way what their thinking was. (Ulla, Interview 2, shifted from B to F to D)

Having an observer perspective. Being able to observe how students were reasoning was also valuable to teachers. In the following quote Faye indicates the value of listening and observing students for an extended period of time to follow the developing argument occurring between the students.

> …I thought it was fantastic to be able to take a step back and observe the conversations that the students were having, which was obviously very important to reasoning because that's what they were using – how they were going to convince each other about their numbers. …So I think being able to observe that lesson allowed all of us to be able to tap into that conversation and hear it as it's happening rather than getting a piece of work and then having to say afterwards, "Can you explain your reasons behind this?" So I think that was really pertinent to understanding what the students' concept of reasoning was, being able to listen to them. (Faye, Interview 2, shifted from A to D to D)

b) Benefits of post-lesson discussions

Teachers reported that sharing their observations during the post-lesson discussion enabled the learning from their observations to be further enhanced. The benefits of the post-lesson discussions included discussing student responses, collaborative observation, raising awareness, contribution of expertise by the demonstration teacher and a collegial environment.

Rich deliberation of student responses. The opportunity to discuss student responses with the benefit of the classroom teacher's insight about the child along with other teachers' affirmation of the knowledge of the children in their classroom, were key benefits for teachers.

> ..I just thought it was good to debrief. And particularly when there were different, say if there were kids that came out with different sort of answers, just to have that opportunity to discuss how we thought they did that …" (Sonja, Interview 2, shifted from A to C to C)

> …because I knew the children and I knew what I expected from the children, hearing other people say, well you know when so and so said this, I thought that was really good ... So it felt good as a teacher … hearing someone confirm that. (Olive, Interview 2 shifted from A to G to E)

These comments indicate the learning, and affirmation, that is possible when the classroom teacher's knowledge of the student is combined with other teachers' observation of the child to analyse the child's responses and reasoning. Having a number of teachers observe students also provided for a richer discussion that included observation of more students than a single observer could accomplish.

Teachers noted the benefit of observing lessons in groups so that observation of students could be shared with colleagues enabling richer discussion with more people contributing through extended observation of more children in the classroom.

> So having everyone there meant that if you didn't get to that pair, somebody else had feedback to provide or how they were reasoning and talking, the language they were using, which was really good. (Ulla, Interview 2, shifted from B to F to D).

Increased awareness of learning issues

The post-lesson discussion also enabled teachers' to identify learning achieved in the lesson and further learning needs to be addressed as evidenced by Faye in the following quotes:

> To have the feedback from others to work out, perhaps, what areas may have been lacking which – an area we noticed was language. The students didn't have the language to say, "A 2 digit number." They weren't using the word "digit" at all. So that's something that was mentioned in that follow-up discussion that we had together, that then we'd each thought about [it] in our own classrooms to see if our individual students – and when I [trialled] the demonstration lesson in the other grade – whether they have that language. Which they don't. So yeah, that all came out because of the discussion that we had. (Faye, Interview 2, shifted from A to D to D)

Gains from demonstration teacher's expert input. Being able to address questions to the 'expert' teacher about the pedagogical strategies used was also valued.

> I just think it's always good after something like that to have that opportunity… to question her [the demonstration lesson teacher] on different things and why she did that… (Sonja, Interview 2)

Collegial extension and reflection of ideas. Others appreciated the collegial and supportive atmosphere of the post-lesson discussion for furthering their understanding. It is likely that teachers with prior knowledge of mathematical reasoning (Group A teachers) enriched the discussion.

> Obviously speaking to colleagues and listening to other teachers' opinions allows you to extend your own ideas, think of some or be aware of perhaps some misconceptions or misunderstandings that you may have yourself. And allows you to discuss with your colleagues in a relaxed and open forum without feeling that you're being judged by anyone, I think that's really powerful. (Ally, Interview 2)

The collegial experience of the post-lesson discussion also enabled the collaboration to continue beyond this discussion into the staffroom as the teachers prepared to trial the lesson in their own classroom. The post-lesson discussion also prompted teachers to continue reflecting on the lesson and discussion of the lesson and to continue this discussion with colleagues later.

> ..I guess just as we talked about what we noticed, then I was sort of reflecting on that as I decided how I was going to deliver it. But I still had to keep thinking about it, I still, I was a bit anxious about the lesson because I just wasn't sure how it was going to go with kindergarten.… And I had a little talk in the staff room of if you were teaching kindergarten, how would you do it, and … And it wasn't so much about the [lesson], it was about the reasoning, but it was how can I deliver a similar lesson with a different audience? (Ida, Interview 2)

c) Benefits of implementing a similar student appropriate lesson

One teacher saw the value of trialling the lesson in her own classroom and seeing how her students reason mathematically.

> … doing it in your own grade and seeing how the kids respond to the content and the structure of the lesson, and how they are going with their reasoning because without that, then none of the conversations would happen and the work wouldn't be evolving, as well as it is, so yeah, it's really rewarding... (Faye, Interview 3)

Trialling the lesson has been valuable but one experienced teacher felt that to develop her understanding and practice further she needed to do more herself to the point of actually constructing her own lesson rather than merely following one that has been developed by the research team.

> Now, when someone else prepares the lesson, that's fine as a demo. Now, I need to actually develop a reasoning lesson and I haven't done that. ... I need to go through the thinking process of actually planning and constructing. So I haven't done that. Then it would have more impact on my understanding and knowledge of it. (Grace, Interview 3)

It is evident that teachers who had shifted in their perceptions about reasoning valued the observations of the demonstration lessons, the post-lesson discussions and actually trying the lesson in their own classroom.

Discussion and implications

The Primary Teachers' Perceptions of Mathematical Reasoning Framework (Herbert et al., 2015) enabled us to track shifts, or lack of shift, in teachers' perceptions and understanding of the reasoning proficiency. We found that 13 of the teachers were better able to articulate their understanding of reasoning after participating in at least one demonstration lesson cycle and by the end of the research project, only 3 out of 24 teachers were unable to clearly define reasoning in a mathematical sense.

The teachers who showed shifts in their knowledge attributed their understanding to having observed the demonstration lessons, participated in the post-lesson discussions and trialled the lessons themselves. This finding is represented in Figure 2.

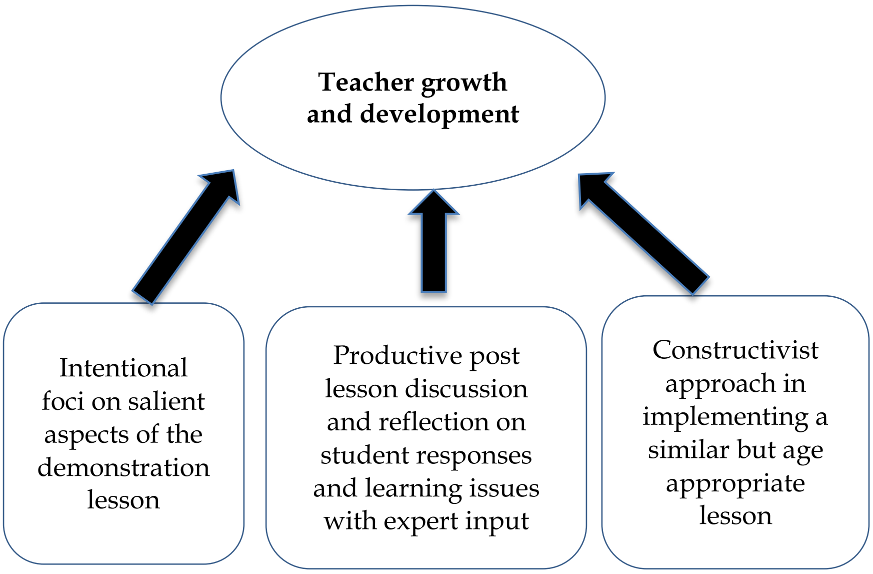

Figure 2. Guiding principles for successful professional learning using demonstration lessons.

Consistent with the findings of others such as Clarke et al. (2013) our research was able to confirm that observation of demonstration lessons conducted by an expert has a positive effect on teachers. These teachers began to acquire an understanding and knowledge of the situations for reasoning to occur and the activities that promote it (Stylianides & Ball, 2008). The teachers confirmed Marshall and Norton's (2011) findings that designing a lesson that allows for initial exploration, aids higher order thinking in students. They also noticed and developed understanding of the types of questions asked and the questioning techniques that enabled reasoning to be drawn out.

Having the opportunity to listen to students' responses was enlightening for many teachers, be it the class teacher or teachers of a different class or year level. Many of the teachers changed their understanding and perception of reasoning after observing and listening to the students explain, justify and convince their peers. Noticing and anticipating the diversity of students' responses observed is fundamental to improving lessons (Huang, et al., 2014; Yang, 2009) as well as contributes to teachers' developing mathematical understanding and their knowledge of students (Bruce et al., 2009; Takahashi, 2011; Murata, 2011).

As reported in the Results section, the benefits of post-lesson discussions were highlighted by the participants. Teachers reported that the post-lesson discussions allowed for rich deliberation of student responses, increased their awareness of learning issues, enabled collegial extension and reflection of ideas, as well as gave them the opportunity to gain from the demonstration teacher's expertise. They were able to openly discuss what they had observed and receive from others those aspects that they have missed out. Another benefit was the opportunity to discuss student responses with the classroom teacher's insight about the child and other teachers' affirmation. They were made aware of new issues and felt affirmed in their current awareness of the various aspects of mathematical reasoning.

It was interesting to note that teachers who did not demonstrate shifts articulated their noticing of the classroom dynamics rather than their noticing of the reasoning pedagogies used by the teacher or incidences of student reasoning. Other studies of demonstration lessons have shown that generally teachers tended to notice teacher actions such as teacher questioning rather than student action (for example, Clarke et al. 2013). Our study has shown that most teachers noticed both teacher actions and student actions. However a small number of teachers were not focusing on the actions of the demonstrating teacher. In this case, not noticing the expert actions of the demonstrating teacher seemed to be detrimental to teacher's understanding of reasoning or ability to define reasoning. It is imperative that productive mathematical noticing (Choy, 2013) be emphasised in future professional development. Some studies such as Clarke et al.'s study specifically asked teachers "… what are you planning to focus on in your observation with regard to teaching? … student learning?" (Clarke et al., 2013, p.213) to help understand what teachers are interested to learn as well as to help them focus on the key elements. The findings in our study imply the need to raise awareness of the importance of shifting teachers' noticing of classroom and student organisation to students' reasoning and the pedagogical strategies that elicit this.

Another possible reason for a lack of shift could be related to teachers' content knowledge. Analysis of teachers' content knowledge was not included in the current study however research has shown that when teachers' specialised mathematical content knowledge is weak, they are less able to respond effectively to students' answers and reasoning (Ball et al., 2008; Doerr & English, 2006). Conducting the two demonstration lessons closer together, as recommended by a number of participants, might have assisted these teachers to focus on developing understanding of reasoning and including this learning goal in their on-going practice.

School culture and practices regarding collaboration and professional learning may have also contributed to differences in the impact of the MRPLRP (Jaworski, 2004, 2008). Some schools have explicit on-going professional learning programs or a culture of collaboration where teacher conversations influence planning of lessons, teacher actions and the vocabulary used. The role participants play in their school such as being in a leadership position and the year levels teachers taught may have influenced their initial perception or contributed to the shifts in understandings but it is not clear in our data and analysis whether these roles were contributing factors. Further research on the impact of school collaborative culture on teachers' perceptions, understanding and teaching of reasoning is recommended.

The other group of teachers who did not show shifts in their learning was the group that already possessed a more complex perception of reasoning. They appeared to possess a depth of knowledge and understanding of reasoning through their philosophy and beliefs in teaching and learning.

Another reason for the lack of shift could be that our Primary Teachers' Perceptions of Mathematical Reasoning Framework, designed through phenomenographical analysis of the variation within the whole group's perceptions of reasoning did not allow for further growth in understanding of mathematical reasoning by participants with prior knowledge of reasoning. Furthermore the demonstration lessons focused on number, and a limited set of reasoning actions. The lessons did not include generalising in other mathematical content domains, and the lesson plans did not progress to the level of proving or deductive reasoning involving infinitely many cases as suggested in the framework by Stylianides and Ball (2008). These aspects of the research design could in some way explain why that group of teachers were perceived as not having more inclusive perceptions of reasoning.

Whilst there were evidence of the program effects on shifts in teachers' ability to articulate their perceptions and understanding of reasoning, we are mindful that their ability to express their perceptions could possibly stem from re-voicing specific vocabulary or phrases used in the demonstration lessons (both in the lesson plans and in the expert's vocabulary). These phrases were readily used by teachers in the second interview. Having said that, although 'conjecture' was used in the lesson plan and on the worksheet and there was evidence that students were forming conjectures in the Magic V lesson, the word 'conjecturing' did not find its way into most teachers' vocabulary in the third interview. Awareness of generalising and forming conjectures was more likely to be identified for the most experienced Group 1 or 2 teachers in the second and/or third interview. It might be that the less experienced teachers did not hear the demonstration teacher using this word, recognise it when it did occur or appreciate its association with generalising and making connections. However this is speculation on our part.

The question remains as to how much modelling do teachers need to observe for them to be able to incorporate reasoning into their mathematics teaching? We have had two demonstration lessons that focused on generalising and justification and wonder if having more or different demonstration lessons that focused on generalising in other mathematics domains or other aspects of reasoning such as proof or generalisable argument, is necessary to support teachers to enact these reasoning actions in the primary classroom.

Conclusion

This professional development research program was prompted by the emphasis on mathematical reasoning and an inquiry into how well prepared teachers are to deliver reasoning in their classrooms. Findings from this study provide us with data about shifts and growth in teachers' perceptions and understanding of reasoning and evidence of the effect of teachers' productive noticing of well-constructed demonstrated lessons on their learning. There seems to be evidence of short term impact in the way teachers define reasoning as they shifted from defining reasoning as thinking, communicating thinking or problem solving to validating thinking, forming conjectures, using logical arguments to validate conjectures or connecting aspects of mathematics. Consistent with Bruce et al.'s (2009) claim that demonstration lessons are useful for participants at the entry level, this study found that teachers who displayed distinct shifts in their reasoning perceptions reported the benefits of observing demonstration lessons for their own understanding and for learning new strategies and techniques to elicit students' reasoning. The collaborative reflections in the post-lesson discussions and the subsequent implementation of the lessons further deepened their understanding and experience.

A limitation of this study is that data were mainly self-reported and teacher trials of the lessons were not captured. Whilst there is evidence of shift in some teachers' perceptions or understanding of reasoning, the longer term impact is less conclusive. Further research is needed to investigate how teachers use this improved understanding of reasoning to plan and teach lessons that develop children's reasoning. The study also showed that teachers who have not shifted may need further support to fully realise the benefits of the program. These have implications for future iterations of the professional development program as well as the mathematical research community in general.

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