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THE WINNING NUMBER: AN HEURISTIC APPROACH WITH THE GEOGEBRA´S HELP

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Abstract: Admittedly, the study of Complex Analysis (CA) requires of the student considerable mental effort characterized by the mobilization of a related thought to the complex mathematical concepts. Thus, with the aid of the dynamic system Geogebra, we discuss in this paper a particular concept in CA. In fact, the notion of winding number $\nu[f(\Gamma), P] = n$ has a profound topological and geometrical meaning. Moreover, we record the effort of some authors in order to emphasize its intuitive description (Needham, 2000). We will show some basics commands that allow too the exploration of geometric and topological ideas related to the Rouché´s theorem and the Fundamental Theorem of Algebra (TFA) (Alves, 2014).

Key words: The winding number, Geogebra, Rouché´ theorem, Visualization.

1. Introduction

Based on an intuitive and heuristic discussion, Needham (2000, p. 353) guides the reader to imagine "walking a dog round and round a tree in a park." The closed path is developed around a tree. Needham consider still a leashes of variable length, "similar to a spring-loaded tape measure". When we keep the leash short, the dog stays closer of your hell. Furthermore, "it is then clear that the dog is forced to walk round the tree the same number of times that you do. On another walk, thought, you decide to let out the leash somewhat so that the dog may scamper about, perhaps even running circles around you. Nevertheless, provided that you keep the leash short enough so that the dog cannot reach the tree, then again the dog must circle the tree the same number of times as you".

This metaphorical and intuitive description promotes the understanding of the classical formalism related to the Rouché´s theorem that we can find in an extensive literature (Cecília & Bernadez, 2008; Conway, 1978; Flanigan, 1972; Gong, 2001; Mitrinovic, 1966; Shokranian, 2011; Soares, 2014). However, Needham (2000, p. 353-354) uses the Rouché´s theorem for to demonstrate the Fundamental Theorem of Algebra (FTA), accepted om intuitive grounds in the seventeenth and eighteenth centuries (Kleiner, 2007, p. 10 - 11). We recognize the effort of this author in other to convey the reader about some complex and conceptual ideas related to the topological notions. In this sense, we indicate the Figure 1 below. Here we see that a point r moves around a loop K.

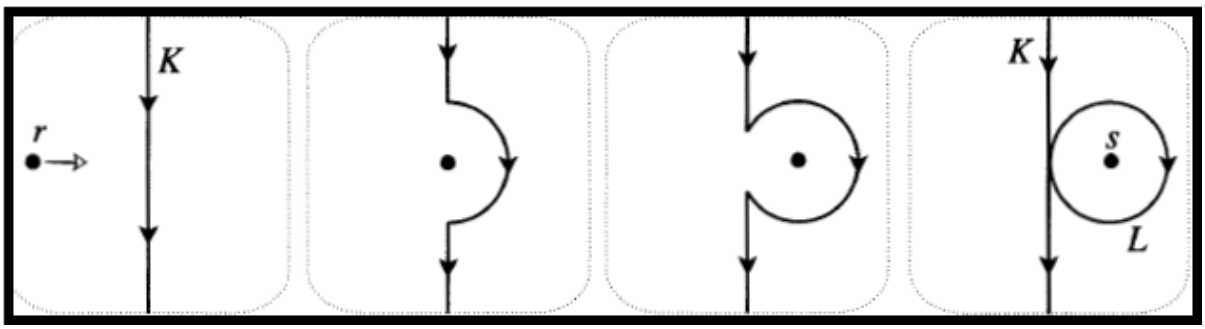

Figure 1. Needham (2000, p. 340) explains a geometric process relatively a moving point and a loop

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On the other hand, we could point others authors that emphasize a geometrical approach (Polya & Latta, 1974; Wegert, 2012). However, in this papper, we will focus the Needham´s perspective. In fact, in the Figure 1, Needham (2000, p. 340) proposes an elegant qualitative method for the visualization of the winding number. From this figures, he establish the "crossing rule" (Figure 2-II). Here, we observed a clear didactical intention of this author in trying to explain, intuitively, an immediate consequence of the idea involving the connection between the number $n = \nu[L, p]$ and the intersections´ points of loop L with a ray emanating from a point p. In the Figure 2, we show two figures proposed still by this author. In the left side we visualize a not simple loop. In fact, we note that the loop L determines a partition of the $C - plane$ relatively to the sets indicated by D_1, D_2, D_3, D_4 (see Figure 2-I). Concluding this preliminar section, we observe the dynamic character and the heuristic bias on these two figures suggested by the author. These elements are a clear indication of the mathematical transmission not limited to algebraic and logical formulations. In the next section, we will indicate some formal mathematical properties in the $C - plane$.

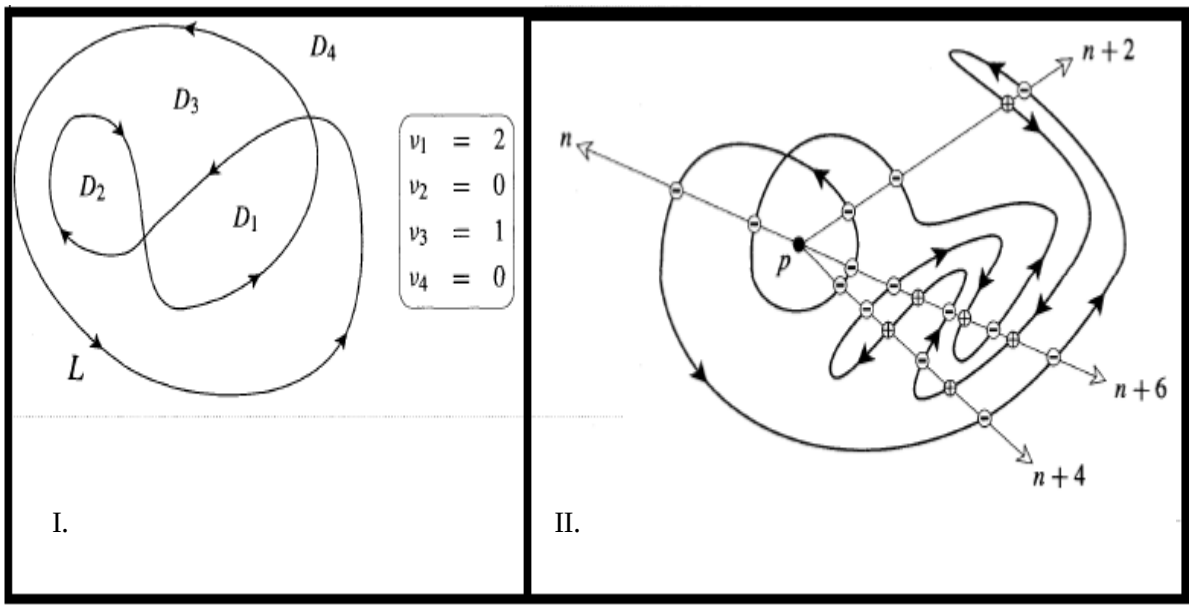

Figure 2. Needham (2000, p. 340-341) describes a heuristic procedure for to calculate the winding number indicated here by $n = \nu[K, p]$ (and the crossing rule)

In the next section, we will bring to the reader some concrete examples of the description for the winning number, but not only through the logical and mathematical ideas and orthodox arguments. On the other hand, we must make some operational considerations about the Rouché´s theorem. Let´s start with some examples!

2. Visual complex Analysis

We consider now the complex polynomial $g(z) = z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1$. When we follow the invariant and formalist style, we get other convenient function $f(z) = 71z^4$ and verify that $|g(z) - f(z)| = |z^{87} + 36z^{57} + z^3 - z + 1| \leq |z|^{87} + 36|z|^{57} + |z|^3 + |z| + 1$. Well, if we choose the condition $|z| = 1$, we will conclude that: $|g(z) - f(z)| \leq |z|^{87} + 36|z|^{57} + |z|^3 + |z| + 1 \leq 40 \leq 71$. So, in the unit circle we have $|g(z) - f(z)| \leq |f(z)|$. By Rouché´s theorem, we can conclude that $g(z) = z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1$ and $f(z) = 71z^4$ has the same numbers of root

inside the unit disk indicated by $D_1(0,0) \subset C$. However, en virtue the TFA, we can still reach the localization of the other $87-4=83$ roots of equation $z^{87}+36z^{57}+71z^4+z^3-z+1=0$.

Indeed, we will chose now $|z|=2$ and another convenient function $f(z)=z^{87}$. Therefore, repeating the same procedure, we get that: $|g(z)-f(z)|=|36z^{57}+71z^4+z^3-z+1| \leq 36\cdot 2^{57}+71\cdot 2^4+2^3+2+1$ $\leq 2^{87}=|f(z)|$. By the TFA again, we know now that the all roots of $g(z)$ are in the region of the $C-plane$, however, we already know that four of the roots are in the unit disk, while the others are in the particular ring $1 \leq |z| \leq 2$. We must be carefull because we do not know its numerical values.

Well, we bring some formal arguments that are naturally opposed to the data discussed in the introduction of this article. Indeed, we indicated a heuristic and metaphorical approach to the Rouché´s theorem in our introduction. However, originating in a formal, inferential and a linear structure, we lose the moment and the opportunity of to stimulate a heuristic reasoning, a tacit intuition and the private perception of the student (Alves, 2012; Alves & Marinho, 2015). Thus, in the following section, we try to show a heuristic and intuitive way for the teaching.

Our methodological approach is supported by the dynamic system Geogebra. From our point of view, when we use the currently technology, we can explore the intuition and the perception of important properties. Moreover, Needham (2000, p. 341) also points out another non-trivial problem that can be studied with this software. Indeed, from the Mathematical Analysis, we know that the Hopf´s theorem can be generalized to any n-dimensional space. In his book, Needham (2000, p. 341) mentioned that "a loop K may be continuously deformed into another loop L, without ever crossing the point p, if only if K and L have the same winding number around p".

We can verify with Geogebra this complex property indicated by this author. On the other hand, we will consider the following problems:

(1) Determine the number of zeros of the equation $z^7-5z^5+z^2-2$ in $|z| \leq 1$;

(2) Determine the number of zeros of the equation z^5+3z^2+6z+1 in $|z| \leq 1$;

(3) Determine the number of zeros of the equation z^4-5z+1 in $1 \leq |z| \leq 2$.

We strongly reject this style of mathematical statement in the teaching context of Complex Analysis. Otherwise, with the aid of the two softwares; we will describe another route of didactic transmission for such content. In this context, we will explore some algebraic operations provided by a Computer Algebraic System (CAS) Maple. Its role is important in virtue that it performs certain algebraic functions which the Geogebra does not manifests the same performance.

We still do not observe in the previous problems any appeal to intuitive reasoning, based in the visualization, for example. Indeed, in these tasks, if the student knows the hypothesis and the results provided by Rouché´s theorem, undoubtedly, it can achieve some result. However, the simple logic implementation of an inferences chain is not always a consequence of a real understanding. In the next section, we try put in evidence the graphic-geometric meaning related to these notions, with the scope to promote an alternative approach.

3. The notion of winning number

Our first example is the function $f(z)=z(z^2+1)(z^2-1))(z^2+4)(z^2-4)$. Quickly, we verify that its degree is nine. So, en virtue the TFA, in the formal point of view, we can state the existence of nine roots. In the Figure 3, we use the command **Curve[<Expression>, <Expression>, <Variable>, <Inicial Value>, <Final Value>]** we describe the two curves below. The first is the

circumference indicate by **Curve[cos(t),sin(t),t,0,2*pi]**. However, we will confront in AC several problems to determine the real and imaginary parts of the function $f(z) = f(x+iy) = \mathrm{Re}(f(z)) + iIm(f(z))$. In this case, the dynamical software GeoGebra actually manifests some limitations (Alves, 2012; 2016).

In the Figure 3, we can promote a geometric meaning related to the winning number. On the left side, we can predict the amount of roots near at the origin, from the dynamic movement in anti-clockwise direction. For this, we indicate the segment $\overline{OQ}$. However, we can observe, on the left side, another segment. In this figure, the point will turns around the origin, an amount equal to the number of the roots inside the curve indicated by $(3\cos(t), 3sen(t))$. On the right side, we consider a parametric plane curve Γ, and the corresponding transformation by the function $f(z)$, designated by $f(\Gamma)$.

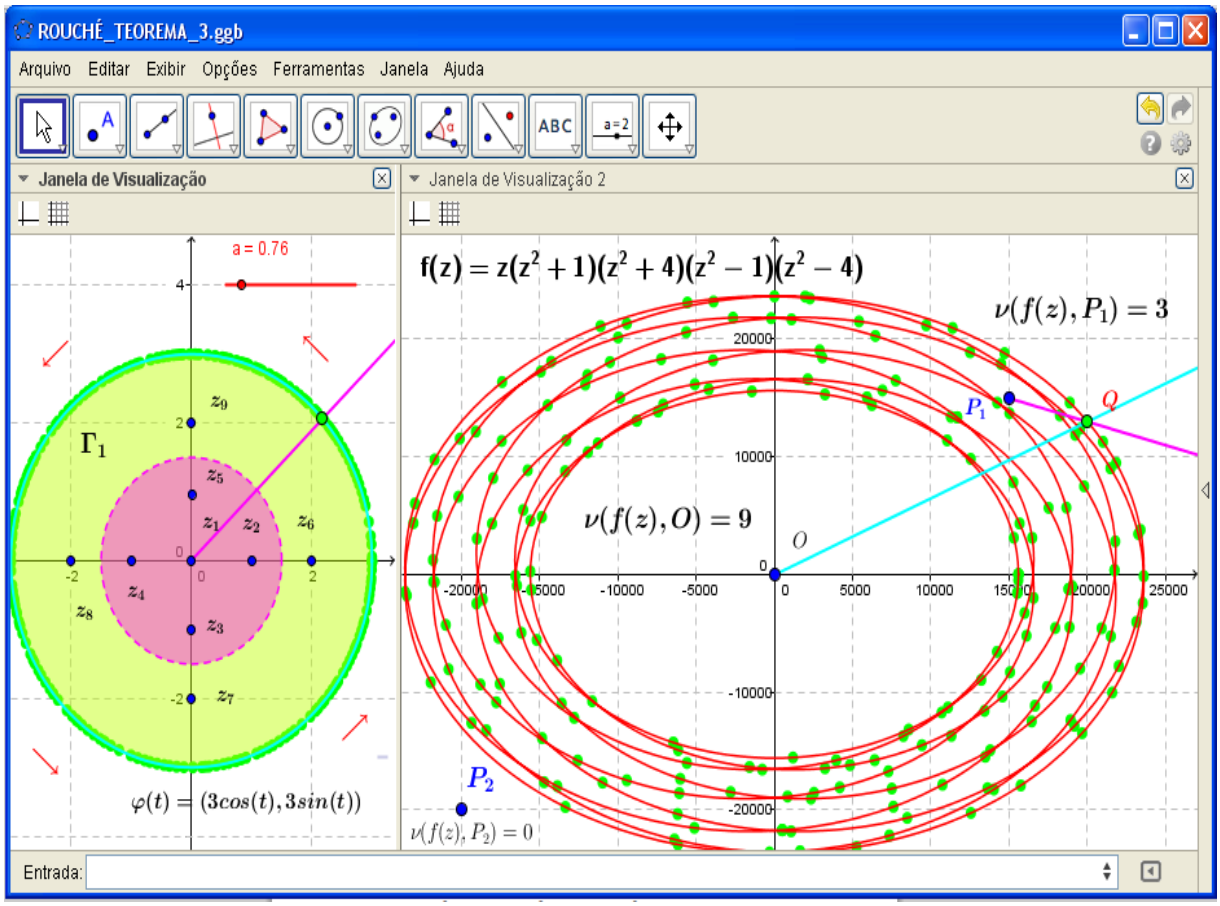

Figure 3. With the Geogebra we describe the set $f(\Gamma)$ in the $C-plane$ that represents the image of the function under certain restrictions indicated by Γ (parametric curve) (Designed by the author)

The advantage of the visual point of view is that we can directly count the winding number related a moving point Q (on the right side) indicated in the Figure 3. We can also check what happens to the winding number when we change the position of the point Q in the $C-plane$. We visually perceive the complexity of the partition determined by the set $f(\Gamma_1)$ unlike the case of the Figure 2-I (on the left side). Let us know consider the first question. In view of the understanding the graphics in the Figure 4, in the plane complex, we will take $f(z) = z^7 - 5z^5 + z^2 - 2$.

By the TFA, we can declare the $f(z) = z^7 - 5z^5 + z^2 - 2 \in C[z]$ has exactly seven roots. Well, in the Figure 4, we indicate two curves (on the right side). Without the technology, we observe in the specialized literature the emphasis in the analytical style (Bottazzini, 1986).

On the contrary, we recall the approach described by Needham (2000, p. 353). In this informal context, we observed a person, a dog and its leash. Mathematically, when Needham declares that "nevertheless, provided that you keep the leash short enough so that the dog cannot reach the tree, then again the dog must circle the tree the same number of times as you" this phrase means that $|g(z)| \leq |f(z)|$. This is a strong hypothesis related to the Rouché´s theorem. For example, in the Figure 4, we described two curves $f(\Gamma)$ and $g(\Gamma)$. The green curve is inside of the purple curve (on the right side).

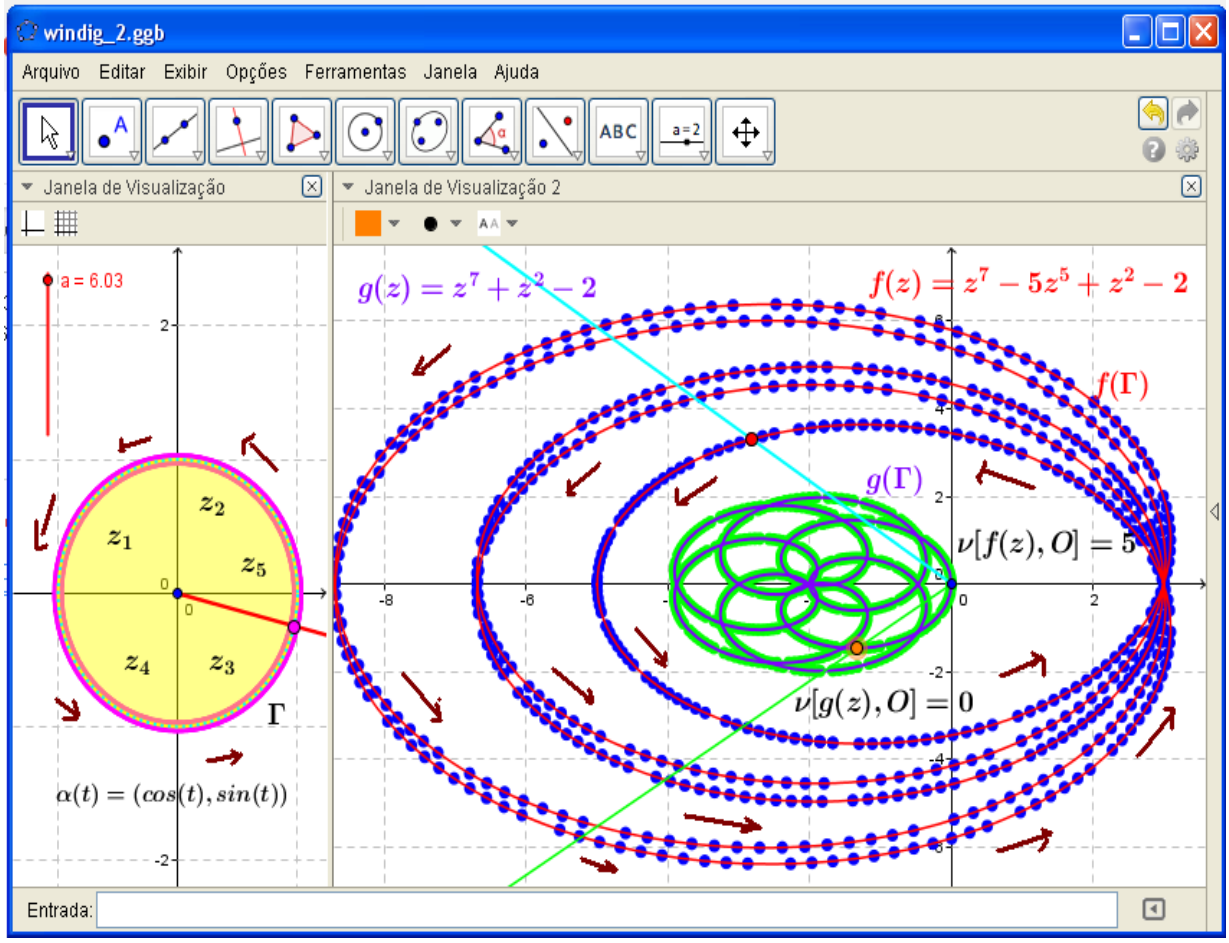

Figure 4. We visualize several elements related to the Rouché´s theorem and a counterclockwise revolution of the moving points under the sets $g(\Gamma)$ and $f(\Gamma)$. (Designed by the author)

The inequality condition is described geometrically by the dynamical software. Indeed, if we take any point in the green curve and compare with a relative distance to the highest curve, we will conclude that $|g(z)| \leq |f(z)|$. However, the functions $f(z) = z^7 - 5z^5 + z^2 - 2$ and $g(z) = z^7 + z^2 - 2$ are not the appropriate choice en virtue the theorem. In any case, the Figure 4, we count the winding number relatively to origin O. We state that $\nu[f(z), O] = 5$ while $\nu[g(z), O] = 0$ according to the condition $|z| \leq 1$. The relevant here is the graphical and geometric description about this numbers.

Moreover, when we compare the number $\nu[f(z), O] = 5$ with the degree of the $f(z)$, we will stimulate the students to reach other neighborhood so that all the roots are analyzed. Thus, we will consider the following parameterized curve $\beta(t) = (2\cos(t), 2\sin(t))$ (on the left side). This last parameterization provides the greater curve that we show in the Figure 4. We advocate a view in order

to promote the exploration of the problem using the software Geogebra. In this way, tacitly we will see graphically and geometrically the hypothesis.

We consider now $f(z) = f(x+iy) = z^5 + 3z^2 + 6z + 1 = \mathrm{Re}(f) + i\,\mathrm{Im}(f)$. With the CAS Maple, we can get the following analytical Cartesian expressions: $\mathrm{Re}(f(x+iy)) = 1 + x^5 - 10x^3y^2 + 5xy^4 + 3x^2 - 3y^2 + 6x$, $\mathrm{Im}(f(x+iy)) = 5x^4y - 10x^2y^3 + y^5 + 6xy + 6y$. On the other hand, we get these expressions en virtue to describe the set $f(\Gamma_i)_{i\in\{1,2\}}$, where we indicate the symbols $\Gamma_1 : \alpha(t) = (\cos(t), \sin(t))$, with $t : 0 \mapsto 2\pi$ and $\Gamma_2 : \alpha(t) = (2\cos(t), 2\sin(t))$. In Figure 5, we observe a small curve inside a larger curve. With the software, we can gradually the disk $|z| = |a \cdot \cos(t) + i \cdot a \cdot \sin(t)| \le r$ and to check the Rouché´s theorem.

Considering these data, we can write the expression $f(x+iy) = \mathrm{Re}(x,y) + i\,\mathrm{Im}(x,y)$. We can verify a complicated analytical expression related to these two curves (on the right side). Analytically, we can get that $|g(z)| = |z^5 + 3z^2| \le |z^5| + 3|z^2| = 4$ and $|f(z)| = |6z+1| \ge 6|z| - 1 - 5 \ge 4 \ge |g(z)|$, we established that $|f(z)| \ge |g(z)|$. From this analytical argument, easily we can determine particularly the root of the equation $f(z) = 6z + 1 = 0$. We indicate the complex number $z_1 = -1/6 + i0 \in \Gamma_1$ in Figure 5. Moreover, from these data, we establish that $\nu[f(\Gamma_1), O] = 1$.

However, we must reach another neighborhood and thus to find the location of the remaining four roots. For this purpose, we take the following functions $f(z) = z^5$ and $g(z) = 3z^2 + 6z + 1$ and finally, we write again $|g(z)| = |3z^2 + 6z + 1| \le 3|z|^2 + 6|z| + 1 = 3 \cdot 4 + 12 + 1 = 25 \le |g(z)| = 32 = |z|^5$, en virtue the condition $|z| = 2$ (Γ_2). Finally, we get a neighborhood in which there are all the roots. In Figure 5 we can visualize some possibilities for its location in the $C - plane$.

We still have another example in which we identify an intuitive notion related to what we call the homotopy paths in the complex plane. With the help of the software, we can gradually make some variations in $1 \le a < 2$. In this case, we see the following behavior of the visual path (on the right side, Figure 6). On the other hand, when we consider other numerical behavior $0 \le a \le 1$, we will find, according to the formal definition, that the number $\nu[f(z), E] = n$ changes according to the location of the moving point E in the $C - plane$ (Figure 7).

We observe in the specialized literature several versions and descriptions to the Rouché´theorem (Krantz, 2007). We will show a classical example. In fact, we consider $f(z) = \sin(z) - z$. And let us choose functions $f(z)$ and $g(z)$ which have roots in the unit disc and which we know satisfy the hypotheses of the theorem. With the software GeoGebra, we can conjecture the limitations of the hypothesis. In this case, we can get that directly from Figure 8, the meaning about to some properties conditioned by the norm function $|\ |$.

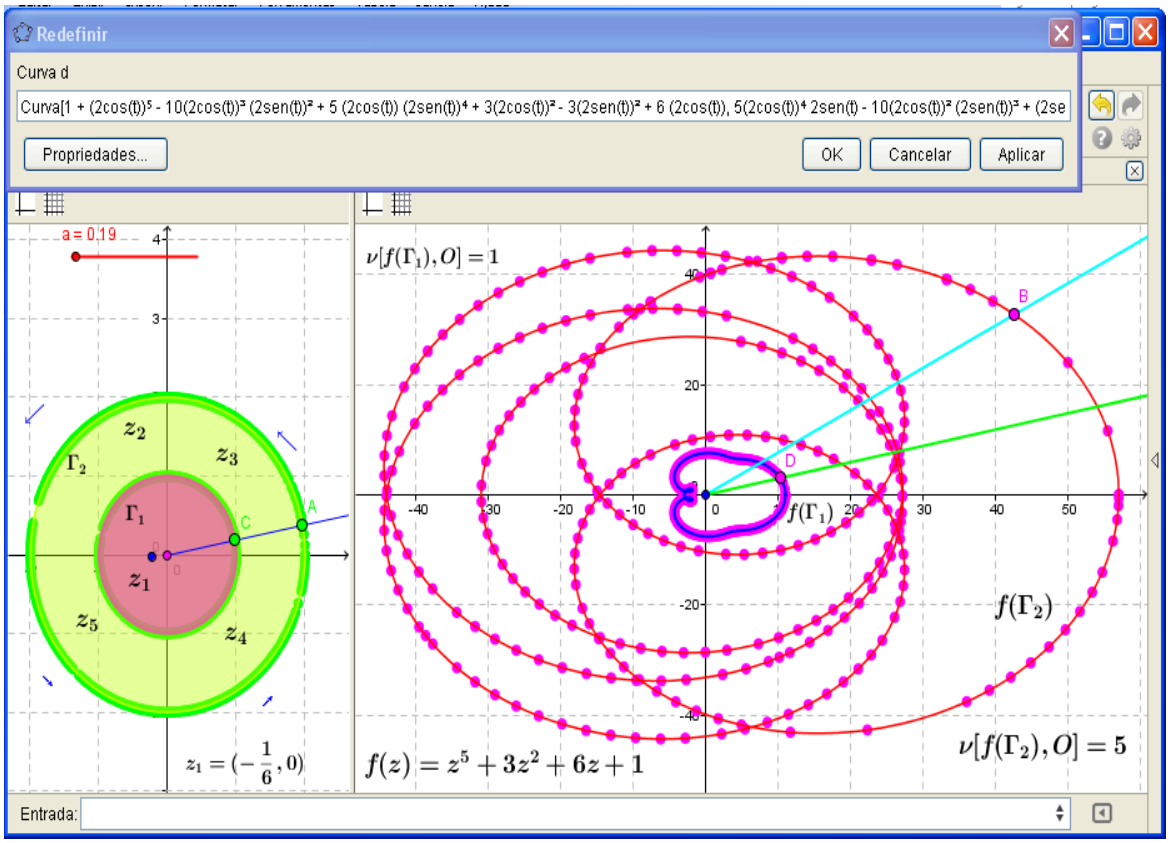

Figure 5. We can adjust the convenient neighborhood Γ_i en virtue the Rouché´s theorem. (Designed by the author)

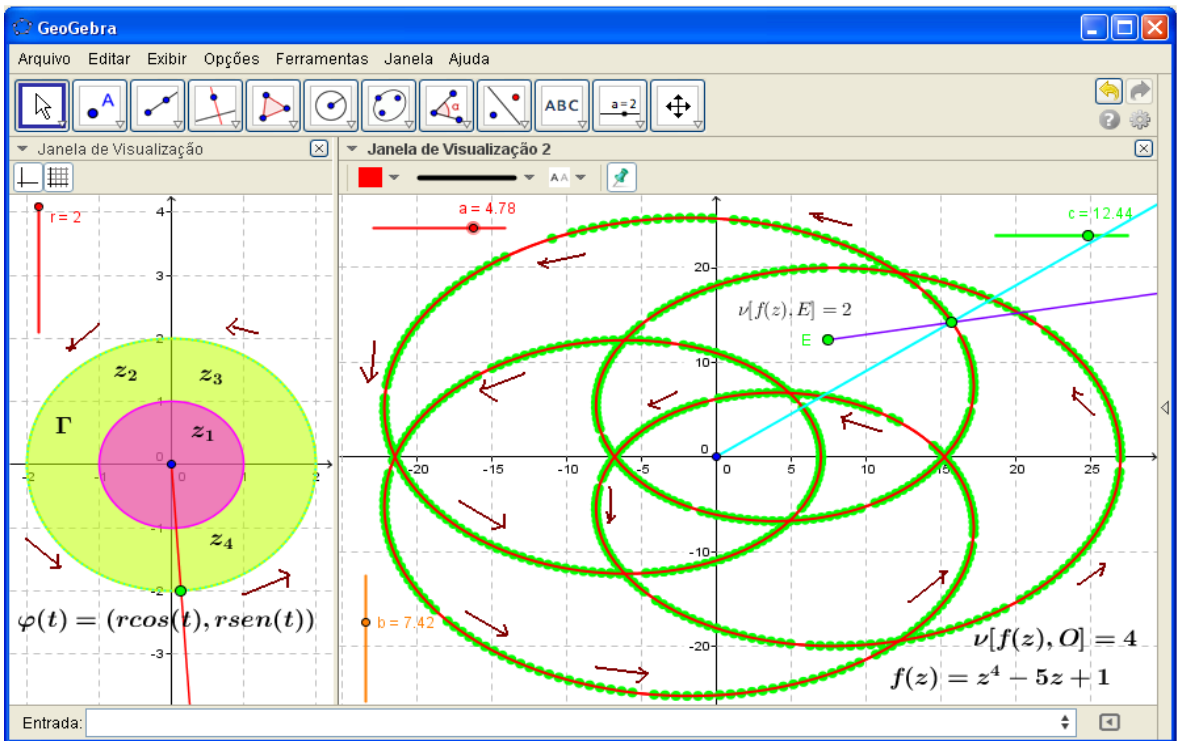

Figure 6. Visual computing of the winding number related to the all polynomial roots with $|z| \leq 2$. (Designed by the author)

Indeed, we write the following functions: $f(z)=\sin(z)$, $g(z)=z$, $h(z)=\sin(z)-z$. In these cases, we obtain without computer's help that $h(x+iy)=(\sin(x)\cosh(y)-x)+i(\cos(x)\sinh(y)-y)=\mathrm{Re}(h)+i\,\mathrm{Im}(h)$. In order to describe the set which we visualized in the figure below as a curve, we need to write $h(\Gamma)$. In this manner, we will establish that $h(a\cdot\cos(t),a\cdot\sin(t))=$
$=(\sin(a\cdot\cos(t))\cosh(a\cdot\sin(t))-a\cdot\cos(t))+i(\cos(a\cdot\cos(t))\sinh(a\cdot\sin(t))-a\cdot\sin(t))$

Moreover, with these conditions, we will produce certain intuitive conjectures in the view of the dynamic nature of this software (see Figures 8 and 9).

In the figures below, we can experience the behavior of the curves according the conditions $0\le a\le 1$ and $1<a\le 2$. In the second case, we indicate tree segments (on the right side). We can visualize the mathematical symbols $|z|$, $|\sin(z)|$, $|\sin(z)-z|$. From the figure, we can state the following inequality $|\sin(z)-z|\le|z|$, for example.

Similarly, we perceive that the inequality $|\sin(z)-z|\le|\sin(z)|$ and we can repeat the same procedure.

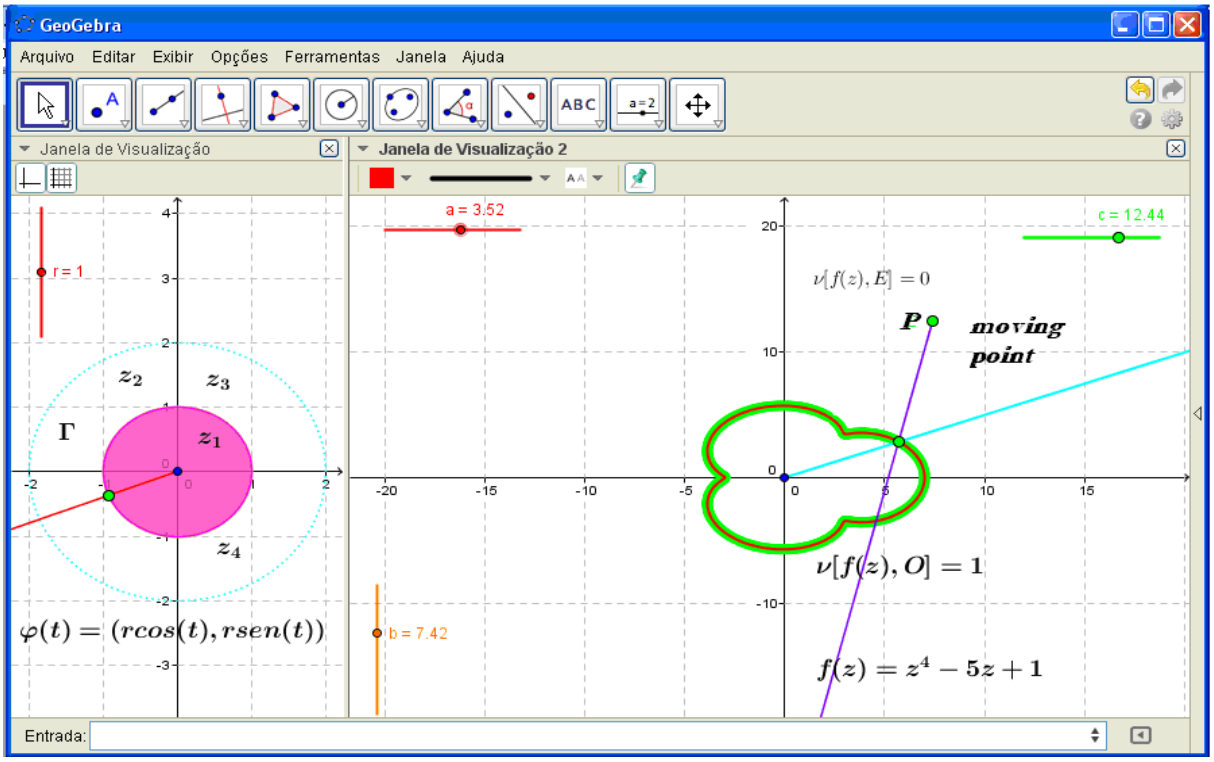

Figure 7. Visual computing of the winding number related to some of polynomial roots under the restriction $|z|\le 1$. (Designed by the author)

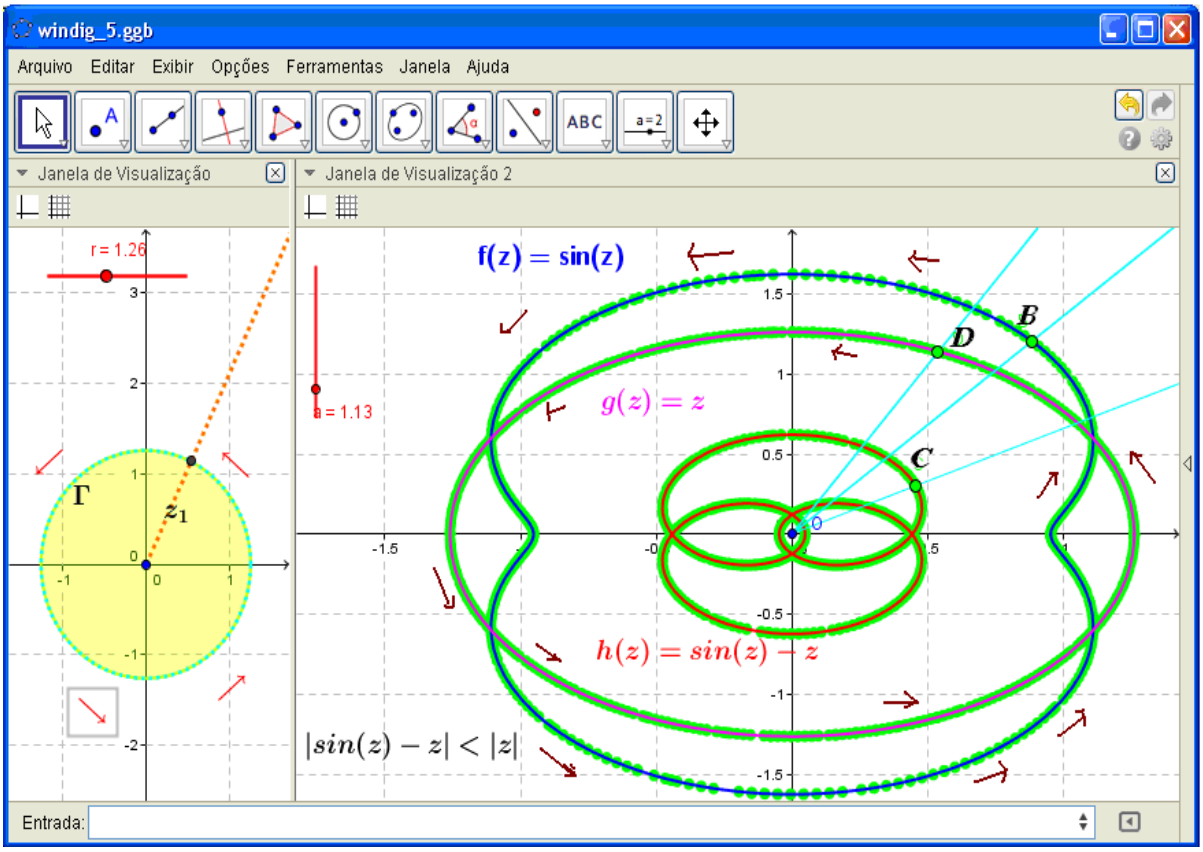

Figure 8. Visualization of some inequalities related to the Rouché´s theorem under the condition $1<r\leq 2$. (Designed by the author)

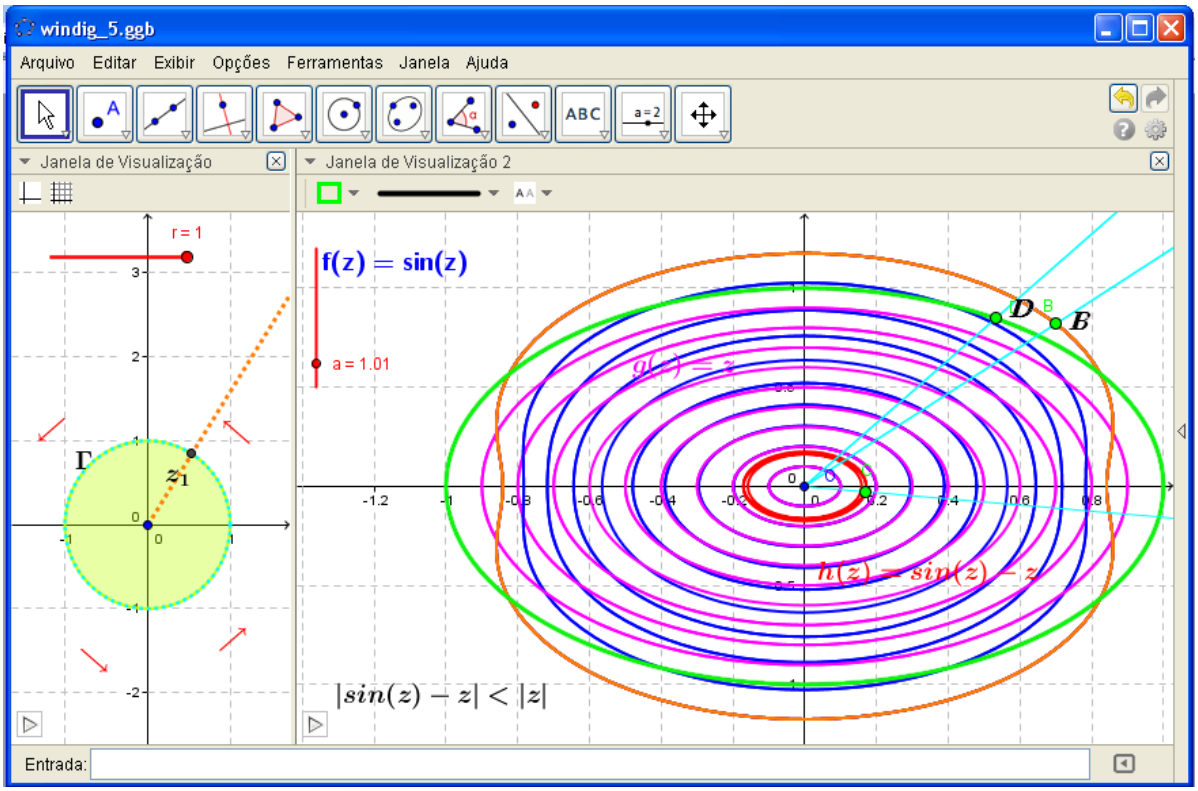

Figure 9. Visualization of some inequalities related to the Rouché´s theorem under condition $0<r\leq 1$. (Designed by the author)

The inequality $|\sin(z) - z| \leq |z|$ means that the images of the unit circle under the holomorphic functions f and g are not too far apart. So they must encircle the origin the same number of times. In the specialized literature (Bottazzini, 1996; Bottazzini & Gray, 2013; Gelbalum, 1992; Krantz, 2007; Shakarchi, 2000; Tauvel, 2006), we can observe the verification of the others equivalent inequalities, like $\left|\frac{f(z)}{g(z)} - 1\right| < 1$. In Figure 9 we can follow a continuously deformation of the two curves determined by these functions. On the right side, we visualize two segments OD and OB. Based on the figures, we can conjecture that $|OD - OB| < 1$. This scenario's research is impossible without the use of the technology. To conclude, we will indicate some elements that need a greater attention in the future work.

4. Final Remarks

We showed in this paper a heuristic and dynamic way to describe the idea of winding number, designated formally by the $\frac{1}{2\pi i}\oint_{\alpha}\frac{dz}{z-P} = \nu[f(z), P] = n \in Z$. Unlike the usual standard approach, we indicated in the previous sections several elements of perceptual nature and intuitive meaning that can be mobilized by the mathematical teacher and the student, with the aid of software Geogebra (Alves, 2012; 2013a, 2013b, 2014; 2015).

Moreover, with this software, we can test several mathematical properties. In fact, we can indicate here some of them: (i) verify the changes of the numerical values of the winding number related to the "crossing rule" (fig. 2-I); (ii) visualize the graphical behavior of the set $f(\Gamma)$ in the $C - plane$; (iii) make counting of the winding number without use of the "crossing rule"; (iv) by the continuously deformation, visualize the behavior of the homotopic paths in $C - plane$ (Hopf´s theorem); (v) conjecture about the choice of the convenient neighborhoods for check the Rouché´s theorem (Figures 5 and 6); (iv) understanding the geometric meaning of certain inequalities related to the Rouché´s theorem (Figures 8 and 9) and the limits of its application in problem situation.

To conclude this article, we question a category of teaching that emphasizes only the formalist mathematical style and disregards its intuitive bias. Still on the issues listed in the second section of this paper, we would like to suggest the following modifications:

(1) From the behavior of the graphics in the Figure 4, identify a region in the $C - plane$ where we have that $\nu[f(z), P] = 2$;

(2) From the construction in Figures 8 and 9, determine a neighborhood where we do not have the condition that characterize the Rouché´s theorem;

(3) Verify the analytical condition $\left|\frac{f(z)}{g(z)} - 1\right| < 1$ related to the Figure 9 based in the graphical and dynamic construction provided by the software Geogebra.

Admittedly, the software GeoGebra enables a dynamic exploration of the local and global properties (Alves, 2016). In this way, we recall the Atiyah´s considerations when he observes that "when we move from the small to the large, the topological features become the ones that are most significant" (Atiyah, 2002, p. 2). Related to this possibility valued by this great mathematician, we indicated several constructions that enable an exploration and a production of tacit conjectures about the certain mathematical concepts. In this regard, when we confront the static figures (Figures 1 and 2) with others dynamic pictures, we can identify relevant situations for the teaching context.

Finally, we reall several obstacles faced by mathematicians in the past centuries, with the goal to building up the theory of variable complex functions (Gray, 2010). Therefore, the technology can

provides an alternative approach and an intuitive thinking related to some complex mathematical concept.

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