Integrating technologies into mathematics: Comparing the cases of square roots and integrals

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Introduction

Although the term is often used to denote electronic devices, the idea of a 'technology', with its origins in the Greek *techne* (art or skill), refers in its most general sense to a way of doing things. The development and availability of various technologies for computation over the past forty years or so have influenced what we regard as important in mathematics, and what we teach to students, given the inevitable time pressures on our curriculum. In this note, we compare and contrast current approaches to two important mathematical ideas, those of square roots and of integrals, and how these have changed (or not) over time.

To locate this work in context, it is important to note that some technologies are made explicit in the Australian Curriculum, perhaps most obviously in the case of arithmetic computation in primary school. Thus, for the latest version of Year 4 (Australian Curriculum, Assessment and Reporting Authority (2016), we have the two statements:

> Recall multiplication facts up to 10 × 10 and related division facts (ACMNA075)
> Develop efficient mental and written strategies and use appropriate digital technologies for multiplication and for division where there is no remainder (ACMNA076)

These statements reflect an agreement over many years now in Australia that students need a variety of technologies for computation, including recall of some facts, mental arithmetic, written strategies and the use of 'digital' technologies, which in this case probably refer to calculators of some sort, including calculators embedded in other devices. (The Australian Curriculum documents consistently refer to 'digital' technologies, presumably to recognise that there are other kinds of technologies as well.)

For other aspects of mathematics, however, the curriculum is much less explicit regarding the technologies involved, due in part no doubt to its

deliberate brevity. Thus, considering the case of square roots, the single reference in the Australian Curriculum appears in Year 6:

> Investigate and use square roots of perfect square numbers (ACMNA150)

Of course, students beyond Year 6, including certainly students in the senior secondary years, will need to understand and deal with square roots of other numbers as well, and thus employ a technology of some kind, but the curriculum documents leave this detail to teachers, recognising that it will be implicit in various tasks.

This paper suggests that a succession of technologies has been available over time for finding square roots, but there has not been a similar succession for finding indefinite integrals. In precisely the same way that a technological solution to finding square roots can save a lot of time spent on routine computation, it is argued that a technological solution to finding integrals can save a lot of time spent on routine symbolic manipulation. In each case, hand-held technologies can support both conceptual development as well as finding answers to question. A balance between sufficient emphasis to understand the key ideas and excessive routine manipulation is needed.

Square roots

The idea of a square root is a useful and powerful one in mathematics, inextricably caught up with the idea of the square of a number. Indeed, the concepts are complementary, so that their understanding is intertwined. It is a practical idea, as some problems require us to find a square root as part of the solution, and to interpret it in some context. Not surprisingly, it has long been a part of our mathematics curriculum, once students get to a certain level of sophistication.

Square roots have always been a bit problematic, however, as some of them are easier to find than others. In some cases, we would expect that students evaluate square roots immediately and mentally, in part as evidence that they understand what the idea means:

$$\sqrt{49} = 7 \text{ (because } 7^2 = 49).$$

When the numbers get beyond typical mental expectations, a more sophisticated technology is needed. Thus, to find the square root of 7056, it is helpful to first find the factors of the number:

$$7056 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7 \times 7.$$

Conveniently, in this case, factors are repeated, so taking just one of each repeated factor provides the square root (and reinforces the concept at the same time):

$$\sqrt{7056} = 2 \times 2 \times 3 \times 7 = 84 \text{ (and } 84^2 = 7056).$$

Unfortunately, the square roots of most numbers cannot be resolved by these technologies, so a more practical method is required. Even numbers

that are perfect squares (such as 29 929 and 17 397 241) do not yield easily to these primitive methods, while numbers that are not perfect squares, such as 52, are more problematic, and negative numbers are even more problematic.

Since most square roots (such as $\sqrt{52}$) are irrational, one possible solution is to represent them exactly. In the case of $\sqrt{52}$, the two most likely choices are clearly equivalent:

$$\sqrt{52} = 2\sqrt{13}\ .$$

For practical purposes in many contexts, a decimal approximation is preferred, however. So technologies for doing this were developed, used and taught in schools. As a young boy, I learned the following method to find $\sqrt{52}$, which will be familiar to many (older!) readers, in which three successive steps have been shown here to illustrate how they were conducted. Each step produces an additional digit in the decimal approximation.

```
        7                 7 . 2                 7 . 2 1
  7 ) 5 2 . 0 0         7 ) 5 2 . 0 0 0 0     7 ) 5 2 . 0 0 0 0 0 0
      4 9                   4 9                   4 9
      -----             1 4 2 -------         1 4 2 -------
        3                 3 0 0                 3 0 0
                          2 8 4                 2 8 4
                          -----             1 4 4 1 -------
                            1 6                   1 6 0 0
                                                  1 4 4 1
                                                  -------
                                                    1 5 9
```

First, the digits in the number are paired (both sides of the decimal point): in this case, there are only two of them (52), so no pairings are evident to the left of the decimal point. The largest integer whose square is less than 52 (the first two digits) is found as the first digit of the approximation (7, in this case). Then the approximation (only 7 so far) is multiplied by the divisor (also 7), and subtracted from 52 as in the long multiplication algorithm.

In the second step, the approximate solution (7) is doubled (to get 14), the next two digits are 'brought down' (to produce 300, ignoring the decimal point) and a digit n to append to the 14 is chosen so that $n \times 14n$ is close to, but does not exceed the number 300. (Here, unusually, $14n$ does not have its normal meaning of $14 \times n$, but is a three digit number, the third digit of which is n.) In this case, $n = 2$ is found via some mental arithmetic, as $2 \times 142 < 300$. So the second digit of the approximation is a 2.

The third step continues this process, beginning with doubling 72 to get 144 and looking for the digit n so that $n \times 144n$ is close to, but less than 1600. In this case, $n = 1$.

And so on, for as many steps as are desired. The process could continue indefinitely, of course, as the number is irrational, although the mental arithmetic quickly becomes tedious. It is interesting that the process does not produce successively better approximations, but rather produces successive digits in the decimal representation of $\sqrt{52}$, so that if one sought an approximation correct to three decimal places, it would be necessary to go first to the fourth place. Furthermore, the process provides no insight into the idea of a square root; it is merely a procedure to follow.

I do not recall anyone ever explaining to me why all these steps worked, even as an exercise. I am not sure that my teachers knew why it worked, anyway. It just did work, and was a solution to the problem, so students were expected to develop expertise at actually doing it. It was a tedious and error-prone process.

A little further into schooling, a superior computational method was introduced, relying on logarithms, which were especially helpful for this sort of task. Almost overnight from their invention by Napier and Briggs early in the 17th century, almost 400 hundred years ago, logarithms transformed the ability of people to calculate, and thus extended the effective working lives of mathematicians, astronomers, engineers and scientists. They were still an important tool for students of my generation to use from around the middle of secondary school in the 1960s. The properties of common logarithms (i.e., logarithms to base 10, often written without the base) helped us to see that, since

$$52 = \sqrt{52} \times \sqrt{52}$$

and so

$$\log 52 = \log \sqrt{52} + \log \sqrt{52}\,.$$

Then

$$\log \sqrt{52} = \frac{1}{2} \text{ of } \log \sqrt{52}\,.$$

So to find $\sqrt{52}$, it was necessary only to halve the log of 52 and then raise ten to the power of the result (which was called finding an antilogarithm, sometimes accessed in separate tables for convenience). An extract from a log table (Abelard, 2015) in this case is shown in Figure 1:

50	6990	6998	7007	7016
51	7076	7084	7093	7101
52	7160	7168	7177	7185
53	7243	7251	7259	7267
54	7324	7332	7340	7348
N	0	1	2	3

Figure 1. Extract from a table of four-figure logarithms to base 10 (Abelard, 2015).

Careful reading of the (four-figure) table, taking account of the fact that $10 < 52 < 100$ established that $\log_{10}52 \approx 1.7160$, from which we get $\log_{10}\sqrt{52} \approx 0.8580$. To complete the historical picture, note that in this formulation, the decimal part (in this case, 0.7160) was referred to as the 'mantissa' and was always positive, while the integer part (in this case, 1) was referred to as the 'characteristic', and could sometimes be negative; as for the earlier algorithm, I do not recall any discussion in my schooling of why these particular terms were used.

To raise 10 to this power, an equivalent antilogarithm table could be consulted, or the same table studied differently, as shown in Figure 2, to find the number whose logarithm is 0.8580 :

N	0	1	2	3	4	5	6	7	8	9	Proportional parts								
											1	2	3	4	5	6	7	8	9
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8579	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8639	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5

Figure 2. Interpreting a logarithm table to find an antilogarithm (Abelard, 2015)

The first three digits of the antilogarithm are found by noting that 8579 is in the '1' column of row 72. Adding a 'proportional part' of 1 gets 8580 (as desired) and the fourth digit (1) of the result of $\sqrt{52} \approx 7.211$, after adjusting (mentally) the magnitude of the result. Some students understood this (especially if we did not use an antilogarithm table, which was rarely interpreted as related to finding powers of ten), but many did not, and just followed the steps. The steps got more complicated when small numbers were involved, too, such as finding $\sqrt{0.52}$, since the logarithm of 0.52 was generally first written with a characteristic of "bar 1" to avoid dealing with some aspects of negative numbers, while leaving the mantissa positive:

$$\log 0.52 = \bar{1}.7160$$

The number $\bar{1}.7160$ represents –1 + 0.7160 (preferred to –0.2840 in order to facilitate table use). The use of logarithms allowed square roots to be found fairly efficiently, provided one had a set of tables handy, and was a great deal easier than the division method. Although they were very useful devices, it was rare for students to be advised how the log tables were actually produced; instead, they were simply purchased or consulted in a textbook.

In senior secondary school, many students were permitted to use the wonderful technology of the slide rule to get approximate values for square roots in the senior secondary years. Although the approximations were quite poor, they were easily obtained (more easily at least than the previous methods), provided one had a slide rule handy. Figure 3 shows that reading a square root off a slide rule typically involved finding the number (in this case 52) on the A scale and using the cursor to find the result on the C scale, mentally adjusting appropriately for the magnitude of the numbers.

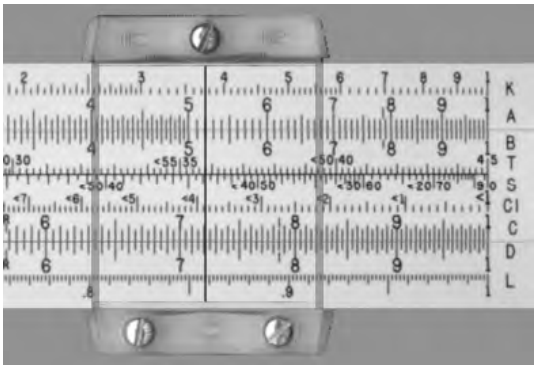

Figure 3. Approximating $\sqrt{52}$ on a slide rule (Ross, 2015).

In this case, it is very hard to do better than two-digit accuracy, $\sqrt{52} \approx 7.2$, although with great care, it is possible to see that $\sqrt{52} > 7.2$. The connection between square roots and squares was nicely visible here. The connections of slide rules with logarithms (which might have explained how they actually worked) were very rarely highlighted, at least in my recollections of my own experience.

And then came the calculator, around 40 years ago. Finding a square root became merely a matter of using a square root key, originally pressed after the number, but on modern calculators such as the one shown in the left screen in Figure 4, pressed before the number, as for the conventional representation. In addition, a quick and reassuring check that the result is what was expected can be obtained by squaring it immediately to see that the square is indeed 52, as shown in the right screen:

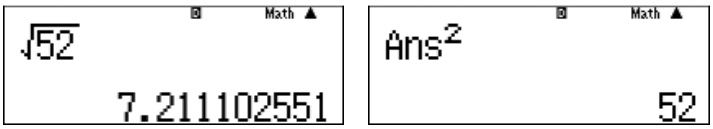

Figure 4. Approximating and checking $\sqrt{52}$ numerically on a Casio fx-100AU PLUS scientific calculator.

The effect of this new technology was that most of the previous technologies were quickly discarded. In some cases, the previous technologies were described, but students were not expected to develop substantial by-hand expertise with them. In a crowded curriculum, when time is short, the focus is on what a square root means, in what circumstances one might want to evaluate one, how they are related to squares, and how to use and interpret them. There was little enthusiasm for engaging students in lengthy, error-prone and frequently problematic by-hand computation, when a single keystroke produced the result in an instant.

Modern calculators might allow us to do better than this, however, helping students to explore for themselves some of the previous technologies, and hence possibly increasing their understanding of the idea of a square root, but without the burden of by-hand calculation, which made the previous technologies problematic. For example, the tedium of finding factors can be replaced by a factor command on a calculator, as shown in Figure 5, and attention can be focused on interpreting the array of factors to see how square roots can be determined from them:

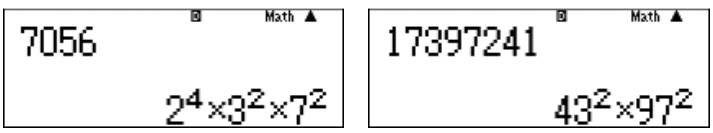

Figure 5. Using a factor command to explore square roots on a Casio fx-100AU PLUS.

The connections between logarithms and square roots can also be explored using a scientific calculator to handle the arithmetic (or the table consultation), as suggested by Figure 6:

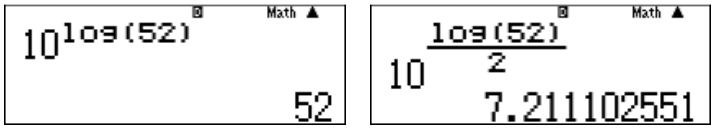

Figure 6. Exploring logarithms and square roots on a Casio fx-100AU PLUS.

Explorations of this kind also provide a way for students to experience and appreciate the relationships between common logarithms and powers of ten, fundamental to understanding the concept of logarithms. Since logarithms are still important, even if they are not routinely required for computation any more, this would seem to be a good context for developing that understanding.

While a scientific calculator these days is essentially a black box for the computation of square roots, anecdotally this does not seem to be of concern to most mathematics teachers. However, a calculator might be used to explore ways in which computations to determine a square root can be undertaken by the calculator, at the very least to dispel any student notion that the calculator just somehow 'remembers' all the square roots. Although the mechanism probably involves power series for logarithms and exponentials, an iterative process to determine a square root can be used as well. After an initial guess g is made, an improved guess g' can be obtained by finding the average of the guess and the result of dividing the number (52, say) by the guess, as follows:

$$g' = \frac{g + \frac{52}{g}}{2}$$

On a modern calculator, iterations of these kinds are readily set up, using the *Ans* variable to refer to the previous result (guess). In this case, after an initial (poor) guess of 7, shown in the first screen in Figure 7, later guesses rapidly converge to a good approximation to the result:

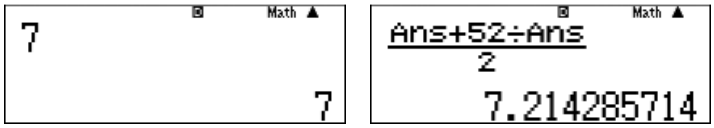

Figure 7. Setting up an iterative procedure to find $\sqrt{52}$ *on a Casio fx-100AU PLUS.*

After constructing the iterative step, each successive iteration requires only that the = key be pressed. Figure 8 shows that this procedure converges very rapidly after only two more steps, to an accurate approximation:

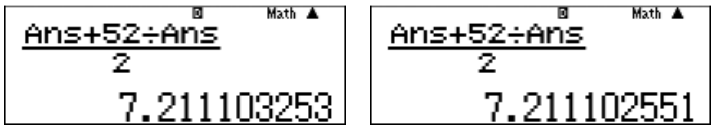

Figure 8. The next two iterations to approximate $\sqrt{52}$.

Figure 9 shows that the result can easily be checked by squaring the most recent iteration, demonstrating in this case, to the accuracy provided by the calculator, that $\sqrt{52}$ has been determined in only three steps, which might account for how rapidly the calculator produces the result. While this might be of interest in a school class, formal study of such procedures, including their convergence properties, might sensibly occur some time after students leave school, in more advanced mathematics courses, such as undergraduate numerical analysis. They might also be the subject of elementary algorithmic design and programming of calculators, computers or spreadsheets.

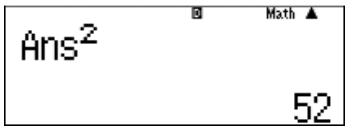

Figure 9. Checking the iterative result to approximate $\sqrt{52}$.

Technology might be used in other ways to unpack the idea of a square root, without the need to spend so much time undertaking the calculations. Figure 10 shows two examples, firstly finding the square root of a negative number (which requires a complex result) on the Casio fx-100AU PLUS and secondly finding an exact square root (for which a different calculator has been used, the Casio fx-991ES PLUS):

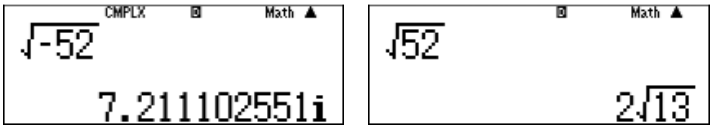

Figure 10. Further opportunities to explore square roots on calculators.

In addition, this technology might be used to illuminate some important properties of logarithms, using the vehicle of square roots. Thus, in Figure 11, natural logarithms are used instead of common logarithms, while a calculator that provides logarithms to bases other than 10, such as Casio fx-991ES PLUS, is used to demonstrate that the idea of halving the logarithm to find a square root (previously illustrated in Figure 6) is a feature of logarithms in general, not only of common logarithms: the numerical square root of 52 can be found with logarithms to any base.

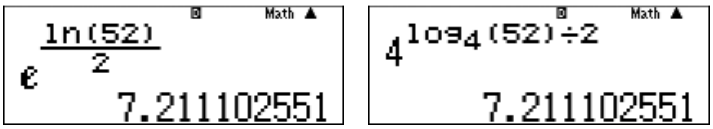

Figure 11. Using calculators to explore logarithms to other bases and square roots.

In summary, the idea of a square root is an important one, previously problematic in practice, as a lot of time was needed to use standard procedures to evaluate it. When an easy alternative technology became available, it was

seized upon, and the previous methods mostly discarded, although they might have helped offer some insight. While the computation of square roots is practically important, their understanding was often not much helped by their means of computation; yet the use of a calculator might permit some insights to be developed about square roots, as the burden of computation is lifted.

With these observations in mind, it is of interest now to turn to the second example of a similar kind, concerning integration of functions.

Integrals

The idea of an integral is a useful and powerful one in mathematics, inextricably caught up with the idea of a derivative. Each of them offers powerful ways of measuring aspects of change, and hence provide the essential machinery for mathematical models of a changing world. In the case of integrals, accumulation is measured (such as the distance travelled over a period of time by a moving object, probably represented as an area), while the derivative measures the rate at which one quantity is changing in relationship with another. The concepts are complementary, so that their understanding is intertwined, through the fundamental theorem of calculus. Integration is a practical idea, as some problems require us to find an integral as part of the solution, and to interpret it in some context. Not surprisingly, it has long been a part of our mathematics curriculum, once students get to a certain level of sophistication and undertake some study of the calculus.

Integrals have always been a bit problematic, however, as some of them are easier to find than are others. In some cases, we would expect that students evaluate integrals immediately and mentally, in part as evidence that they understand what the idea means and how it is associated with derivatives:

$$\int (2x+7)\,dx = x^2 + 7x + C$$

since

$$\frac{d}{dx}\left(x^2 + 7x + C\right) = 2x + 7$$

In this case, a constant C is included; however, in the following cases it will be omitted, on the assumption that readers are aware that indefinite integrals should always include a constant.

When the antiderivatives involved exceed typical mental expectations, a more sophisticated method is needed, however. Thus, to find the following integral, it is helpful to recognise its form:

$$\int (2x+5)e^{x^2+5x-1}\,dx$$

If the chain rule is recognised to be relevant here, some students might see immediately how this has occurred and so be able to write down the integral, while for others a suitable substitution such as $u = x^2 + 5x - 1$ will facilitate

the integration process. Although this is a relatively straightforward example, significant—although routine—algebraic manipulations are involved after deciding on the substitution. These manipulations are essentially algebraic manipulations, and are not an intrinsic part of the calculus. If we are not careful, students can spend a great deal of time devoted to learning about calculus to developing by-hand expertise with symbolic manipulation—which of course is something different from the calculus; while it is sometimes a means to an end, it is important to not confuse it with the ends. While it is a good practice after obtaining a definite integral to find its derivative as a check that the manipulations were carried out correctly, and a nice reminder of the relationship between the functions concerned, this can sometimes be tedious and at other times can be extremely tedious, so is frequently not done by time-poor or symbolic manipulation-averse students.

In recent years, students and others have begun to use computer algebra systems (CAS) on calculators and computers, which handle the routine work of integration, analogous to the way in which the scientific calculator handles the routine work of finding a square root. In this paper, we use one example of a hand-held CAS, on Casio's Classpad II, a popular example of this technology.

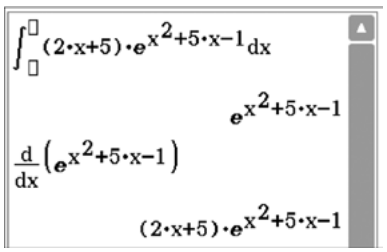

Figure 12. Evaluating and checking an indefinite integral on Casio's Classpad II.

Figure 12 shows the (indefinite) integral just mentioned (again, shown without the constant, as the calculator does not routinely provide it), as well as a quick and reassuring check that the derivative of the result reproduces the original integrand. As the screen suggests, this particular device represents an indefinite integral by leaving blanks where the limits of a definite integral might appear. Expressions are entered with a keyboard and a stylus is used for drag-and-drop movements of expressions to limit the need for typing. Although it concerns a different aspect of mathematics, there are clear similarities here with the use of the scientific calculator to find and understand square roots, as shown in Figure 4.

Using a CAS calculator as in Figure 12 does not require students to deploy the symbolic procedure of integration by substitution, including choosing what to substitute, and completing all the associated manipulations. However, a suitable discussion, with the help of the teacher, might be expected to help them to see how the chain rule is related to the results.

As for the case of square roots which can be recalled from memory or readily seen when factors are sufficient, some integrals can be handled easily

without recourse to technology. Similarly to the case of square roots, a number of methods for evaluating indefinite integrals have been developed for more difficult situations. Unlike the case of square roots however, the methods rely on the structure of the integrand, and the corresponding differentiation processes, or on helpful relationships between mathematical expressions, such as double angle formulas in trigonometry or other trigonometric identities. A current example of the suite of possibilities is listed in Topic 1 of Unit 4 of the senior secondary Mathematics syllabus, Specialist Mathematics, developed by the Australian Curriculum, Assessment and Reporting Authority (ACARA) (2015), concerned with integration techniques. The suite includes several methods of integration, such as:

Integrate using the trigonometric identities (ACMSM116)

Use substitution $u = g(x)$ to integrate expressions of the form $f(g(x))g'(x)$ (ACMSM117)

Integrate expressions of the form $\frac{\pm 1}{\sqrt{a^2 - x^2}}$ and $\frac{a}{a^2 + x^2}$ (ACMSM121)

Find and use the derivative of the inverse trigonometric functions: arcsine, arccosine and arctangent (ACMSM120)

Use partial fractions where necessary for integration in simple cases (ACMSM122)

Integrate by parts (ACMSM123)

These various methods in some cases involve clever adaptations of algebraic and trigonometric relationships (almost in the form of symbolic tricks). In other cases, the methods are derived from properties of differentiation (such as substitution, from the chain rule, or integration by parts from the product rule) and in still other cases are developed from complicated symbolic transformations (such as the use of partial fractions). They are all essentially algorithmic and symbolic, although at times needing various levels of ingenuity to decide which to use. However, they do not contribute significantly to learning about integration as an important mathematical idea. Indeed, the time spent devoted to developing by-hand expertise with them seems luxurious in a crowded curriculum, at least when there are efficient alternatives.

CAS calculators certainly provide alternatives. When a CAS calculator is used to obtain integrals, the mechanism used to do so is not shown; in parallel, this is also the case for the scientific calculator finding square roots. A difficult (by hand) example is shown in Figure 13; as previously, the calculator is also used to verify that the derivative of the result reproduces the original integrand.

Similarly, Figure 14 shows an example (on the left) involving integration by partial fractions and another example (on the right) involving integration by parts, together with the associated verifications of the results by differentiation.

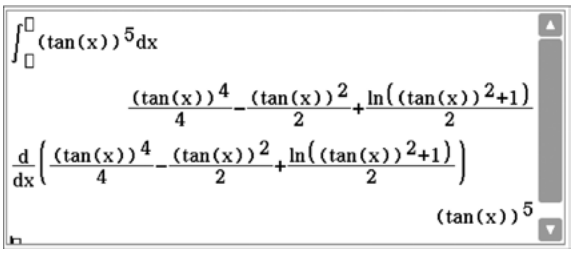

Figure 13. Integrating using trigonometric identities on Casio's Classpad II.

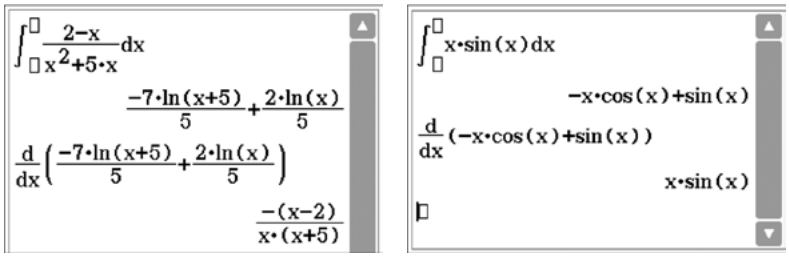

Figure 14. Integrating by partial fractions and by parts on Casio's Classpad II.

Figure 14 also illustrates that it is sometimes necessary for students to rearrange calculator results in order to see equivalences; although the final result obtained in the first case is equivalent to the integrand, the denominator has been factorised and the numerator written differently from the original. CAS calculators often present challenges of these kinds, ideally stimulating students to look to see simplifications for themselves, as shown in the two examples in Figure 15, which also illustrate the handling of one of the methods of integration suggested by the ACARA curriculum. In this case, the calculator provides results that are easier to see in relation to the original integrand in the case of a particular value of a being chosen.

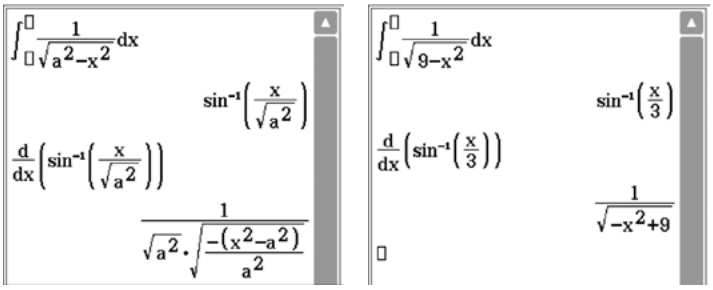

Figure 15. Interpreting the form of results for standard integrals on Casio's Classpad II.

These examples do not exhaust the list of integration methods proposed by ACARA (2015), but hopefully serve to illustrate two main points: that detailed knowledge of the symbolic manipulation mechanisms associated with various methods are not necessary, if one has access to a modern CAS calculator, and that the calculator might be used to explore some relationships evident in the results, probably relying on the teacher to stimulate that process.

A comparison

The ways in which technology is typically used for these two concepts of square roots and integration are clearly different. Integration is a central component in introductory calculus. Students need to learn what it is, decide when to use it, know how to use an integral to represent a relationship of interest, be able to evaluate the integral and finally interpret what the result means (in some context, often). The CAS calculator addresses only one of these five aspects (that concerned with evaluation), but it seems not unreasonable that more time might be spent on the other four aspects, which are frequently neglected, if less time needs to be spent labouring on extensive by-hand symbolic manipulation. While the case of the integral is perhaps different in significance, it is not fundamentally different in kind from the case of the square root on the scientific calculator, where we seem collectively more comfortable with leaving the details of the process to a machine.

The challenge in the case of integrals is to get an appropriate balance between by-hand and by-machine methods; this is not usually regarded as a challenge in the case of square roots. One way of 'resolving' this challenge is not to use CAS calculators at all, which of course runs the risk that students will continue to regard calculus as heavily reliant on symbolic manipulation. Furthermore, some by-hand manipulation is necessary: few (if any) would argue that a CAS calculator alone is sufficient to help students develop an understanding of integrals. In addition, most teachers would expect students to evaluate some integrals mentally, just as they would expect students to evaluate some square roots mentally. When CAS calculators are used in Australian schools, typically some assessment permits use of the devices and other assessment prohibits its use; while this seems a good idea, care is needed with the calculator-free assessment that we do not continue to have essentially the same expectations as previously for by-hand symbolic manipulation, lest nothing is gained, and a new burden for students created.

If less time is spent in school developing extensive by-hand symbolic manipulation, those who are either interested in doing so, or who need to do so for some purpose (such as studying the actual algorithms used) can devote time and energy to doing so at the appropriate time—after they leave school—when the need is apparent to them. In the same way, those who would like to, or who need to, know the finer details of how the scientific calculator finds square roots can do so at a later time and in more depth.

Although this paper has not explored the detail, technologies such as calculators offer teachers and students opportunities to explore many mathematical concepts more deeply. Some possibilities were outlined for the case of square roots, while for the case of integrals, CAS calculators can provide a means of representing integrals as areas under curves, or as limits of Riemann sums, as well as symbolic expressions, providing access to richer meanings.

Conclusion

Technologies have the potential both to save time and to improve students' understanding of mathematical ideas. Two examples have been used in this paper, contrasting the acceptance of one example (finding a square root on a calculator) with our reservations about the other (evaluating indefinite integrals on a CAS calculator). A balance between sufficient mental, by-hand work and use of digital technologies is needed to help students understand the key ideas, while avoiding excessive routine manipulation. While teachers working with curricula that prohibit the use of CAS to students have no alternative to devoting a lot of time to symbolic manipulation to evaluate integrals, those working in curricula that permit its use need to seek a reasonable balance.

Two decades ago, in an award-winning paper, Dan Kennedy (1995) likened learning mathematics to climbing a tree, for which there was only one way to climb: up a large and solid trunk. In the limited time that is available, many students give up the climb, impede others, fall off the trunk or fail to climb the tree sufficiently well. In the case of integration, the solid trunk seems to be heavily laden with algebraic manipulation. Kennedy suggested that technology might provide help in the form of ladders to climb the tree in other ways. Just as the use of technology allowed us to bypass the numerical requirements to calculate square roots (and other aspects of basic mathematics), it now seems time to look carefully at the use of computer algebra to reconsider how much of the algebraic trunk is really needed to help students climb the tree, look around and start to explore the branches of the tree that look interesting to them.

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