Using a framework for three levels of sense making in a mathematics classroom

Diana L. Moss
Appalachian State University, USA
<mossdl@appstate.edu>

Teruni Lamberg
University of Nevada, Reno, USA
<terunil@unr.edu>

This discussion-based lesson is designed to support Year 6 students in their initial understanding of using letters to represent numbers, expressions, and equations in algebra through making thinking explicit, exploring each other's solutions, and developing new mathematical insights.

Meeting the needs of all students during a mathematics lesson can be a challenge. Many teachers ask themselves: "How can I do all this in one hour? I have to meet the individual needs of my students while teaching the whole class at the same time". The *Australian Curriculum: Mathematics* (ACARA, 2014), and the *Common Core Standards for Mathematical Practice*, a set of academic standards for mathematics in the United States (NGA/CCSSO, 2010), emphasise that students must actively be engaged in sense making and be creative communicators when learning mathematics. *The Learning Principle* of the National Council of Teachers of Mathematics (NCTM, 2000), the largest mathematics education organisation based in the United States, posits that, "students can learn mathematics with understanding" (p. 21). This understanding develops through classroom discourse and social interaction. To deepen student understanding of mathematical ideas, students need to generate and evaluate knowledge, communicate their own thinking, and be able to reflect on and critique others thinking (ACARA, 2014).

Researchers contend that understanding variables is fundamental in learning algebra (Kaput, 1995; Kieran, 1996; Moseley & Brenner, 2009). The concept of variable is multifaceted, has many possible definitions, referents, and symbols, and has changed over time (Usiskin, 1999). In school algebra, variables are thought of as (1) quantities that have the feel of knowns (for example, $A = LW$ where A, L, and W represent the quantities area, length, and width); (2) unknowns (for example, $20 = 4x$); (3) arguments of a function (for example, $f(x) = \sin(x)$; (4) parameters (for example, $y = kx$ where k is the parameter); and, (5) unknowns that vary (for example, $y = kx$) (adapted from Usiskin, 1999). In the *Australian Curriculum: Mathematics* number and algebra strand (ACARA, 2014), students in Year 7 are introduced to variables as a way of representing numbers using letters (ACARA, 2014). Furthermore, students in Year 7 are also expected to create algebraic expressions and evaluate them by substituting a value for each variable.

The Year 7 *Australian Curriculum: Mathematics* number and algebra strand (ACARA, 2014), is very similar to the *Grade 6 Common Core State Standards for Mathematics* (CCSSM) in the United States (NGA/CCSSO, 2010). In a Grade 6 American classroom, students (ages 11 to 12) explored the concept of representing quantities with letters (see Figure1) as part of a classroom teaching experiment research project on expressions and equations. These students had minimal exposure to variables, expressions, and equations before their involvement in this teaching experiment. In previous school years, these students had learned to graphically represent categorical data and used letters to identify the categories. For example, b represented boys in the class and g represented girls in the class. This lesson engaged students' prior knowledge of categorical variables and was taught before the formal definition of variable as having no fixed quantitative value. The teacher used the *Three Levels Of Sense Making Framework* (Lamberg,

2013). This framework consists of: Making Thinking Explicit, Exploring Each Other's Solutions, and Developing New Mathematical Insights. The first author designed the lesson and worked with the Year 6 teacher (whom we will refer to as Ms James), with input from the second author to the implement the lesson using the framework. Ms James taught the lesson.

The soccer ball problem

Ms. James began the lesson by saying, "Remember, you and your family want to go to the World Cup. The World Cup would not be possible without soccer balls. So, we are going to look at what a soccer ball actually is." We provided students with soccer balls and asked them to name the patterns they observed (for example, black and white, pentagons and hexagons). First, they individually thought about the problem. Then they discussed in small groups how they could count and represent the number of hexagons and pentagons on the soccer ball. We chose the soccer ball problem because it provided students with a context that promoted more sophisticated thinking about representing numbers using letters. The context is important to support students to understand a meaning of variables as quantities that have the feel of knowns, and become familiar with using a letter to represent a number (Earnest & Balti, 2008; Friedlander & Arcavi, 2012; National Research Council, 2001). We wanted students to come up with an expression or equation to represent the quantities of the number of hexagons and the number of pentagons using single letters. For example the letter to represent the number of hexagons on a soccer ball could be the single letter, h, and the letter to represent the number of pentagons the letter p, making an expression $h + p$ or an equation $h + p = 32$. Once students had individually thought about possible ideas for representing these quantities of the total number of hexagons and pentagons with expressions or equations, they shared their thinking with their small groups.

Four students with different solutions were asked to share their thinking with the whole class in order for everyone to think about how a number can be represented using a letter in an expression or equation, to provide students with experience of working with a quantity that has the feel of a known. We have found that when students are asked to solve mathematical problems, there are typically three to five types of solutions or ways of thinking. This makes discussion more manageable to address the diverse thinking of students. By sharing the four different ways, the thinking of many of the students in the class was addressed.

Making thinking explicit

Students shared their group thinking with the whole class so as to establish a foundation for further exploration and discussion. These students were carefully chosen to share their different representations of the number of hexagons or pentagons as either equations or expressions. Figure 1 shows the responses from four of the students.

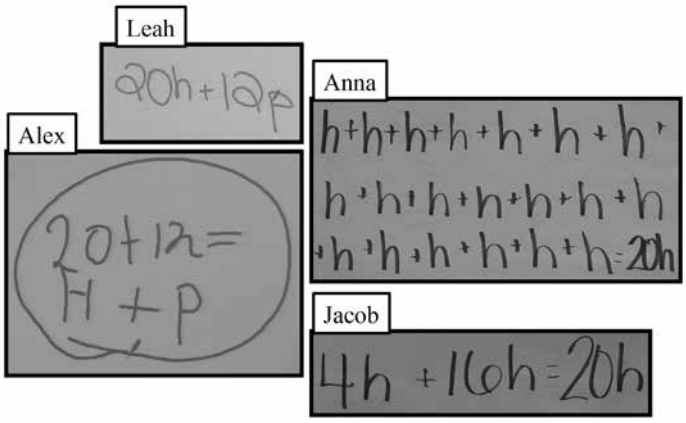

Figure 1. Four student examples representing the number of hexagons and pentagons on a soccer ball.

The student answers in Figure 1 represent four different ways of thinking about the number of hexagons, and the number of pentagons with letters. Each student explained how they kept track of the number of hexagons and pentagons.

Analysing each other's solutions

Rather than validating or rejecting an answer, Ms James asked the class to compare their answers to the answers on the board and to pose clarifying questions. To initiate the whole class discussion, Ms James asked Alex, Leah, Jacob, and Anna to defend their representations, making their thinking the focus of the discussion instead of the correct answer.

Alex's solution

Alex began by explaining that he counted 20 hexagons and 12 pentagons, writing "20 + 12" and underneath this, "$H + P$". He then posed a question to the class, asking them if this is an expression or an equation. He called on another student in the class who said, "expression because it doesn't have an equal sign in the problem". Alex wrote the equal sign and then asked the class to tell him what the H and P meant in the equation. Another student in the class said that they are variables because hexagons are being replaced with an H and pentagons are being replaced with a P. This student did not yet understand the idea that a variable varies, a distinction of which teachers need to be mindful. In this level of the discussion, Alex made his thinking explicit through questioning his classmates to see if they could figure out his reasoning. When Alex put the "=" sign in his answer, his number sentence $20 + 12 = H + P$ represented the total number of hexagons and pentagons on the soccer ball. The H represents 20 and the P represents 12. In this situation, the letters represent numbers. The expression of $H + P$ represents the total number of hexagons and pentagons on the soccer ball. Alex now viewed the relationship between 20 + 12 and $H + P$ as representing an equation.

Leah's solution

Leah then shared that she wrote $20h + 12p$ because there are 20 hexagons and 12 pentagons on the soccer ball. She explained that this is similar to Alex's expression because they both found 20 hexagons and 12 pentagons. In Leah's expression, the h is representing categorical data or h for hexagons as opposed to the quantity, 20 hexagons. She used a constant coefficient to show the amount of hexagons in the category.

Jacob's solution

Jacob noticed that his equation $4h + 16h = 20h$ was similar to Leah's expression because they both had $20h$ for 20 hexagons.

Anna's solution

When Ms James asked Anna to explain her solution the following discussion ensued. Jake also had some input to this discussion.

Anna:	I have 20 h's or 20 hexagons.
Ms. James:	Why did you write it 20 times?
Anna:	There are 20 hexagons on the ball.
Ms. James:	So, if I write a single h, how many hexagons do I have?
Jake:	One hexagon
Anna:	I wrote it 20 times to show 20 hexagons.
Jake:	Or you could just write $20h$.
Ms. James:	Why?
Jake:	$h + h + h$.... 20 times is the same as $20h$.

Developing new mathematical insights through scaffolding student thinking

Based on these discussions and students solutions, Ms. James assessed that some students understood that a letter could be used to represent the number of hexagons, a quantity that has the feel of a known (for example, Alex's answer of $h = 20$). Others had the misconception of a letter being used as a label for a category (for example, Leah's answer and Anna's answer that $20h$ is 20 hexagons). Ms. James posed the following question to the class: "I want you to think about these answers and in small groups talk about what the letters mean in Alex's expression and what the letters mean in Leah's expression. What is different about these expressions?" This question prompted students to think about the meaning of the letters in Alex's expression and in Leah's expression, and was an effort to encourage them to identify their misconceptions. In a small group discussion, Eric directed his group to look at Leah's expression and stated, "All of it is variable as label." Other students in the group agreed that $20h$ means 20 hexagons and $12p$ means 12 pentagons. Then, this group analysed Alex's expression and discussed that H means 20 hexagons and P means 12 pentagons. Another student in Eric's group stated, "This is a variable that is an amount." During these small group discussions, Ms. James was circulating around the room and informally assessing her students. She made a mental note of how the students were thinking about the letters in each of these expressions. As students reiterated the expressions for the number of hexagons and pentagons on the soccer ball, Ms. James wrote them on the board (Figure 2). These big ideas noted during the walkabout and listening to student discussions were recorded to provide a guide and reminder of the objective of the lesson, the focus of which was on understanding that a letter can represent a quantity that has the feel of a known.

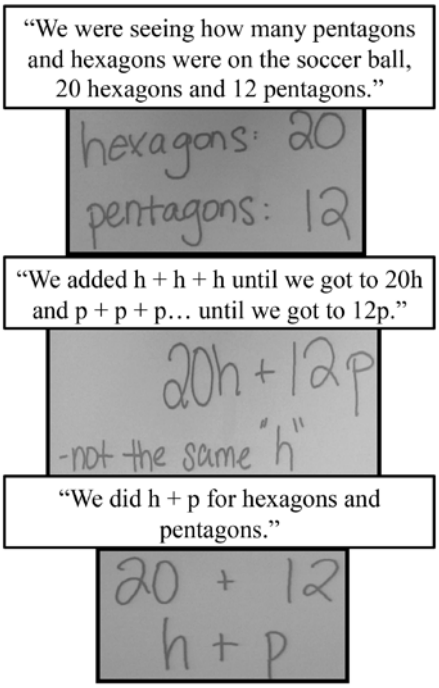

Figure 2. The big ideas: a guide for the objective of the lesson.

After summarising these expressions for the number of hexagons and number of pentagons, Ms. James posed this question to the class, "When we are looking at $20h + 12p$, what do h and p mean?"

Izzy:	This means 20 hexagons and 12 pentagons. I think you use variables to explain what they are, like y means yard.
Linda:	Yeah. I think that we use variables to use the first letter of a word to be a variable and it will be a simpler way to represent something.
Ms. James:	Anyone else?

Dan:	h is hexagon and p is pentagon
Ms. James:	Does anyone know what operation is occurring between $20h$?
Chris:	Multiplication.
Ms. James:	So when we are looking at $20h + 12p$, what do h and p mean?
Heather:	h is one hexagon and p is one pentagon.

At this point in the discussion, Ms. James asked Alex to explain his expression again. Alex told the class that $H + P$ stands for 20 + 12, so H is 20 and P is 12. Ms. James reiterated this statement to the class saying, "h and p are quantities", and called for students to discuss in small groups and write down their present understandings of using letters to represent quantities that have the feel of knowns. After giving students time to develop their ideas, the lesson closed with the teacher and students creating these generalisations.

Figure 3 shows Ms James' summary of the shared understanding of a letter used to represent a quantity that has the feel of a known.

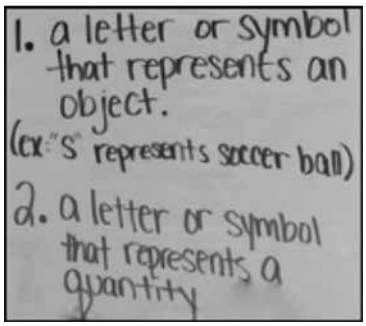

Figure 3. Summarising the shared understanding of a letter used to represent a quantity developed.

Using three levels of sense making in class discussions

This lesson illustrates how a discussion progressed through the three levels of sense making and analysis. This *Framework For Three Levels Of Sense Making* (Lamberg, 2013) can be used as a strategy to facilitate meaningful mathematical discussions to support students' learning. In each level of the discussion, students are connecting prior knowledge to new knowledge and making new mathematical connections. The first level, Making Thinking Explicit, occurred when students shared their reasoning and checked that other students in the class understood. The second level, Analysing Each Other's Solutions, required students to study each other's answers to make connections between the different ways of writing expressions and equations for the quantities, pentagons and hexagons, that had the feel of knowns. The third level, Developing New Mathematical Insights, not only involved summarising what was learned during the lesson, but also generalising the collective understandings of using a letter to represent a number. The purpose of a whole class discussion is to develop 'big mathematical ideas' that students can transfer to other problems and situations. To end the discussion, the teacher and students summarise what was learned as recommended in the framework. In this lesson, Ms James asks the class, "What were the big ideas of this lesson?" Figure 4 displays student descriptions of the big ideas and these provided Ms James with starting points for discussion to help students develop new mathematical insights.

The big idea was...

- Expressions and equations
- We were seeing how many pentagons and hexagons were on the soccer ball, 20 hexagons and 12 pentagons.
- We had to figure out how many whites and how many blacks on the soccer ball.
- We did $h + p$ for hexagons and pentagons.
- We added $h + h + h$ until we got to $20h$ and $p + p + p$... until we got to $12p$.
- We were coming up with different ways of how to represent the number of hexagons and pentagons. So we had $h + p$ and $20h + 12p$.

Figure 4. Summary of ideas from students about what they thought were the big ideas in the Soccer Ball Problem.

During all levels of the discussion, misconceptions and errors were addressed. For example, the idea that the letter h stands for hexagon was reasonable given the experiences these students had to date where the letter h could be an abbreviation for the word "hexagon." When Ms. James encouraged students to share their initial reasoning, she understood, and addressed through teaching, how students were thinking about letters that represent numbers, and also encouraged students to consider different ways to represent the number of hexagons and the number of pentagons in an expression or equation. The open-ended nature of the questions gave students opportunities to work together and struggle with mathematical ideas, shifting their understanding from a letter that labels a word to a letter that represents a number. The next lesson in this progression is the concept of unknowns with a feel of variability. Teachers need to carefully attend to their students' understanding of variables because student thinking must shift from a letter that represents a quantity with the feel of a known to a variable with a feel of variability.

Teacher questioning is essential for scaffolding the discussion among students (Lamberg, 2013). In this lesson, Ms. James generated class discussion by inviting students to contribute and comment on the ideas being presented. Appendix A provides some strategies for teachers to prepare for a whole class discussion. It also expands on how teachers might get students to think about their solutions and create representations. In addition, it might help teachers decide how to focus the discussion based on student reasoning. Ms. James also enabled students to elaborate on their thinking, which benefitted all the students in the class. When students are encouraged to participate in a class discussion about mathematical ideas, they are challenged to think more deeply about the concepts and make connections between these concepts. Appendix B provides the framework for teachers to assist them with facilitating a discussion to help students make deeper mathematical connections. Whole class discussion can efficiently address the thinking of the whole class by looking at types of solutions and allowing students to individually think about their solutions and how others are thinking.

References

Australian Curriculum, Assessment and Reporting Authority [ACARA]. (2014). *Australian Curriculum: Mathematics*. Sydney, NSW. Retrieved 4 September 2015 from http://www.australiancurriculum.edu.au/mathematics/curriculum/f-10?layout=1 Earnest, D., & Balti, A. (2008). Instructional strategies for teaching algebra in elementary school: Findings from a research-practice collaboration. *Teaching Children Mathematics, 14*(9), 518–522. Retrieved 4 September 2015 from http://ase.tufts.edu/devtech/courses/readings/earnestbalti2008.pdf

Friedlander, A., & Arcavi, A. (2012). Practicing Algebraic skills: A Conceptual Approach. *Mathematics Teacher, 105*(8), 608–614. Retrieved 4 September 2015 from http://www.nctm.org/Publications/mathematics-teacher/2012/Vol105/Issue8/Practicing-Algebraic-Skills_-A-Conceptual-Approach/

Kaput, J. J. (1995, October). *A research base supporting long term algebra reform?* Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Columbus, OH. Retrieved 4 September 2015 from http://files.eric.ed.gov/fulltext/ED389539.pdf

Kieran, C. (1996). The Changing Face of School Algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde, & A. Perez (Eds.), *8th International Congress on Mathematical education (ICME): Selected lectures* (pp. 271–290). Seville, Spain: S.A.E.M. Thales.

Lamberg, T. D. (2013). *Whole class mathematics discussions: Improving in-depth mathematical thinking and learning*. New York, NY: Pearson Education.

Moseley, B., & Brenner M. E. (2009). A comparison of curricular effects on the integration of arithmetic and algebraic schemata in pre-algebra students. *Instructional Science: An International Journal of the Learning Sciences, 37*, 1-20. doi: 10.1007/s11251-008-9057-6.

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics. Retrieved 4 September 2015 from http://www.nctm.org/standards/

National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO]. (2010). *Common core state standards initiative: Mathematics standards.* Washington, DC. Retrieved 4 September 2015 from http://www.corestandards.org/Math/

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* In J. Kilpatrick, J. Swafford, & B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Usiskin, Z. (1999). Conceptions of school algebra and uses of variables. In B. Moses (Ed.), *Algebraic thinking, grades K–12*: Reading from NCTM's school-based journals and other publications (pp. 7–13). Reston, VA: National Council of Teachers of Mathematics.

Appendix A

Preparing for a whole class discussion

(Adapted from Lamberg, 2013, p .60)

Getting reading for discussion	
What is the teacher doing?	**What are the students doing?**
• Posing a Problem for students to solve. • Encouraging students to explain their thinking and to create representations to record their thinking. • Walking around the room, looking at student work, and listening to the small group discussions. • Noticing misconceptions and noting which solutions to share with the class (low-level to more sophisticated thinking and does not necessarily have to be the correct solution). • After gaining perspective on the various solutions in the room, decide which student solutions and in what order might be helpful to generate class discussion about a mathematical issue/idea.	• Thinking and writing about the problem individually. • Discussing how they are thinking about the problem in small groups. • Reflecting on their individual thinking and what others are doing in small groups.

Appendix B

The whole class discussion: three phases of sense making framework

(Adapted from Lamberg, 2013, p.116)

1. Making thinking explicit		
What is the teacher doing?	**What are the students doing?**	**Teacher questions**
• Encouraging students to explain their thinking by sharing strategies or posing a problem or issue to discuss.	• Listening to each other's solutions and reflecting on own thinking. • Asking questions if unclear about the idea presented.	• Ask questions to help students share and clarify their thinking so that the whole class can follow students' explanation. • What are you thinking? • Can you explain what you did here? • How did you figure that out?
1. Analysing each other's solutions		
What is the teacher doing?	**What are the students doing?**	**Teacher questions**
• Encouraging the class to ask the student at the board "How do you know?" • Facilitating the discussion by expanding on why some ways of solving the problem may be more efficient than others. • Scaffolding student thinking.	• Asking each other questions that clarify different ways of doing the same problem. • Comparing their answers. • Actively listening to the discussion. • Connecting the mathematical ideas that are presented.	• Ask questions to promote analysis and reflection. • Help students look for structure in mathematics by comparing answers. • How are answers similar and different? • Address errors/misconceptions by having students justify answers so that others can understand their reasoning. • Pose questions to reflect on efficient strategies and modelling strategies.
3. Developing new mathematical insights—making generalisations		
What is the teacher doing?	**What are the students doing?**	**Teacher questions**
• Recording the 'big idea' that was generated by the class as a written record. • Scaffolding student thinking, by introducing vocabulary, formulas that build on the discussion when it makes sense to do so.	• Answering the teacher's questions to come up with a rule or summary of the mathematics learned. • Applying this rule to a new, related problem that the teacher poses. • Connecting new information with what was previously discussed. • Recording the new insight as a tool in their math journals so that they can use this information to solve future problems.	• Ask questions to promote mathematical insights. • Asking students questions to come up with a conjecture, rule, or definition that summarises the learning that took place.