Look down from the sky: Is it a bird? Is it Superman? No, it's a plane.

Helen Chick

University of Tasmania

<helen.chick@utas.edu.au>

Introduction

The imagery from Google Maps (maps.google.com.au) can reveal some curiosities if you know where to look. Years ago someone discovered what looked like a jet plane sitting incongruously in an Adelaide park. Further analysis revealed the truth: the plane only appeared to be on the ground; it was, in fact, tracking into Adelaide airport and passing over the park when the photograph was taken.

That particular image has now disappeared from Google's satellite images, but careful searching of current imagery (as at 7 November 2016) reveals another plane, caught in flight over Melbourne's suburbs. Figure 1 can be found using the following coordinates in Google Maps, with satellite view turned on: -37.6680 145.0028 (or S 37.6680 E 145.0028).

Figure 1. Jet plane over Melbourne's northern suburbs. Source: Google maps (maps.google.com.au)

My first question when I saw this image was: "How far above the ground is the plane?" Is it so close to the houses that it will shake their foundations as it skims across their rooftops, or is it sufficiently high that it's leaving a contrail and there's no point waving to the pilot?

Before I present a solution to this problem—a solution which is accessible to secondary students—I want you to think about how you might go about solving it. So, put down this issue of AMT, go to your computer, enter those coordinates into the Google Maps search bar, look at the plane, and think about a way to answer the question. Seriously. I mean it. Come on! You know you want to figure it out for yourself! Are you still here, or have you really gone away, solved it, and come back?!

Why the plane problem is a good one

Numerous authors highlight the power of effective problem-solving activities. Jo Boaler (Boaler, 2008), writing about the frequent disconnect between school mathematics and real world uses of mathematics, points out that students who learn with a problem-solving approach are better able to connect, apply, and see as relevant the mathematics that is learned at school, because they view maths as evident in, and applicable to, their lives. Sullivan (2011) argues that it is important to engage students with challenging tasks that require decision-making, foster communication, and promote fluency and transfer of skills. These authors, among many, suggest that extended, engaging, and relevant problems are essential for students' mathematical development.

The plane problem is a real-world problem, presented without any suggestion as to how it might be solved. It arose unexpectedly as I was messing around on the internet, not thinking about maths at all. I did not encounter the problem in a maths lesson, nor as homework in the middle of a unit on a particular topic, and so I had no clues about what method was going to be useful, or even if I would be able to solve it. All I had—initially at least—was the image of the plane and a mystery: at what altitude is it flying?

With regard to the *Australian Curriculum: Mathematics* (ACARA, 2016), this problem is firmly in the realm of the problem solving proficiency. Since we are targeting height, this suggests that some aspects of measurement are likely to come into play, and, since the only source of measurements seem to be from a scaled map, geometry, ratio, and proportion might be involved as well.

The author's first solution

As I looked at the plane and wondered how I might work out its altitude, I noticed shadows from the trees and buildings. Having experienced for myself the view of a plane's shadow from a window seat, I wondered if I could spot the plane's shadow somewhere over suburbia. By observing the direction of the tree shadows I deduced the sensible search area, and eventually spotted the shadow of the plane (see Figure 2 and Figure 3).

I was curious about how far the shadow was from the plane. Using a printout of Figure 2, I measured the distance between the plane and the shadow; it was 14.40cm (to the nearest half millimetre). The 50m scale graticule in the bottom right corner of Figure 2 was about 1.75cm long, and from this I deduced that the distance between the image of the plane and the image of its shadow was $14.40 \div 1.75 \times 50 \approx 411$m.

By now a solution idea was bubbling in my brain, based on a memory of working out heights of tall things from shadows and known smaller things, using the idea of similar triangles. I looked more closely at Figure 3 to see other objects casting shadows, including an electricity pole.

I zoomed to the shadow of the electricity pole, and discovered a difficulty. Sometimes Google's satellite images change as you magnify, giving different photographs for the close-ups instead of enlargements of the original imagery. I could not use these alternative close-ups, because they

Figure 2: The plane and its shadow. Source: Google maps.

Figure 3. A close-up view of the plane's shadow, to prove that I am not imagining it. The plane's shadow and a nearby electricity pole are highlighted. Source: Google maps.
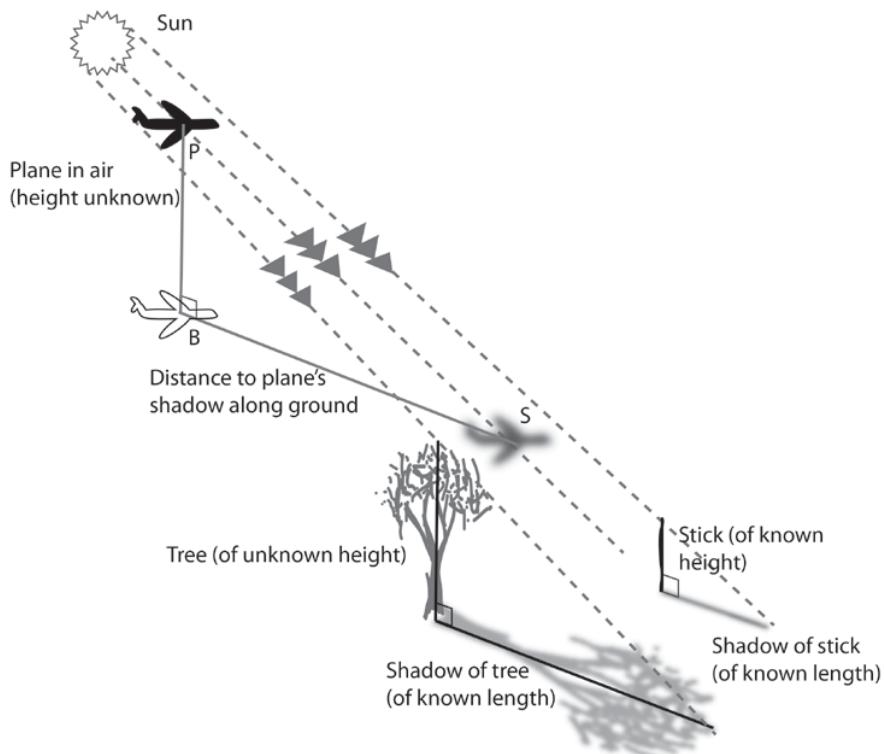
Figure 4. Finding the height of an unknown object, using shadows and an object of known height.

were taken at a different time of day when the shadows pointed in another direction. My similar triangles plan relied on the sun being in the same place for both of the shadow-casting objects.

After zooming as close as I could using the original photo (until just before it changed to the unhelpful, wrong-time-of-day close-up), I measured the length of the shadow of the electricity pole. It turned out to be 0.70cm long on the image, but now the map scale was different: the 10m graticule in Figure 3 was 1.60cm long. I deduced that the shadow was about 4.4m long in real life ($0.70 \div 1.60 \times 10 \approx 4.38$m).

I now had two measurements and a hunch about similar triangles. I remembered a method for estimating the heights of tall objects by using a small known object, similarity, and shadows, as seen in Figure 4. Since the sun is so far from earth its rays are parallel, and because heights are perpendicular to the ground, we can make two similar right triangles: one involving our known object and its shadow (a stick in Figure 4), and the other involving our object of mystery height and its shadow (for example, the tree in Figure 4). If we measure the height of our known object and the length of the shadows, similarity implies that:

$$\frac{\text{height of known object}}{\text{length of shadow of known object}} = \frac{\text{height of tree}}{\text{length of shadow of tree}}$$

...thus allowing us to find the unknown height of the tree.

I wanted to apply this idea to the plane and its shadow, using the electricity pole as a known object. The plane and its shadow also make a triangle that is similar to the one formed by the electricity pole and its shadow (see Figure 4, with the electricity pole taking the place of the stick). Here, however, we need to know what is equivalent to the shadow length for the plane: this is the distance, along the ground, between the plane's actual shadow, *S*, and the projection of the plane onto the ground, *B*. Provided the satellite is overhead, this means we need the distance from the image of the plane in the photo and its shadow. With this information I thought I could determine the height of the plane.

Of course, now I had to find the height of the electricity pole. This time Google Street View came to the rescue. I "popped down" to the street containing the electricity pole, and rotated my

Figure 5: Google Street View image of the street containing the electricity poles near the plane's shadow, showing how to estimate the height of the pole. Source: Google maps.

view until I could see it (see Figure 5). Very conveniently there were some visual clues I could use to estimate the height of the pole: a man, his van, some workplace fencing, and a ladder.

I estimated the fencing to be about a metre high, and used this as a benchmark to estimate the height of the pole at 8m (see Figure 5). In doing so, I knew that my 1m increments on the pole should not really have been equally spaced because the angle changes as you look upwards. Some Googling revealed that ladder rungs are between 25cm and 30cm apart, making the ladder about 3m tall, which gives an approximate confirmation of our 8m estimate. Other estimation benchmarks may occur to you.

With the electricity pole estimated to be 8m tall with a 4.38m shadow (this is the stick and its shadow in Figure 4), and the distance from the plane to its shadow being 411m (this is equivalent to BS in Figure 4) I could estimate the height of the plane (equivalent to PB in Figure 4). We have:

$$\frac{\text{plane height}}{411} = \frac{8}{4.38}$$

...and so the plane's height is $8 \div 4.38 \times 411 \approx 751$m.

The author's improved solution

This was initially quite satisfying, but I had a concern. Close examination of the satellite imagery made it obvious that the satellite taking the photograph was not directly overhead.

You can actually see the full electricity pole in Figure 3 instead of just the top framework coinciding with the base as you would expect to see from directly above. This implies the image of the plane on the map was actually not directly below its actual position in the air, and so when I measured from the plane to the plane's shadow, BS, I was not measuring the plane's "shadow length" at all. Consequently, this length did not correspond, by similarity, to the pole's shadow length which was measured from the base of the pole. I needed to think about what the satellite was seeing since it was the satellite's view that was making the map image I was examining. I was no longer sure I had any similar triangles with which to work.

With care and thought I drew Figure 6, which shows what is going on. The parallelogram is the ground and also what the satellite produces as the satellite image view in Google maps. Everything drawn in grey is what is on the ground or map. The objects in black are above the ground in the third dimension: the plane, up in the air, and the electricity pole pointing upwards. Let P denote the middle of the plane, and P' the top of the pole. The base of the pole is denoted by B' and this can be seen in the image. The plane, on the other hand, is flying directly over B, but there is nothing on the ground that tells us where this is.

The sun, as in Figure 4, casts shadows on the ground. The shadow of the plane is at S, and the top of the pole's shadow is at S'. The triangles PBS and $P'B'S'$ are exactly like the two triangles from Figure 4, and are similar. Unfortunately, although we can measure $B'S'$ (which I did above, to get 4.38m) we cannot measure BS, because we don't know where B is. This is because we are not viewing the map from directly overhead; what we see in the image as the plane is not sitting on top of B. (In my first solution I thought I was measuring BS, but, as discussed, I was wrong because I hadn't realised that I could not actually see B.)

Thus we have to think about what the satellite observes. It views the scene from an angle, since we actually see the electricity pole partly from the side and not directly above. Since the satellite is relatively far away from the earth I made the assumption that lines of sight from it to the objects below are close to parallel. The satellite's view of the electricity pole creates the image $G'B'$ on the map; its view of the plane puts an image of the plane at G. (Note: apparently some of Google's "satellite" imagery also comes from aerial photography, taken from a much lower altitude than satellites, and here the "parallel viewing" assumption may not be totally valid.)

This argument produces two right triangular pyramids, $PBSG$ and $P'B'S'G'$. Because of all the parallel lines, there are many similar triangles and corresponding angles between the two. In fact, the pyramids are geometrically similar to each other. If I could find some corresponding measurements I would be able to find the height of the plane.
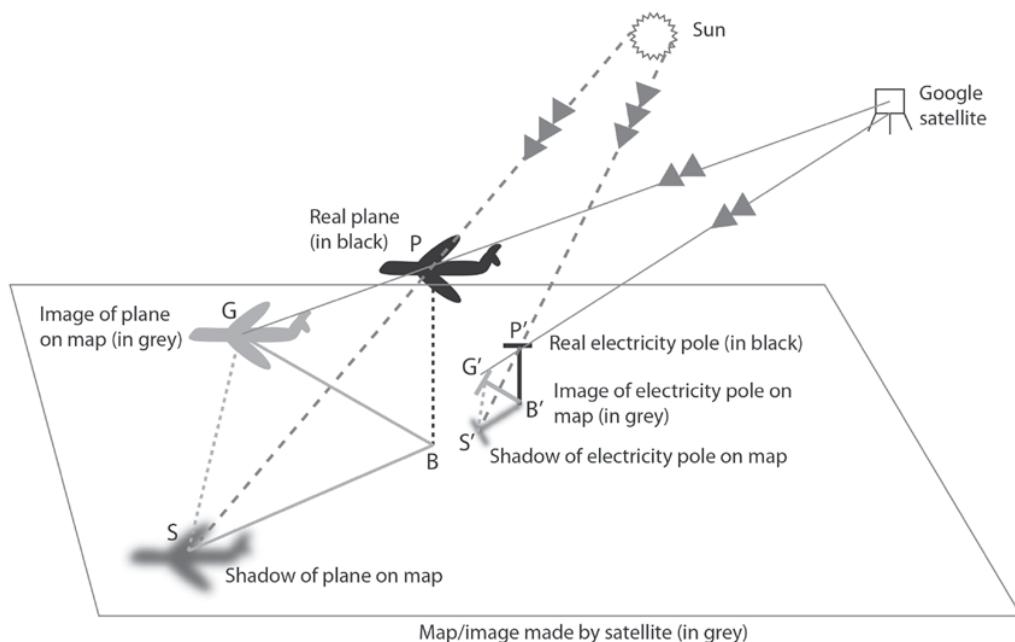

Figure 6. The geometry of the plane and the electricity pole in three dimensions.

In figure 6, shapes that end up on the two-dimensional surface of the map/image that Google shows are in grey. The two rays of light from the sun (that cast shadows) are parallel to each other since the sun is so far away. The satellite, also a long way away, is assumed to view the scene using parallel lines too. This implies that *PBSG* and *P'B'S'G* are similar pyramids.

I had already measured *GS*; that's what I calculated first of all when I measured the distance between the (image of the) plane and its shadow. This distance is 411m. The corresponding length for the electricity pole configuration is *G'S'*, the distance between the top of the pole's shadow and the top of the pole in the photograph. I measured this on a printout of Figure 3 and got 0.80cm, so after using the scale (10m is represented by 1.60cm) I obtained 5.0m as the distance *G'S'*. Since we want to find the height of the plane, or *BP*, the corresponding length of interest is *B'P'*, which is the height of the electricity pole. This measurement was determined in my original solution to be *B'P'* ≈ *8m*.

The similarity between the pyramids means that ratios of corresponding pairs of sides are equal. In particular,

$$\frac{BP}{GS} = \frac{B'P'}{G'S'}$$

and so

$$\frac{BP}{411} = \frac{8}{5}$$

With this I obtained my better estimate, concluding that *BP*—the height of the plane— is about 658 metres (8 × 411 ÷ 5). [I have chosen not to discuss issues of measurement error and accuracy here, but these are also important ideas that could be addressed with a class.]

It should be noted that the plane was about 14km away from Melbourne airport at the time (see Figure 7). Aircraft descend at a range of angles, although about 3° is a suggested estimate for commercial aircraft on a standard approach (see http://www.ifly.com/blog/from-the-cock-pit/approach-and-landing/). Using trigonometry, our computed altitude of 658m when 14km from landing gives a descent angle of about 2.7° which suggests that our height estimate may not be too bad.

Figure 7. Position of the plane in relation to Melbourne Airport. Source: Google maps.

An approach that sounds promising but isn't

I presented this problem to a friend of mine, and he suggested that if you have an estimate for the actual wingspan of the plane (it looks to me like an Airbus A320) and compare this with the amount of ground that is spanned by the wings using the scale, then you should be able to deduce the height. Unfortunately, this method would require more information than is available from just the imagery. The two images in Figure 8 show a drink bottle and a tower made of blocks. In both images the bottle appears as the same height as the tower, and yet in the first photo the bottle is only 20cm from the tower, while in the second photo the bottle is 60cm from the tower (the solid coloured blocks on the left side of the bottle are spaced at 10cm intervals from the tower, and since more are visible in the second photo the bottle is further from the tower). So, why does the bottle appear to be the same size as the tower in both photos? Simple: I adjusted the zoom setting on the camera. This suggests that looking at the size of the plane and the ground alone is not enough to tell us how high the plane is.
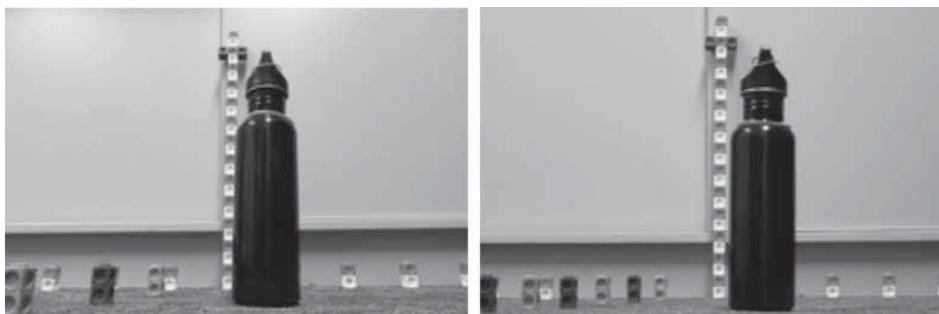
Figure 8. A water bottle photographed at a distance of 20cm (on the left) and 60cm (on the right) from a block tower. The photo on the left was taken with a zoom lens and the second with a wide angle lens.

In the classroom

The mathematical content used in my solutions coincides with Year 9 of the *Australian Curriculum: Mathematics* (ACARA, 2016). My first solution, while not totally correct, involves a straightforward application of similar triangles and an answer that is not far off my improved solution. Some teachers may regard this solution as being adequate enough, although its shortcomings should be discussed.

The mathematical techniques in the second, better solution are the same as the first—since similarity is still involved—but the level of visualisation required is more challenging. With care, and perhaps the building of a physical model, this should still be accessible to upper secondary students too, but may require more guidance than the primitive solution.

It is strongly recommended that the problem be presented to students as has been done here: show students the image of the plane (or just give them the coordinates) and challenge them to find and justify an estimate for the plane's altitude. Working in small groups may be a productive approach. The notion of the "zone of confusion" (e.g., Roche & Clarke, 2015) is a useful one, in that teachers can sanction the importance of facing and dealing with uncertainty when students have a mathematical challenge, and of being prepared to wrestle with a problem to figure out what is required.

After allowing students to explore the imagery, it may be useful to brainstorm ideas for tackling the problem, reassuring students that they have the skills and the information to solve it. Although prompting may by appropriate, this should not be done in haste; students can learn much from unsuccessful ideas and dead ends. Nevertheless, some suitable prompts might include "What measurements can you make?", "Would looking for shadows help?", "Are there things with heights that can be measured?". Finally, students should be encouraged to prepare a formal reasoned solution to the problem, since effective mathematical communication is a vital proficiency.

Conclusion

There is something satisfying about seeing how mathematics can solve a mystery that may appear, at first sight, to be intractable. Students need the opportunity to attempt such problems, and to see not only that mathematics offers powerful tools for solving problems, but that the very process of learning to problem-solve is worthwhile as well.

Author's note about plane spotting

It is strongly recommended that this activity be conducted with Google Maps itself rather than my screen captures. Unfortunately, however, Google's images will disappear eventually, as Google updates its satellite data and Street View imagery. If the Melbourne plane disappears you will need to find another one (as I had to, when the Adelaide plane disappeared). To do this, you will need to visit a busy airport on Google maps (Sydney or Melbourne make good candidates for Australia but may not always be useful; the busier Heathrow and Los Angeles airports have also provided me with successful finds, but are not Australian). Turn on the satellite view, and look at the airport apron to see the size of typical planes. You may need to zoom in so that the planes are just big enough for you to be confident you will be able to spot the same shape and colour while looking out over paddocks and suburbs, but not so big that you will have to spend ages following back along a flight path. Keep the image of one of these planes in your mind. You should also check to see if it is a sunny day in the imagery of the nearby areas, since my suggested solution for the "work out the altitude" problem depends on shadows.

With the target plane image in your mind, look at the runways of your chosen airport, and imagine extending them out across the surrounding countryside. These imaginary lines will be the flight paths for incoming and outgoing planes. The tedious bit involves scrolling the satellite image along one of these flight paths, watching out for a plane image as you go. I have not thought of an easy way of ensuring that you remain on the flight path, but you get better at it with practice. Track out for 10-15km, and then try again with an opposite or different runway direction if unsuccessful in your search.

When (if!) you find a plane, you will need to locate its shadow. You may need to zoom in to find some objects that give you shadow direction, and then use this knowledge to search for the plane's shadow. You also need to ensure that there are some nearby objects—such as electricity, telegraph, or light poles—which are casting shadows, and for which there is enough information to allow you to estimate their heights and shadow lengths.

Finally, in doing all of this, you need to make sure that all your shadow measurements are made from images that were taken at the same time (shadows shorten and lengthen through the day, remember!). As mentioned in the article, Google Maps will sometimes change the set of images that it uses as you zoom in, which results in inconsistent shadows. To work out the heights of the reference objects (e.g., electricity poles), you shouldn't need to worry about the time of day or shadows, so Street View imagery is fine, even though it will have been taken on another day to the satellite imagery.

Acknowledgements: The use of images from the Google Maps website maps.google.com.au (captured in April 2016) is permitted by Google's license to publicly display content with proper attribution in print, as found at https://www.google.com/intl/en_au/help/terms_maps.html and https://www.google.com/intl/ALL/permissions/geoguidelines.html.

References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2016). *The Australian Curriculum: Mathematics.* Retrieved 6 September 2016 from http://www.australiancurriculum.edu.au/mathematics/curriculum/f-10?layout=1

Boaler, J. (2009). *What's math got to do with it? How parents and teachers can help their children learn to love their least favorite subject.* New York: Penguin.

Roche, A., & Clarke, D. (2015). Describing the nature and effect of teacher interactions with students during seat work on challenging tasks. In M. Marshman, V. Geiger, & A. Bennison (Eds.). *Mathematics education in the margins* (Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia, pp. 532–539). Sunshine Coast: MERGA.

Sullivan, P. (2011). *Teaching mathematics: Using research-informed strategies.* Camberwell, VIC: ACER Press.