Counter conjectures:

Using manipulatives to scaffold the development of number sense and algebra

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How the use of concrete materials in the form of counters allows students to model increasingly complex number problems is explored. The use of counters scaffolds students' development of number sense and algebra skills leading to an understanding of quite abstract concepts.

Introduction

This article takes the position that teachers can use simple manipulative materials to model relatively complex situations and in doing so scaffold the development of students' number sense and early algebra skills. According to McIntosh, Reys, and Reys (1997), number sense refers to an intuitive feel for numbers and their various uses. While students' early experiences are usually dominated by the cardinal aspect of number (i.e., counting the number of items in a set), over time students broaden their understanding to include increasingly abstract concepts, such as negative numbers.

Numbers are also used "in the ordinal sense, as labels for putting things in order" (Haylock & Manning, 2014, p. 68). Once students have established the ordering of numbers they can begin to develop the skill of locating numbers on a number line (Anghileri, 2007). As children become comfortable with the ordering of numbers they can begin to talk about their positions in relation to each other (Anghileri, 2014). Number lines feature prominently in the *Australian Curriculum: Mathematics* (ACARA, 2015) and have a variety of applications including graphing, estimation, probability and statistics. In Year 1, students use number lines to locate numbers "up to at least 100" (ACMNA013). By Year 4, students are using number lines to locate and represent "quarters, halves and thirds, including with mixed numerals" (ACMNA078). In Year 5, students "Compare and order common unit fractions and locate and represent them on a number line" (ACMNA102), and in Year 6, students "Investigate everyday situations that use integers… and represent them on a number line" (ACMNA124).

The number line provides a flexible representation that allows students to extend their work to incorporate negative numbers, which occur in a range of conventional contexts including temperatures, banking and elevation below sea level. While there is evidence to suggest that "Quite young children can grasp the idea of the temperature falling below zero" (Haylock & Manning, 2014, p. 197), many international curricula introduce negative numbers earlier than in Australia (Siemon et al., 2011). According to Strogatz (2012), "Subtraction forces us to expand our conception of what numbers are… you can't see negative 4 cookies and you certainly can't eat them—but you can think about them" (p. 15). While "complexities arise with the notion that some numbers are 'smaller than nothing'" (Anghileri, 2007, p. 25), the number line provides a "straightforward image for us to associate with positive and negative integers" (Haylock & Manning, 2014, p. 197).

Here it is argued that manipulatives such as counters may be used as a means of stimulating thinking about abstract mathematical concepts, such as the relative position of consecutive numbers and variable quantities on the number line (e.g., n and $n + 1$). This appears to contrast with the way in which manipulatives have been used, with teachers tending to abandon them as mathematics becomes more complex (Swan & Marshall, 2010). While students can also use the counter representation described here to explore a range of simple number patterns, Swan and Marshall (2010) caution that "simply placing one's hands on the manipulative materials will not magically impart mathematical understanding. Without the appropriate discussion and teaching… children may end up with mathematical misconceptions" (p. 19).

The activities described here provide a context within which students can investigate number patterns and formulate and test conjectures.

According to Haylock and Manning (2014) "the experience of conjecturing and checking is fundamental to reasoning mathematically" (p. 39). Students can be encouraged to refine their conjectures as they work towards establishing a general result. In this way, ideas that emerge from working with manipulatives "give way to more concise, symbolic arguments that will eventually develop into algebra… and carry meaning independently of the investigations from which they were established" (Booker, Bond, Sparrow, & Swan, 2014, p. 93).

Number line representation

Counters are a familiar resource for students learning about counting numbers. While students frequently use counters to explore processes such as counting on, here it is argued that the use of physical manipulatives can greatly facilitate the transition from concrete and more abstract representations, such as the number line. The number line representation can be used to develop the concept that counting numbers continue indefinitely and that it is possible to count on from any starting point.

Figure 1 shows how students' understanding of the counting numbers can be extended from concrete to more abstract examples involving symbols, numerals and counters on the number line. The convention I have followed here is to use a black counter to represent the unknown quantity (i.e., n) and white counters to represent units. Larger numbers, such as $(n + 2)$, may be represented by adding the appropriate number of white counters.

A thermometer is a perhaps the most familiar physical analogue for a number line that allows students to explore negative numbers. The Celsius scale allows a meaningful interpretation to be applied to temperatures such as –5˚C, which can be understood in relation to their positive counterparts. Nevertheless, such as representation is not without its limitations. Here I have avoided referring to –5˚C as the opposite of 5˚C in deference to the absolute temperature scale. It is also impossible to replicate a number line which extends indefinitely in either direction.

Research suggests that an overemphasis on the cardinal aspect can lead to problems with students acquiring an appropriate concept of number" (Haylock & Manning, 2014). Student understanding needs to be extended beyond the point at which physical representations break down. For example, while the opposite of a $20 note has no direct physical analogue, the concept of credit remains a useful one with direct application in the real world. In the abstract sense, '– 20' can be seen as a point on the number line that is twenty units to the left of zero. In this context, '– $20' can be meaningfully interpreted as the amount needed to offset a $20 deposit.

The use of two-colour counters allows us to extend the model so that students can count both forwards and backwards from any point on the number line (see Van De Walle, 2007, pp. 500–501). For consistency, we continue to use the white side of the counters to represent units, while the red side is used to represent the opposite of a unit (i.e., –1). This allows us to represent numbers in the abstract sense such as n and $(n – 1)$. Either a white or a red counter can be turned over to create its opposite, with a white counter and a red counter together having a net value of zero (see Figure 2).

Here we note a small but significant parallel between the use of counters and Roman numerals. Exploring the Roman numerals reveals a useful pattern in the representations used for consecutive numbers such as IV, V and VI. Notice that IV represents a quantity that is one less than five or one before five (i.e., 5 – 1) while VI represents one more than five (i.e., 5 + 1). The same pattern exists for other Roman numerals (e.g., IX, X and XI) and can be extended to any set of consecutive numbers such as $(n – 1)$, n and $(n + 1)$.

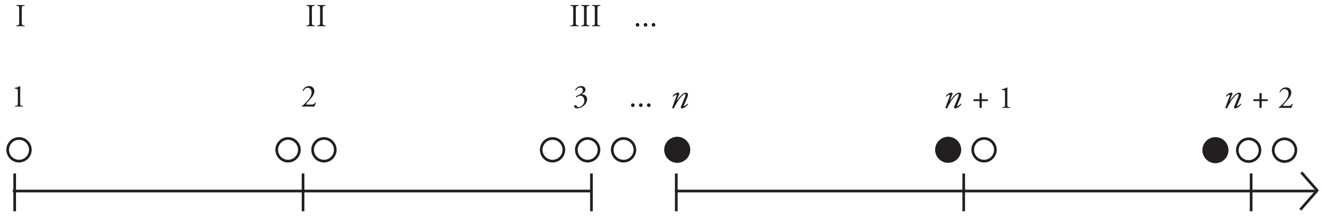

Figure 1. Using symbols, numerals and counters to represent the counting numbers on the number line.

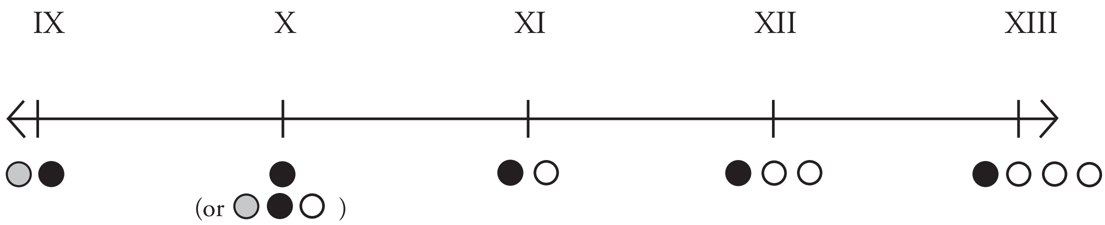

Figure 2. Roman numerals and counters showing the relative position of consecutive numbers on the number line.

Number conjectures

Students can also use counters to explore a range of simple number conjectures in the classroom context. Clearly it is essential to ensure that students have sufficient opportunities to work with concrete examples before exploring number patterns in a more abstract sense. This provides students with time to develop and formulate conjectures, a skill which is essential in mathematical reasoning.

Consecutive number sums

A handful of examples (such as $1 + 2 = 3$ and $6 + 7 = 13$) are usually all that is needed for most students to conjecture that the sum of two consecutive numbers is odd. Indeed, many students are willing to accept at face value that this pattern will continue indefinitely. The use of prompts such as "Can you show this is true for 127 and 128?" and "Why?" may be needed to restore an appropriate degree of scepticism to the classroom.

Having established an abstract representation for consecutive numbers, it is a simple matter to show that the sum of any two consecutive numbers is odd. Using a black counter to represent n and one black and one white counter to represent $(n + 1)$, we see that the sum (i.e., two black counters and one white counter) is clearly not divisible by 2 and therefore odd. Moreover, this remains true regardless of the value of n. This can be verified by substituting any set number of white counters for each black counter (e.g., $n = 2$). This process has been illustrated in Figure 3.

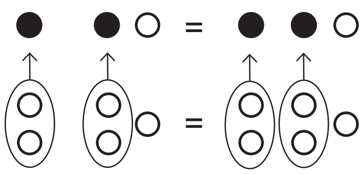

Figure 3.The sum of two consecutive numbers is not divisible by 2, shown here for the general case and $n = 2$.

The algebraic representation provides a concise way of describing the sum of two consecutive numbers (i.e., $2n + 1$). Students may, however, reason that the sum is one more than twice the smaller number or one less than twice the larger number. Alternatively, students may reason that consecutive numbers must consist of an odd and an even number and therefore that the sum is necessarily odd. Such reasoning is equally valid and should be encouraged.

Using the algebraic representation also allows us to consider other potentially interesting cases. What would happen, for instance, if n represented a rational number such as 3.5? In this case, $2n + 1 = 8$, which is clearly even, but n and $(n + 1)$ no longer represent consecutive numbers.

Arithmetic mean

Counters may also be used to prove some simple conjectures about the arithmetic mean. For example, it may be readily observed that $(n + 1)$ is the arithmetic mean of n, $(n + 1)$ and $(n + 2)$. I have deliberately avoided referring to n, $(n + 1)$ and $(n + 2)$ as consecutive numbers, since there is no requirement for n to be an integer. Indeed the result holds regardless of the value of n.

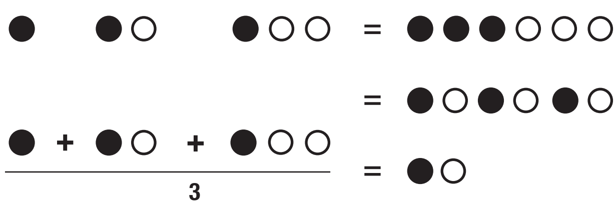

Figure 4. Using counters to find the arithmetic mean of n, $(n + 1)$ and $(n + 2)$.

Similar results hold for other sets of numbers that are evenly spaced on the number line. The counter representation can also be used to establish, for example, that $(n + 2)$ is the arithmetic mean of n, $(n + 2)$ and $(n + 4)$. Students can explore similar conjectures with larger or smaller sets of numbers.

For instance, it can readily be observed that the arithmetic mean of n and $(n + 2)$ is $(n + 1)$. It is also apparent that the mean of four consecutive numbers is not an integer, since $n + (n + 1) + (n + 2) + (n + 3) = 4n + 6$ (represented by four black and six white counters) is clearly not a multiple of four.

Odd and even numbers

Counters also provide an effective way of exploring the patterns inherent in odd and even numbers. While many students are inclined to begin their exploration of odd and even numbers with statements such as 'Let n be an even number', it is more advantageous to create a representation in which n may represent any counting number. Multiplying any counting number by two necessarily produces an even number, and this fact can be exploited to represent any even number as $2n$. Since odd and even numbers alternate, $2n$ is immediately followed by $2n + 1$ (which is odd) and $2n + 2$ (which is even).

Using this representation, it is clear that the sum of two consecutive even numbers (e.g., $2n$ and $2n + 2$) is even (i.e., $4n + 2$). It can also be shown that this result extends to the sum of three (or more) consecutive even numbers, since $2n + (2n + 2) + (2n + 4) = 6n + 6$. While it is also true that the sum of two consecutive odd numbers is even, since $(2n + 1) + (2n + 3) = 4n + 4$, the sum of three consecutive odd numbers is not.

It is also possible to establish that the arithmetic mean of two consecutive odd numbers is even. Using $2n + 1$ and $2n + 3$ to represent consecutive odd numbers gives a sum of $4n + 4$, which is clearly divisible by 2. In contrast, the mean of two consecutive even numbers is odd, since the mean of $2n$ and $(2n + 2)$ is $2n + 1$. The mean of three consecutive even numbers is even, as is apparent in Figure 5.

Think of a number problem

Counters can also be used to model a variety of number puzzles (see Booker et al., 2014, p. 391; Reys et al., 2012, p. 363). Table 1 illustrates how a word problem can be represented verbally, using counters, symbols and algebra. Students may begin with a concrete example such as $n = 2$. Doubling and adding three produces seven. Adding the number that we started with (i.e., 2) gives nine. Finally, dividing by three gives us an answer of three. A second example (such as $n = 5$) produces a result of six, suggesting that the number problem appears to produce an answer that is one more than the number that we started with.

The general result can be established by using a black counter to represent n. Doubling produces two black counters (i.e., $2n$) and adding three gives $2n + 3$, which is represented by two black and three white counters. Adding the number we started with (i.e., a black counter) gives $3n + 3$, which is represented by three black and three white counters. Finally, dividing into three groups leaves one black and one white counter in each group (i.e., $n + 1$). Thus the counter representation (and the associated algebraic reasoning) allows us to confirm that the number problem always produces a number that is one more than the number we started with.

Conclusion

The counter representation described here allows students to model increasingly complex number problems and conjectures using concrete materials. The approach was trialled with approximately 550 pre-service primary teachers in their final mathematics education unit. The feedback received from students revealed that very few had previously encountered this approach.

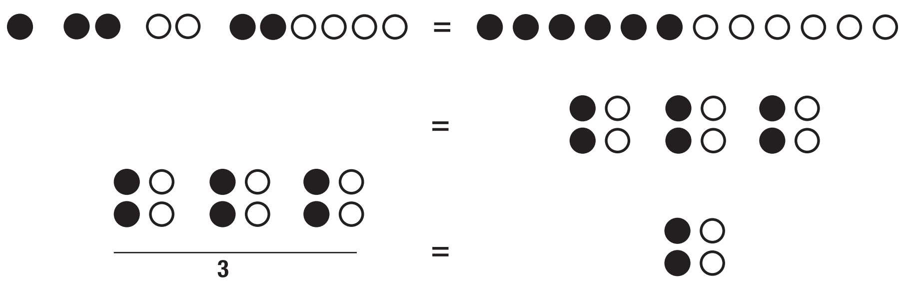

Figure 5. Using counters to show that the mean of three consecutive even numbers is even.

Table 1.Verbal, physical, symbolic and algebraic representations of a word problem.

Verbal representation	Physical representation	Symbolic representation	Algebraic representation
Choose a number	●	Δ	n
Double it	● ●	$2 \times \Delta$	$2n$
Add 3	● ● ○ ○ ○	$2 \times \Delta + 3$	$2n + 3$
Add the number you choose	● ● ● ○ ○ ○	$3 \times \Delta + 3$	$3n + 3$
Divide by 3	● ○	$\Delta + 1$	$n + 1$

While some students initially found the counter representation confronting, it may be that this was due to the perceived novelty of the approach. With appropriate support and guidance, it took approximately 20 minutes before most students were reasonably proficient with this approach. Students then quickly began to pass on their skills to other members of their group.

Of particular interest was that often students who appeared to have the strongest grasp of algebra found it most difficult to understand an alternate representation. Conversely, a number of students who self-report difficulty with algebra experienced significant 'A-ha' moments. Further research is required to explore the potential benefits of such an approach.

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