Fostering students' development of the concept of angles using technology


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The use of dynamic geometry environments, such as GeoGebra, can support students' development of geometric concepts. Tasks and investigative ideas for the development of the concept of angle are shared.

Abstract

We have used *GeoGebra*, a dynamic geometry software environment, to explore how Year 4 students understand definitions of angles. Seven students defined angle and then completed several activities adapted for the dynamic environment. Afterward, students again shared their definitions of angles. We found that even a short investigation using dynamic geometry software could support students' development of angle definitions. We share tasks and investigation ideas using *GeoGebra* to support students' learning about angles.

Introduction

In a sing-song voice, David chants, "A right angle is 90 degrees, a straight angle is 180 degrees, an acute angle is less than 90 degrees, and an obtuse angle is greater than 90 degrees and less than 180 degrees." David's approach to remembering facts about special angles illustrates the challenges some teachers face in supporting children's conceptual understanding of angle concepts.

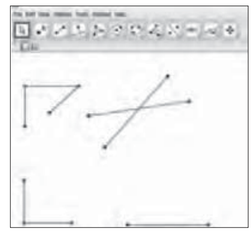
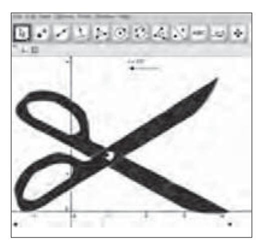
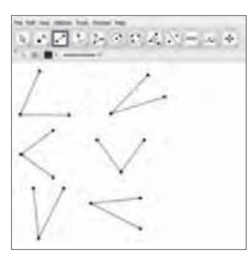
In the *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2014), the concept of angles is the foundation for many mathematics concepts in both primary and secondary schools (Host, Baynham, & McMaster, 2014). Students who do not fully learn the concept of angle will be ill-equipped to learn basic trigonometry and more advanced geometry tasks (Weber, Knott, & Evitts, 2008). Researchers have found that the difficulties experienced by students who struggle with angle related concepts, are related to learning multiple definitions of an angle (Keiser, 2000; White & Mitchelmore, 1998). Mitchelmore and White (2000) indicated that three common representations are used to define an angle in mathematics education: an amount of turning between two lines—rotation; a pair of rays with a common point —wedge; and the region formed by the intersection of two-half planes—interior region between the intersection of two lines.

After hearing David's initial idea, we wanted to further understand how David and his classmates were thinking about angles. Clements and Battista (1989, 1990) and Browning, Garza-Kling and Sundling (2008) had success exploring Year 4 and Year 6 students' ideas about angles using the *Logo* programming environment and graphing calculators, respectively. In these studies, students explored geometrical concepts by understanding and directing a turtle or angle's movement using numerical inputs. We wondered how dynamic geometry software (DGS) could help build understanding of the definition of angle. We chose *GeoGebra* software to explore with the children because the inputs are spatial rather than numeric, allowing students to manipulate objects in a dynamic spatial environment that translates to the numeric representation. Emerging evidence indicates that opportunities to manipulate objects in a spatial environment can improve children's mathematical development (Cheng & Mix, 2014). And for teachers, this software is free and easy to access and use.

Investigating students' ideas about angles

Seven Year 4 students, labelled as "high ability" by the school, volunteered to solve angle-related tasks using technology. The students worked in pairs during 30 to 45 minute video recorded think-aloud interviews. We asked the students to define an angle in their own words, and then asked them to complete five tasks related to angle definition and measurement that we adapted for *GeoGebra.* We developed the tasks and additional probing questions to align with existing studies (Browning et al., 2008; Clements & Battista, 1989, 1990; Host et al., 2014), as well as the *Australian Curriculum: Mathematics* (ACARA, 2014), and to address issues found in the literature about students' definitions of angle (Mitchelmore & White, 2000; Keiser, 2000; White & Mitchelmore, 1998). We focused on characteristics from Mitchelmore and White's (2000) definition: two rays that share a common endpoint or intersection, angle as a geometric shape or wedge, and the amount of turn in a circular arc or rotation (see Table 1). Tasks were aligned with the content descriptions "Identify angles as measures of turn and compare angle sizes in everyday situations" and "Compare angles and classify them as equal to, greater than, or less than, a right angle", content descriptions for Year 3 and Year 4 taken from the *Australian Curriculum: Mathematics* (ACARA, 2014). After completing the tasks, we asked the students again to share their definitions of an angle.

Table 1. Angle tasks students completed in GeoGebra.

Characteristics of angles	Task description & probing questions	Image of task from *Geogebra*
How many angles? (Intersection, wedge)	How many angles do these figures have? How do you know? (Adapted from Clements & Battista, 1990) Manipulate figures to determine how students are thinking about angle definitions.	
Find angles (Intersection, wedge)	Find the figures that are angles. How do you know? (Adapted from Clements & Battista, 1990) Put a third point on first line segment and ask if it could then be an angle.	
Scissors tips touching (Intersection, wedge)	Compare the angles found in a short and a long pair of scissors whose tips touch. (Adapted from Tzur & Clark, 2006) Probe for relationship students see between the lengths of the angle sides and measures.	
Rotating scissors (Rotation)	(Students move a slider allowing one blade of the scissors to rotate). What do you see? Probe responses about scissor blade rotation in relation to the rotation definition of an angle.	
Which angle is bigger? (Intersection, wedge, rotation)	Which angle in each pair is bigger? How do you know? (Adapted from Clements & Battista,1990) How can you make the first angle match the second angle (for each pair)?	

Students' initial angle definitions

We began by asking the students to share their definitions of angles. Students either said that an angle is a way to measure degrees or drew two lines that intersected at a point (see Table 2), which corresponds to the literature about upper primary students' conceptions of angles (Keiser, 2000). Like our participants, the sixth-grade students in Keiser's study defined angles as vertex, corner, point, or intersecting lines, and struggled to identify which component of an angle is being measured (i.e., the length of the rays, the linear measure between rays).

Changes in students' angle definitions

While some students' initial definitions focused on angle as a vertex or intersection, after discussing the tasks from Table 1 in pairs for 30 to 45 minutes, six of seven students' definitions changed. Their post-task definitions contained a dynamic element, represented either by arrows drawn on paper or swiping hand motions during verbal descriptions (see Table 2). This evidence corresponds to angle as rotation, one of the three definitions of an angle that was missing in all of the pre-task definitions (ACARA, 2014; Mitchelmore & White, 2000). For example, one student initially drew two intersecting lines and said, "I do not know why it is an angle." After working on the tasks, she drew an angle with an arrow indicating "turning," and said it is two line segments which are "turned," indicating the rotation definition. Another student's definition included evidence of angle as wedge, when he swept his hand to indicate the space of an angle and said that an angle includes "the length of the inside of a shape" (see Table 2).

The act of dragging (e.g., with a cursor) was a key component in the shift in the students' understanding and reasoning about angle definitions, as dragging helps primary students determine the effects, differences, and properties of objects such as angles (Gonzalez and Herbst, 2009; Hollebrands, 2007). This is not possible when numeric inputs are used to move objects or angles, as in Clements and Battista (1990) and Browning et al.'s (2008) studies. As the students reflected on their work in *GeoGebra* they developed new images of angles (Driscoll et al., 2007). These results indicate that angles should be presented in environments that incorporate a physical angle context.

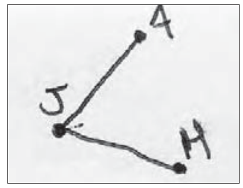
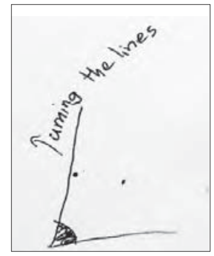
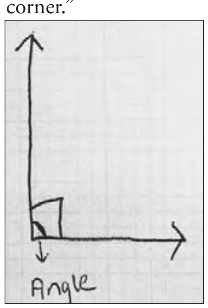
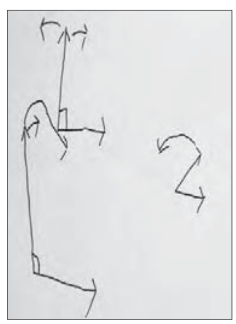
Suggestions for teaching

Browning, et al. (2008) wondered if it is possible for students to develop a broader understanding of angle before Year 6 or in fewer than the seven lessons conducted in their study. In this study, we showed that it is possible for upper primary students to build broader reasoning and understanding of the definitions of an angle using the dynamic spatial environment found in *GeoGebra.* Even a short investigation of angles using dynamic geometry software can support Year 4 students' development of angle definitions.

To launch the lesson, teachers can encourage students to locate angles in the classroom environment. In a quick discussion afterward, students can share an item they chose and state characteristics that make it an angle. Next, students can explore the *GeoGebra* tasks such as; "Find the angles as many as you can", "How many angles can you see?", "Which angle is bigger?", and "Rotating scissors" (see supplemental materials). Finally, in the lesson summary, students can define an angle and share with a classmate. The technology can be incorporated in different ways depending on the technology available in individual classrooms. For example, this lesson could be done by the whole class on a set of tablets or laptops with the *GeoGebra* tasks pre-loaded and students working in pairs. For classrooms with an electronic white- board, student pairs could come to the front to manipulate the figures in *GeoGebra*. This lesson could also be demonstrated by the teacher using a computer connected to an overhead projector.

Because more abstract geometric reasoning is on the horizon for Year 4 students, as students work, the teacher would look for instances of students reasoning across relationships, investigating invariants, and making generalisations, and then ask targeted students to share their work and thinking during a whole class summary discussion (Driscoll et al., 2007). Specifically, teachers would focus on geometrical thinking and different definitions of angles such as those asked in the task descriptions and additional questions (Table 1). The questions could focus on relationships within and across figures; instances of sometimes, all, and never; phenomena that change or stay the same; and considering what if…? For example, a teacher might ask students to describe what they think will happen when one ray of the angle is manipulated while the other side remains fixed. Students would be given a final opportunity to refine their angle definitions after the whole class summary discussion. A recording sheet for students, including probing questions for each task, is provided in the supplementary material, along with links to the tasks. An additional lesson and *GeoGebra* activity link are included in the online materials. This lesson and activity, adapted from Millsaps (2012),

Table 2. Students' angle definitions before and after using *GeoGebra*

	Initial definition of an angle	Post-tasks definition	Change in angle representations
Student 1	She drew two intersecting lines, and then said, "I do not know why it is an angle". Initially, she did not define the angle.	"An angle is two line segments turned like this:  An angle has to have three points to been an angle. Like I showed you in the example, point J is where you look to see what type of an angle it is. There are 4 main types of angles: right angles, obtuse angles, acute angles and straight angles."	Rotation added to intersection, indicated by her drawing and words saying "two line segments turned like this:" 
Student 2	"A right angle is 90 degrees, a straight angle is 180 degrees, an acute angle is less than 90 degrees, and an obtuse angle is greater than 90 degrees and less than 180 degrees."	"The length of the inside of a shape." (He used a swiping hand motion to indicate an angle.)	Named angles changed to rotation indicated by his swiping hand motion. Wedge indicated by "The length of the inside of a shape" along with his hand motion.
Student 3	He drew a right angle and explained, "This is a right angle because it is 90 degree. It makes square corner." 	"An angle is what two lines form when they meet." 	Angles as corners changed to intersection definition: "An angle is what two lines form when they meet". Rotation indicated by the curved arrows showing motions.

supports students' ability to recognise angle as measures of turn and as the intersection of two rays sharing a common endpoint that is the centre of a circle (ACARA, 2014).

Conclusion

In his post-task definition, David swiped his hand in an arc across his paper to show an angle as he said an angle "…is the length of the inside of a shape." What David was suggesting with the explanation and the gesture was a broader understanding including angle as wedge and rotation, in addition to angle as the intersection of two rays. This definition now more closely aligns with the understandings required by the content descriptions in the *Australian Curriculum: Mathematics* (ACARA, 2014). His experiences with varied angle contexts have supported him to see angles as more than the intersection of two rays (Mitchelmore & White, 2000). As David's thinking about angles continues to evolve, he is in a better position to make connections between his angle definition and other academic experiences. His understanding of the definition of an angle as the intersection of two rays, a wedge, and rotation by using dynamic geometry software should support him now and in his future explorations of geometry.

Tasks

How many angles can you see?
http://www.geogebratube.org/material/show/id/212671

Find the angles as many as you can.
http://geogebratube.org/material/show/id/212653

Which angle is bigger?
http://www.geogebratube.org/material/show/id/212701

Rotating scissors
http://www.geogebratube.org/material/show/id/212635

Scissors tips touching
http://www.geogebratube.org/material/show/id/212689

How many wedges?
http://www.geogebratube.org/student/m255687

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Hollebrands, K. F. (2007). The role of a dynamic software program for geometry in the strategies high school mathematics students employ. *Journal for Research in Mathematics Education, 38*(2), 164–192.

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Keiser, J. M. (2000). The role of definition. *Mathematics Teaching in the Middle School, 5*(8), 506–11.

Weber, K., Knott, L., & Evitts T. (2008). Teaching trigonometric functions: Lessons learned from research. *Mathematics Teacher, 102*(2), 144–150.

White, P., & Mitchelmore, M. (1998). Sharpening up on angles. *Australian Primary Mathematics Classroom, 3*(1), 19–21.

Mitchelmore, M., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics, 41*(3), 209–238.

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Supplementary materials

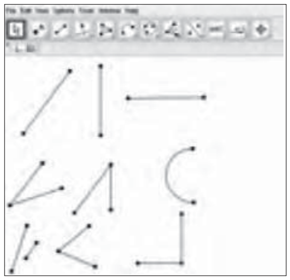
GeoGebra launch

Look around our classroom. Find 3 examples of an angle and write the characteristics that make it an angle:

Angle example	Characteristics that make it an angle

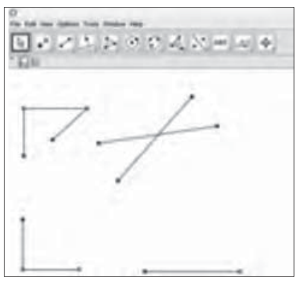
Exploration in *GeoGebra*

Find angles



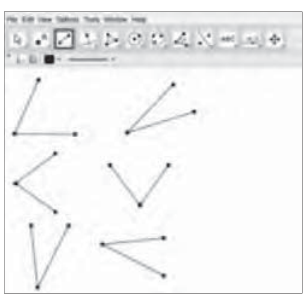
Label the figures that are angles. How do you know? For those figures that are not angles, change them so they are angles. Describe what you did:

How many angles?



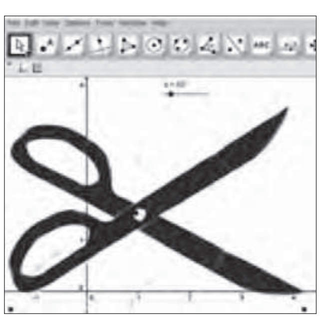
How many angles do these figures have? How do you know?
For each figure, describe a change to make one *more* angle:
1.
2.
3.
4.
For each figure, describe a change to make one *less* angle:
1.
2.
3.
4.

Which angle is bigger?



Which angle in each pair is bigger? How do you know?
Describe how you can make each pair the same size:
1.
2.
3.

Rotating scissors



Move the slider and describe what you see.

GeoGebra summary

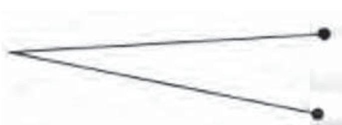
Write your definition of angle:

How many wedges?

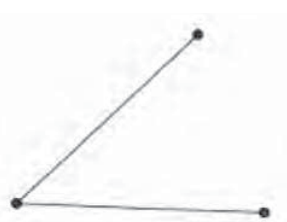
1.  Angle measure: Wedges used:

2.  Angle measure: Wedges used:

3.  Angle measure: Wedges used:

4.  Angle measure: Wedges used:

5. Angle measure: Wedges used:

6.  Angle measure: Wedges used:

7. 

Angle measure: Wedges used:

8. 

Angle measure: Wedges used:

9. Now make the circle bigger or smaller. Which wedges did you use to cover the circle? Describe any changes you had to make to the wedges.

10. What happens to the angle measure?

11. How many wedges do you use in your new circle?

12. How does the size of the circle affect the measure of the angle and the number of wedges you used to cover your circle?