Developing a kindergartener's concept of cardinality

Jennifer Throndsen
Utah State University
<jennifer.throndsen@schools.utah.gov>

Beth MacDonald
Utah State University
<beth.macdonald@usu.edu>

Jessica Hunt
North Carolina State University
<jhunt5@ncsu.edu>

Building students' understanding of cardinality is fundamental for working with numbers and operations. Without these early mathematical foundations in place, students will fall behind. Consequently, it is imperative to build on students' strengths to address their weaknesses with the notion of cardinality.

Building a student's understanding of number is critical in establishing a solid foundation for future mathematics learning. In fact, the *Australian Curriculum: Mathematics* for the foundational year expects such student proficiency to be demonstrated through their understanding of connecting names, numerals, and quantity. Most kindergarten students understand the fundamental ideas related to number, such as the relationships between quantity and unit, as well as basic counting and its role "in integrating aspects of number, such as sequence, cardinality, order, and measure" (Dougherty, 2010, p. 7). When a student acquires the meaning of counting and makes the connection between the set of objects they counted and the quantity of objects, they are said to possess cardinality (Sarnecka & Wright, 2013).

Cardinality is an essential construct of developing mathematical proficiency, as it is foundational for later number and operations. Children's ability to operate on number is grounded in their ability to understand number as a mental object. Yet, children who do not have cardinality consider number as a set of counting actions. Furthermore, students with cardinality understand that when an object is added and subtracted from a set the amount changes, but just moving the objects around or spacing them out does not have an effect on the quantity. In most cases, a child develops the cardinality principle by the age of four and a half years old. Unfortunately, this is not true for all young children.

Even as students first enter kindergarten the achievement gap is already present. Some students enter knowing how to count to 10 and beyond, understand the language and principles of "more" or "less", and perceive a quantity as unchanged regardless of changes to space between the objects. Other children are not yet capable of engaging with number in this way. Such dissimilar starts for children provide a challenge for kindergarten teachers who have a set of mandated standards that each student is expected to achieve by the end of the academic year.

Many teachers search to identify tasks that support children's understanding of foundational mathematics and provide multiple entry points to effectively engage students with a variety of readiness levels. For some students, teachers must more frequently engage in explicit, targeted practice opportunities to support struggling students in their mathematical development.

Ms Stapleton, a kindergarten teacher, planned her mathematics teaching for the beginning of the school year. Her plans involved regular opportunities for students to engage with reasoning and problem solving involving number as designated by the *Australian Curriculum: Mathematics.* Ms Stapleton began with an initial check of her students' foundational mathematics, as she knew such understandings support more advanced concepts during the school year. As she observed one child, Ed, she noticed that he was able to identify two, three, four, five and six items without counting, perceive as many as 10 items, and count up to three items consistently. Yet, it seemed as if Ed's understanding of number was relatively weaker than many of his peers. For instance, he inconsistently counted beyond three accurately, struggled to grasp the idea of combining subgroups to make a total amount (e.g., 2 and 1 makes 3), and relied on the orientation, or arrangement, of objects to determine quantity. As Ms Stapleton continued to work with Ed, she realised he had not yet developed cardinality. For instance, Ed was generally successful in counting a set of three items, but was very inconsistent in counting beyond three. In fact, even after one-on-one instruction, he often would subitise or estimate how many objects were present and then change his counting to match his guess:

T: (Lifts the top mat to show Ed four red counters in a square orientation.)
How many counters did you see?
E: Five.
T: Would you like to count them?
E: (Nods his head in agreement). 1, 2, 3, 5.
T: Five? Let's count them together.
T & E: 1, 2, 3…
E: 4.
T: So how many are there?
E: 4.

Similarly, Ed would count an inaccurate sequence, like 1, 2, 3, 7 for four or 1, 2, 3, 7, 8 for five.

Ms Stapleton, reflecting on her observation of Ed, wondered how she might use tasks to support further development of the Ed's fundamental ideas of number. What can educators do to equalise levels of competency and support students who demonstrate an early lack of readiness?

Ed is just one example of the many students found in today's classrooms who struggle to acquire the basic foundational mathematical competency needed to understand, perform, and apply more advanced mathematical concepts. The purpose of this article is to describe the learning activities we have used in our research (MacDonald, Throndsen, & Hunt, forthcoming) to develop children's counting and grouping, in hope to assist other educators when intervening to support early years students' understanding of number. First, we will describe how early number concepts are built up through children's subitising activity and notions of cardinality. Next, we provide the case of Ed as a means to illuminate tasks teachers may use to support and extend children who demonstrate an early lack of number readiness. Finally, we conclude with the importance of utilising kindergarten students' reasoning and problem solving when designing tasks.

Embedded within the construct of cardinality (and a student's understanding of number) is the student's ability to subitise. Subitising, a perceptual activity whereby young children quickly attend to the numerosity of a small set of items, has been shown to change to provide support for students' initial use of subgroups when composing a total amount (MacDonald & Shumway, 2016).

There are two main categories of subitising: perceptual and conceptual (Clements, 1999). Students who rely on the orientation to subitise a set of five objects or fewer are engaging in perceptual subitising; whereas, students who use subgroups to determine the composite are involved in conceptual subitising. Conceptual subitising is a more organised and evidence of more advanced thinking (Clements, 1999). For example, when a student looks at a five-dot domino and uses the subgroups of 2, 2, and 1 to determine that there are five dots, the student is using conceptual subitising. Clements (1999) posits that "children who cannot subitise conceptually are handicapped in learning arithmetic processes" (p. 401). Thus, children capable of conceptual subitising would rely on part-whole number understandings to serve them when operating on number in later primary years.

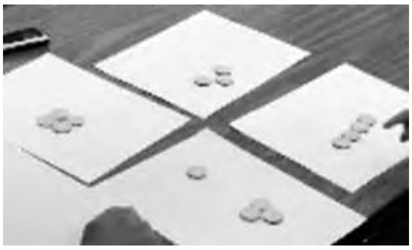
Subitising and cardinality are intertwined in children's early understandings of number. For students who are able to conceptually subitise, this is an indicator that they are beginning to develop their initial conceptualisation of cardinality and is fundamental in their development of understanding number (Clements, 1999). In this way, the instructional pairing of subitising to promote cardinality may help to facilitate children's understanding that counting is more than a rote verbal process.

Accessible subitising and counting activities to support Ed's number reasoning

Like many other students with deficits in early numeracy, for Ed to build a more robust number understanding, he needed to engage in tasks that required flexible understandings of part-whole relationships within numbers, regardless of orientation. To this end, we used subitising games (see MacDonald & Shumway, 2016) that engaged Ed in subitising and then counting (see Table 1). Table 1 displays two sets of counting games we used in instruction to build cardinality through subitising, which then afforded Ed more successful counting activity.

The subitising games proved to be an accessible platform from which Ed could advance his reasoning. For example, we would place counters on a blank mat in different orientations, flash the counters to Ed, quickly cover up the mat, and ask Ed to identify the number he saw. Ed would respond in a variety of ways to communicate his thinking from attempting to count, recreating the orientation before describing the amount seen, using figural representations to express

Table 1. Counting games used to bridge to Ed's grouping.

Task name/description	Materials	Aspects of the game to consider
Hiding bears (adapted from *Math Solutions*, 2013) Place one counter under the paper or cup and three to five counters on top of the paper outside of the cup. Ask the student to how many bears altogether. 	• Counters • Cup or cave image printed on paper	When designing the game, begin with one counter hidden and increase by one each time, or begin with three to five counters under the paper or cup and one counter visible. Increase the visible counters by one each time Ed counted the visible counters, then the hidden counters, which was more difficult because it required him to visualise the hidden counters. . When Ed struggled with coordinating three, four, or five hidden counters and three visible counters, it helped Ed if the counters were arranged like that of a typical die face. These patterned arrangements allowed Ed a scaffold when counting.
Making "four" Place three counters on the mat and ask the student how many he or she sees. After counting or subitising, ask the student to make four on the mat. 	• Counters • White paper mat	This requires the student to engage his or her counting and grouping. When Ed first did this, he added four to the mat to make a total of seven counters. This told us that he could not consider four in groups of three and one. When young children struggle with this activity, they will add all of the counters visible to the mat or add four counters to the mat. The first struggle tells us that the student is relying only on counting to know number. The second tells us that the student is relying on four as a whole, but not composed by subgroups.
Note: Counting games adapted from MacDonald, Throndsen, and Hunt (forthcoming) study.		

the amount seen, or stating the number words before relying on his fingers or counters. For instance, when quickly shown a mat with two sets of two counters diagonally in the top left and bottom right corners, Ed began counting the counters, but was unable to because he was only shown the counters for two seconds. So, Ed reconstructed the image to help him respond.

T: How many did you see?
E: (Recreates the orientation shown to him). Two. (Hold up his middle finger and ring finger).
T: You saw two? Can you show me?
E: (Points to each set of two). Two. Two.
T: How much altogether?
E: These make four.
T: Would you like to count them?
E: (Points to each counter). One, two, three, four.

This type of thinking was a significant transformation of Ed's thinking, as he was now composing subgroups to understand four, described by MacDonald and Shumway (2016) as "perceptual ascending subitising". However, his reliance on the visual material indicated his inability to abstractly understand how these subgroups relate to the total group of items. These initial learning tasks helped develop a basic level of understanding that could be built upon with subsequent subitsing games.

To push Ed further, we used more subitising embedded games from MacDonald and Shumway (2016) to help Ed transition from simply subitising subgroups towards using those subgroups to create a composite group. For example, when playing the Ice-cream Game (see Table 2), we designed the faces of the die with large space between clusters of dots and with subgroups of dots of different colors. On the game mat we designed orientations with patterned sets of four dots. On the die face, we designed orientations that elicited Ed's attention to subgroups two and two. We found that Ed's engagement in this game provided him the ability to (de)compose four with two and two, and provided him quantitative relationships when subitising that transitioned to his counting. Thus, as Ed understood 4 in a more robust and flexible manner, he began to change his counting to align with how 4 was perceived.

Table 2. Most common subitising games used to bridge number understanding to Ed's counting.

Task name/description	Materials	Aspects of the game to consider
Ice-cream Game To be the first to colour all the ice-cream scoops, students are required to match an arrangement of dots on a die face to an arrangement of dots on an ice cream scoop.	• Die • Ice-cream Game mat • Markers	When designing the game, begin with sets of dots that suggest part-whole relationships (i.e. space between subgroups of items, different coloured items) on the die faces, and sets of dots that suggest total sets of dots (i.e., patterned dots with the same amount of space between the dots) on the ice cream scoops. This promoted Ed's part-whole relationships of number.
Penny Bug Game To be the first to have your penny bug reach the finish (indicated by a flower), students roll the die and either hop their penny bug forward the number of dots that came up on the face of the die, or they can skip ahead to a space where their cube face matched a space.	• Die • Penny • Bug playing piece	When designing the game, begin with sets of dots that suggest novel and "known" part-whole relationships (i.e. space between subgroups of items, different coloured items) on the game board, and sets of dots that suggest total sets of dots (i.e., patterned dots with the same amount of space between the dots) on the die. This promoted Ed's flexible part-whole relationships of number.

Counting

Early on in this intervention, Ed engaged in three different types of counting:

1. He aligned his counting with the intended result;
2. He skipped 4, 5, and/or 6 in the counting sequence; or
3. He counted accurately.

Often we found that Ed was accurate in his counting for one of two reasons. First, Ed would count up to four or five items accurately when attempting to reach an intended result after subitising or estimating sets of items. Second, with teacher prompting, he was able to count up to six objects correctly, but was unhindered by the fact that his initial count and his subsequent counts didn't match. Steffe and Cobb (1988) would describe such thinking as pre-numerical counting because Ed had a general understanding of the counting sequence from a procedural standpoint, but did understand number is relative to his actions, not the materials, so counting the same set of objects with two different end results left him unflustered. Such experiences are common for young children who understand number as a set of corresponding actions with a corresponding number sequence. However, some students, like Ed, have difficulty with composing an accurate number sequence until they can coordinate their grouping and/or subitising activity with their counting to achieve cardinality.

Additionally, Ed could not add onto a previously counted quantity if more were added, but rather counted all the objects again. So for example, if he was shown a set of three objects and he counted them and found that there are three, and then another counter was added, Ed would recount the entire group to determine that there were four counters. These types of responses told us that he relied on either grouping or counting, but was unable to coordinate both simultaneously. Ed's technique is inefficient and is commonly demonstrated in students who lack the ability to conserve number.

To press Ed's counting to change, we used Ed's subitising activity with two sets of activities. First, we did this by showing Ed four items arranged in a square-like orientation (at this point Ed knew this represented "four"). We then asked him to count the orientation, which he could do because he used his subitising activity to inform his counting activity. We then rearranged the orientation so there was a large space between the two sets of items. This pressed Ed to attend to the subgroups, two and two. Now when counting Ed had to perceptually chunk and compose (with perceptual subitising) and count to know these sets of items as four. If Ed was inaccurate, the items were rearranged to show him the original square-like orientation so he could reconsider the orientation of items as "four." Second, we asked Ed to make "four" and "two and two" and then compare the sets of items by orientation and counting. Ed began to describe the two sets of items as both being "four" and "two and two."

Conclusion

Through the activities described above, Ed's thinking structures became more abstract and he relied less and less on reconstructing the image, but used either his language (number words) or his figural patterns to communicate the set quantity. He transitioned from unitising individual counters in a set to using subgroups and composite groups to identify the quantity of larger sets.

Our instructional focus was to increase Ed's efficiency and effectiveness in counting by helping him to advance his ability to coordinate his ordering thinking structure with his grouping thinking structure. As a result, Ed's understanding of cardinality and the connection to counting was solidified with real meaning that is foundational for other number concept development. For example, when he identifies 2 and 2 in a set of 4, he will be able to hold 2 in his mind and then count on from there to say 2, 3, 4. Working on this concept would be an essential accomplishment as it would signal "a grasp of part-whole relations which are critical to quantitative reasoning." (Maclellan, 2012, p.7).

Kindergarten is often our first formal opportunity to influence students' learning trajectories with early intervention opportunities. This means that educators must adapt their instruction to the mathematical reasoning and problem solving of their students to maximise learning and successfully build their mathematics understanding. Without these early mathematics foundations in place, students' achievement gaps continue to grow, preventing more abstract mathematics and critical number concepts to be understood in the future. Further, interventions should be utilising students' strengths to support particular areas of weakness. This would provide opportunities for students to perceive themselves as successful mathematics learners when reasoning and problem solving in subsequent activities.

References

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