Laying the foundation for multiplicative thinking in Year 2

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In order for students to move from using concrete materials to using mental strategies and from additive to multiplicative thinking, the use of arrays and visualisation is pivotal. This article describes a lesson in which students are taken through a CRA approach that involves noticing structure, using strategies and visualisation.

Within the first two years of school, many children are able to combine models of numbers where all items are represented with skip counting to find solutions to simple multiplication problems (Sullivan, Clarke, Cheeseman & Mulligan, 2001). However, moving children beyond recognising the group structure to abstract thinking can present challenges for both students and teachers alike. This has become a focus for instruction in my Year 2 mathematics classroom as I aim to provide the types of experiences, models, strategies and discourse that assist students' development of multiplicative thinking.

Multiplicative thinking is the ability to see the individual items of a collection as a composite unit (Sullivan, et. al, 2001: Clark & Kamii, 1996). For example, four items are seen as one four as opposed to four ones. In this way it is distinctly different from repeated addition as it "involves the formation of two kinds of relationships not required in addition: … an ability to think simultaneously about units of one and about units of more than one" (Clark & Kamii, 1996. p. 43). Furthermore, multiplicative thinking involves an understanding of and ability to apply the distributive property of numbers in order to solve problems. For example, 4×23 can be thought of as $(4 \times 20) + (4 \times 3)$.

Downton (2010) provides teachers with a hierarchy of strategies that children draw upon to solve multiplication problems (see Figure 1). The first three strategies are considered additive thinking. The final two strategies are considered to be multiplicative strategies as they reflect both an understanding of the composite unit and distributive property.

In assisting students to move from the concrete to the abstract, and from using additive to multiplicative thinking, the use of arrays and visualisation is recommended (Downton, 2008; Young-Loveridge, 2005). In making this transition, students first need to be able

Table 1. Solution strategies for whole number multiplication problems.

Strategy	Definition
Transitional counting	Visualises the groups and can record or verbalise the multiplication fact but calculates the answer using a counting sequence based on multiples of a factor in the problem.
Building up	Visualises the groups and the multiplication fact but relies on skip counting, or a combination of skip counting and doubling to calculate an answer.
Doubling and halving	Derives solution using doubling or halving and estimation, attending to both the multiplier and the multiplicand. For example, "4 times as many as 18. Double 18 is 2 times, double 36 is 4 times, so 72 stamps".
Multiplicative calculation	Automatically recalls known multiplication facts, or derives easily known multiplication facts.
Wholistic thinking	Treats the numbers as wholes—partitions numbers using distributive property, chunking, and/or use of estimation.

to see, then be able to visualise the various structures within an array. They then need to develop the ability to partition the array in order to calculate the total quantity represented: "Division and multiplication should be derived from equipartitioning/splitting, and coordinated with, not derived from counting, addition and subtraction." (Confrey, 2011). Gervasoni (2013) extends this idea by asserting that not only is it important for children to be able to visualise and describe arrays, but to also be able to manipulate these mental images.

Furthermore, classroom mathematics must provide students with opportunities to engage in higher order thinking. Findings by Downton (2010) suggest that "some students use more sophisticated strategies when presented with challenging problems, involving numbers considered beyond the factor structure determined by the curriculum for that particular grade" (p.169).

How might these ideas impact upon the thinking in a Year 2 classroom?

During a six week period, my Year 2 students explored a range of multiplication and division problems. The tasks incorporated a variety of question structures, allowed students to use a range of mathematical language and to produce models that demonstrated their understandings and solution strategies. Many of the tasks were open-ended questions that allowed for multiple correct answers and for students to use a variety of strategies. The following examples are some of the students' responses to one task in the fifth week.

> The Task: I had some swap cards. My friend had 4 times as many as me. How many cards might we each have had?

Elise built up her collection of possible answers by beginning with two cards, then adding a card to create each new possibility thereafter. In doing so, she did not need to use counters, but could draw her new array by repeating her previously drawn model and adding one more to each row. This also provided a structure for her to quantify each new collection efficiently by subitising and using calculative strategies. Having seen 20 (an array of 4 rows of 5) as 2 tens frames in her 5th solution, Elise partitioned her 6th array (Figure 1) into two smaller arrays; 4 rows of 5 and 4 rows of 1. (Distributive property: 24 = (4 × 5) + (4 × 1)). In this way, she could instantly recognise 20, and knowing 4 more, she was able to calculate the total efficiently.

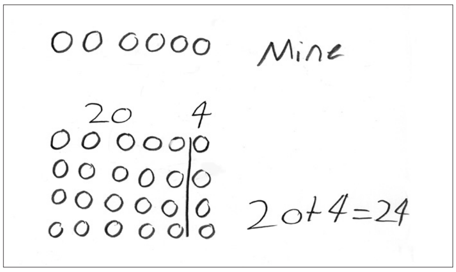

Figure 1. Elise's 6th solution.

When asked to explain how she worked out the total of her 8th solution (Figure 2), Elise stated that she "knew that (4 × 5) was 20, and this [pointing to the 4 × 3 section on the right] was 6 and 6 which is 12." Elise instantly recognised two smaller arrays of 2 rows of 3, and doubled these, thus employing the halving strategy. "Then 12 is made up of 10 and 2 so 20 plus 10 is 30 and plus 2 more is 32."

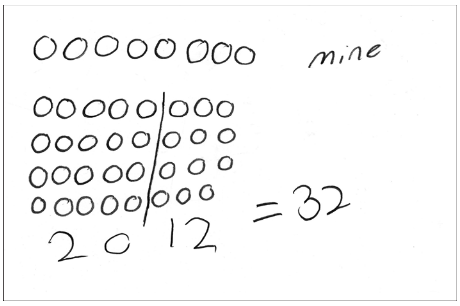

Figure 2. Elise's 8th solution.

As Elise continued to add one new card to her collection with each solution, her friend's collection of four times as many was becoming larger and increasingly cumbersome, and time consuming, to draw. By increasing the number of items with which a child has to deal, they are being pushed to find simpler, more efficient ways of representing (Downton, 2010).

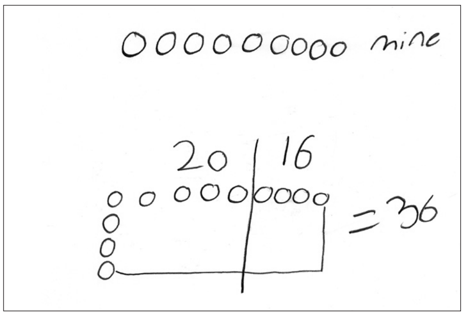

Figure 3. Elise's 9th solution.

By her 9th solution (Figure 3), Elise partially represented the array, including circles for only the first row of the array and for only the first item in the remaining three rows. She then drew two lines to show where the outline of remaining sections of the array would be. In class we refer to this as "using invisible counters". This assists children to develop mental images of multiplicative situations (Gervasoni, 2013). I pointed to the space in the middle of the array.

Mrs W: What do you want me to imagine is here?
Elise: There's 4 rows of 9.
[I pointed to the smaller part labelled 16.]
Mrs W: How do you know there are 16 in here?
Elise: 4 and 4 is 8, and double 8 is 16.

Elise's visual strategy and explanation can be summarised as 4 × 9 = (4 × 5) + (4 × 4). By the end of the lesson, she was able to create mental images of the multiplicative situation. Her drawn models began by drawing all items, but progressed to drawing a partial representation. Although for most of her solutions all items were drawn (direct modelling), she was able to use knowledge of how numbers are structured to assist her to quantify each collection using higher order strategies. These included doubling and halving (strategy for multiplication), knowing 10 more (strategy for addition) and the flexible use of partitioning (place value understandings).

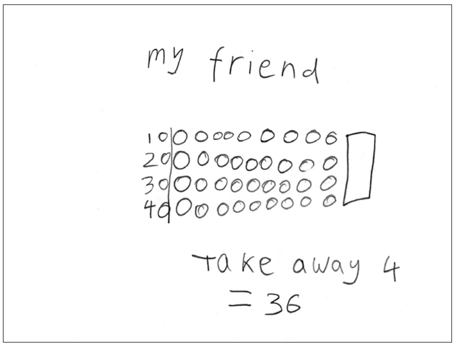

Figure 4. Pippa's 3rd solution.

Like Elise, Pippa also sought to quantify 4 rows of 9, however she drew all items in the collection (Figure 4). Unlike Elise, she did not partition her array but visualised an additional item in each row so that she could skip count by 10s. This was a strategy shared by Lara on the previous day, and is considered a sophisticated image due to its' dynamic properties (moving or changing) (Mulligan, 2002).

Mrs W: Why have you drawn a rectangle on the end of your array?
Pippa: There's 4 counters in there. It's a row, no, a column of 4. They're invisible, so you can't see them.
Mrs W: Why have you got invisible counters there?
Pippa: Because it's easy to count by tens then I can just take them away because they're not really there.
Mrs W: What is 40 take away 4?
Pippa: [confidently without hesitation] 36 because 6 + 4 = 10 so take away 4 will be 36 —tens facts!

Pippa's visual strategy and explanation can be summarised as 4 × 9 = (4 × 10) – 4. Although she directly modelled all items in the collection and calculated the total by skip counting, both of which are counting strategies, she was able to mentally manipulate this image to skip count efficiently. In doing so, Pippa demonstrated an understanding of the composite unit (1 group of 4 rather than 4 ones) and the beginning of more abstract strategies.

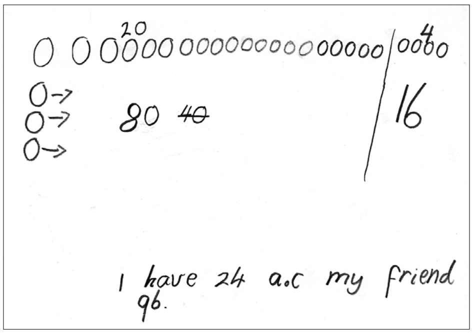

Figure 5. Macy's 2nd solution.

Macy combined a partially modelled array with doubling strategies to find the total of 24 × 4 (Figure 5). She split 24 into 20 + 4, and then multiplied each part before recombining them.

Macy: I wanted to work out 4 × 20. I know double 20 is 40, and that's the same as 2 times, so if I double that again, it will make 4 times which is 80. Then I know that 4 × 4 = 16, and if you add them together, that's 96.

Macy's visual strategy and explanation can be summarised as 4 × 24 = (4 × 20) + (4 × 4).

At the conclusion of the lesson selected students explained their strategies to the class. A list (Table 2) was then created showing some of the possible answers, along with the responses of students who were working on the more challenging, 8 times as many, variation of the question. Students were encouraged to make generalizations through discussion prompted by the questions: "What do you notice about all the responses? What reason can you think of as to why this has happened?"

Supported by the organisation of the responses on the whiteboard, students were quick to notice a skip counting by fours pattern. When asked why this might be, they were quick to relate this back to the context of the problem, articulating that, if their friend always had four times as many as them, then you could always count by fours if you could not work it out a quicker way. They were also quick to note that all the responses in both columns were even numbers. A lively discussion then ensued about even and odd numbers and the consequence of multiplying various combinations of these.

Oscar: That's because if you times two even numbers you get an even number.
Archie: But I had one where there were 7 rows and it was an even answer.
Zac: So did I, so that can't be right. [pause]
Mrs W: What combinations of numbers did you have that resulted in an even numbered answer?

After some further sharing of examples where both the multiplier and the multiplicand were both even numbers and where the multiplier was even and the multiplicand was odd, the students concluded that an even numbered answer would be found if an even number was a factor in the situation. This then gave rise to the question of why?

Macy: What would happen if both of the numbers were odd? [pause]
Oscar: You would get an odd answer because 3 × 5 = 15.
Levi: And 3 × 3 = 9.
Mrs W: Is this true of every example?

This question revealed some uncertainty in their minds and created a long silent pause in the conversation. After a time, I asked Oscar what he thought.

Oscar: I don't know. Can I try some other numbers to see?
Macy: Can we try that tomorrow?

This became the focus of the investigation for the next day.

Although further thoughts were shared by some students, coming up with a concise explanation was not forthcoming, and was noted as a discussion to come back to in the future. By engaging in such discourse, the opportunity to pose further questions was also encouraged.

Table 2. Possible solutions found by the children during the lesson.

Possible number of cards my friend has if he has:	
4 times as many as me	8 times as many as me
4	
8	
12	
16	16
20	
24	24
28	
32	
26	
40	40
44	
	48
56	56
68	
	108
	1056

Concluding thoughts

Over the six week period, three particular aspects of the experiences, an understanding of strategies, structure, and visualisation became important threads to the overall development of students' ability to think in more sophisticated ways. During the course of this lesson, students were given the opportunity to explore the problem using concrete materials, create models or partial models, represent these as drawings and utilise these to calculate the total of their collection using more sophisticated strategies than counting alone. In doing so, they were encouraged to attempt strategies that other children had shared on previous occasions, explain their strategies to others, and reflect upon the efficiency of alternative strategies used within the class.

Understanding the range and hierarchy of strategies that children bring to multiplicative word problems allowed me to notice the type of thinking that was occurring and to frame this within the context of Downton's (2010) strategy hierarchy. I was then able to ask explicit questions that scaffolded students' thinking and encouraged deeper understandings by challenging them to explain and reason their answers and conclusions. In order to apply and discuss these strategies, they needed to attend to the array structure of the problems. More than simply viewing the array as a collection of items organised into rows and columns, they were challenged to seek useful structures within the array. That is, attend to the distributive property by carefully considering how the array might be partitioned in order to calculate the partial products efficiently. The combination of these three aspects (strategies, structure, visualisation) greatly enhanced student thinking that evolved from each problem they engaged with; firmly laying a foundation upon which more sophisticated multiplicative thinking can be scaffolded.

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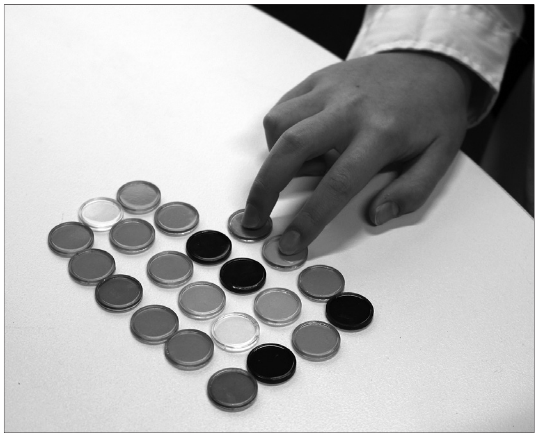

Figure 6. Modelling all items in the array.

Figue 7. Partitioning an array.

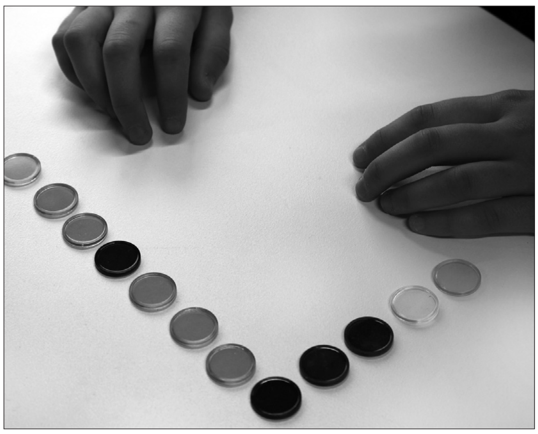

Figure 8. Partially modelling an array.