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Investigating Mathematical Knowledge for Teaching Proof in Professional Development

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Abstract

Research documenting teachers' fragile understanding of proof and how it is advanced suggests that enhancing the role of proof in school mathematics will demand substantial teacher learning. To date, there is little research detailing what mathematical knowledge might support the teaching of proof or how professional development (PD) might afford such learning. This paper advances a framework for Mathematical Knowledge for Teaching Proof (MKT for Proof) that specifies proof knowledge across subject matter and pedagogical domains. To explore the utility of the framework, this paper examines data from four different PD settings in which teachers and leaders worked on the same proof-related task. Analysis of discussions across these four settings revealed similarities and differences in knowledge of proof made available in each setting. These findings provide a means to detail the MKT for Proof framework and demonstrate its usefulness as a tool for designing, analyzing and evaluating proof experiences in PD.

Key words: Proof; Mathematical knowledge for teaching; Professional development

Introduction

Mathematics educators worldwide have advocated that proof be a part of all students' mathematical experiences (Hanna & deVilliers, 2008; Hanna & Jahnke, 1996; Stylianou, Blanton, & Knuth, 2009). Yet, research indicating students' limited proficiency with and opportunities to engage in proof suggest we are a long way toward achieving this aim (Bieda, 2010; Harel & Sowder 2007; Healy & Hoyles, 2000; Hiebert et al., 2003). Not surprisingly, many of these deficiencies have been attributed to teachers own fragile understandings of proof and its role in mathematics (Dreyfus, 1999; Harel & Sowder, 2007; Jahnke, 2007). Teachers often limit proof to verifying formulas in high school geometry, neglecting the explanatory role proof can play in the learning of mathematics at all levels (Hanna, 2000; Knuth, 2002). Moreover, teachers often focus on the form rather than substance of a proof and have difficulty evaluating proofs presented verbally or pictorially (Barkai, Tabach, Tirosh, Tsamir, & Dreyfus, 2009; Dickerson & Doerr, 2014; Knuth, 2002). Teachers, like their students, need to appreciate that some proofs ('proofs that explain') are more elucidating than others ('proofs that prove') as to why something must be true (Hanna, 1989).

These findings suggest that enhancing the role of proof across grade levels and mathematical domains will demand substantial teacher learning. However, there is little research detailing what teachers need to know about proof or how professional development might afford such learning (Ball, 2003; Stylianides & Silver, 2009). Frameworks are needed to characterize this knowledge base in order to provide productive learning opportunities for teachers. To that end, this paper elaborates a framework for Mathematical Knowledge for Teaching Proof (MKT for Proof) and examines the framework's utility in analyzing proof activity in a variety of professional development settings.

Conceptualizing Mathematical Knowledge for Teaching Proof

In order to conceptualize mathematical knowledge for teaching proof, a common understanding of proof is needed. While there is much debate over what counts as proof in mathematics teaching and learning (Hanna & deVilliers, 2008), there is consensus on the meaning of mathematical or formal proof. A mathematical proof involves the validation of propositions through the application of specified rules, as of induction and deduction, to assumptions, axioms, and sequentially derived conclusions (Balacheff, 2008). Taking a more developmental approach to proof, the definition adopted for this paper is that a proof is a mathematically sound argument that

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demonstrates the truth or falsehood of a particular claim. This simplified statement is meant to capture proofs that exist outside of the two-column format prevalent in high school geometry as well as proofs that may lack the rigor of those produced by mathematicians. However, such arguments would still be recognizable as proof within the mathematics community in that they rely on valid modes of argumentation and secure methods of validation (Stylianides & Stylianides, 2009). Mathematics education researchers advance that the mathematical knowledge needed for teaching is different than what one typically acquires as a student of mathematics (Adler & Davis, 2006; Ball, Lubienski, & Mewborn, 2001). Teachers need a flexible, “unpacked” understanding of the mathematics content that they teach, coupled with pedagogical knowledge and skills to apply this understanding to specific teaching and learning situations (Ball, Hill & Bass, 2005; Stylianides & Stylianides, 2009).

This applied knowledge perspective is consistent with situative learning theories that highlight the inextricable relationship between cognition and the social or physical contexts in which knowledge is obtained or accessed (Brown, Collins & Duguid, 1989; Putnam & Borko, 2000). Two examples related to teacher knowledge of proof illustrate this complexity. In a 2004 study, Peressini and colleagues documented how Ms. Savant, a beginning teacher, displayed a sophisticated understanding of proof in her university course work. However, in a teaching situation, Ms. Savant displayed a much different view of proof, allowing students to “prove” the formula for the area of a circle through a construction activity. Similarly, Tsamir and colleagues (2009) describe how a high school teacher, Ms. T, despite being able herself to construct a valid proof, discounted students’ proofs because they were not minimal. Focusing on these teachers’ actions out of context, one might conclude they did not recognize essential components of a valid mathematical proof. A different interpretation would take into account how teachers’ mathematical knowledge interacts with knowledge of teaching and of students and how teachers draw on different but related knowledge in teaching situations.

MKT for Proof Framework

The MKT for Proof framework attempts to detail these differing knowledge bases. To advance the construct of MKT for Proof, research on student and teacher difficulties with proof as well as studies of classroom proving activity was reviewed to identify mathematics content and pedagogical knowledge central to the work of teaching proof (Lesseig, 2011). These ideas were then coordinated with four domains of mathematical knowledge for teaching (MKT) identified by Ball, Thames and Phelps (2008). The resulting MKT for Proof framework specifies ways in which teachers hold their knowledge for proof across two subject matter domains of Common Content Knowledge and Specialized Content Knowledge and two pedagogical domains of Knowledge of Content and Students, and Knowledge of Content and Teaching.

Subject Matter Knowledge for Teaching Proof

Common Content Knowledge (CCK) is subject matter knowledge held in common with others who use mathematics. Obviously, teachers need to have a firm grasp of the mathematical ideas they are expected to teach. This implies that teachers should be able to construct a valid mathematical proof as well as understand the basic components of a proof. Thus CCK reflects essential proof knowledge and skills desired of students (e.g. McCrone & Martin, 2009; National Council of Teachers of Mathematics, 2000) including knowing that a proof is based on accepted statements, follows a logical sequence, and demonstrates truth for all objects within the domain. These elements of CCK are similar to what Steele & Rogers (2012) categorize as creating and defining proof. In their research Steele and Rogers (2012) offer four components of subject matter knowledge for proof (defining, identifying, and creating proofs, and understanding the roles of proof) but do not distinguish between common and specialized knowledge.

The final elements of CCK incorporated in the framework (see Table 1) are related to the role of proof in mathematics. Several roles of proof have been identified and discussed in the literature. Among these are to verify that a statement is true, to explain why a statement is true, to communicate mathematical knowledge, to discover or create new mathematics and to systematize statements into an axiomatic system (deVilliers, 1990; Hanna, 1990). To more closely align with disciplinary practices, proof in school should encompass all these roles. Those functions of proof commonly understood and utilized by mathematicians are categorized as CCK. Specifically, CCK includes recognizing proof as the mechanism through which mathematical knowledge is verified, established and communicated.

To support the work of teaching, fundamental understandings of proof must be developed and made explicit. This more unpacked understanding, or *Specialized Content Knowledge* (SCK), is mathematical knowledge

uniquely needed for teaching proof. For example, while others who use mathematics also know that proofs rely on previously known facts, unlike teachers, they do not necessarily need to know multiple definitions or how an argument might be structured differently depending on the definitions that are accepted. Supporting students' proving ability also demands that teachers be fluent with a variety of visual, symbolic or verbal methods that can be used to express a general argument. To evaluate student-generated proofs, teachers must also be armed with knowledge of a variety of argument types including knowing how a claim about a finite number of cases can be verified through a systematic list, or how a generic example can be used to prove a more general claim. The three criteria for proof (accepted statements, modes of representation and modes of argumentation) outlined in Stylianides' (2007) definition of proof provide a useful way to catalogue these SCK components. In an attempt to provide a consistent meaning of proof across grade levels, Stylianides (2007) defined proof as a mathematical argument that fulfills the following criteria:

- 1) Uses statements accepted by the classroom community that are true and available without further justification (*set of accepted statements*).
- 2) Is communicated with forms of expression that are appropriate and known to, or within the conceptual reach of the classroom community (*modes of representation*).
- 3) Employs forms of reasoning that are valid and known to, or within the conceptual reach of the classroom community (*modes of argumentation*). (Stylianides, 2007, p. 291)

This definition purposely attends to the disciplinary tenets of mathematical proof while remaining responsive to the educative needs and abilities of students (Stylianides, 2007). As such, it encompasses the developmental aspects of proof and provides a means to detail teacher knowledge that would support proof and proving activity across grade levels.

A second category of SCK again attends to the many roles of proof. Much has been written about the important role proof plays in building mathematical understanding (e.g. Ball & Bass, 2003; Hanna, 2000). However, research shows that teachers often have a limited view of proof as merely a tool for verification and reserve proof as a topic of study for a select group of students (Knuth, 2002). To capitalize on proof as a tool for learning, teachers need to better understand how proof can serve as a vehicle for explaining why something is true and providing insight into underlying mathematical concepts (Hanna, 1990).

To reiterate, both CCK and SCK are meant to describe subject matter knowledge that would be useful for and useable in teaching. What distinguishes SCK is not whether mathematicians or others who use mathematics might also hold similar understandings (e.g. most mathematicians are aware of multiple proof techniques and appreciate the explanatory role of proof), but that such knowledge is not necessarily critical to their everyday work (Herbst & Kosko, 2014). While I recognize that there is certain overlap and the distinction is not always clear in practice, highlighting proof knowledge unique to teaching allows us to better address the content needs of mathematics teachers and those in training to become teachers (Ball, Lubienski, & Mewborn, 2001). A summary of subject matter knowledge for teaching proof is provided in Table 1.

Table 1. Subject matter components of the MKT for proof framework

Subject Matter Knowledge for Teaching Proof	
Common Content Knowledge	Specialized Content Knowledge
<p><i>Ability to construct a valid proof</i></p> <ul style="list-style-type: none"> • Understand and use stated assumptions, definitions & previously established results • Build a logical progression of statements • Analyze situations by cases • Use counterexamples <p><i>Essential proof understandings</i></p> <ul style="list-style-type: none"> • A theorem has no exceptions • A proof must be general • A proof is based on previously established mathematical truths • The validity of a proof depends on its logic structure <p><i>Functions of proof</i></p> <ul style="list-style-type: none"> • To establish the validity of a statement • To communicate and systematize mathematical knowledge 	<p><i>Explicit understanding of proof components</i></p> <ul style="list-style-type: none"> • Accepted statements • Range of useful definitions or theorems • Role of language and defined terms <p><i>Modes of representation</i></p> <ul style="list-style-type: none"> • Variety of visual and symbolic methods to provide a general argument <p><i>Modes of argumentation</i></p> <ul style="list-style-type: none"> • Recognize which methods (e.g. proof by exhaustion or counter-example) are sufficient and efficient • Identify characteristics of empirical and deductive arguments (including generic examples) <p><i>Additional Functions of proof</i></p> <ul style="list-style-type: none"> • To provide insight into why the statement must be true • To build mathematical understanding

While not meant to encompass the full range of subject matter understandings teachers may draw upon in their work, these elements of CCK and SCK for teaching proof provide a synthesis of agreed upon understandings and specifically address teacher knowledge deficiencies cited in the literature. This includes research demonstrating that teachers often accept empirical justification as proof of a general statement (Harel & Sowder, 2007; Simon & Blume 1996), have difficulty evaluating non-symbolic proofs (Barkai et al., 2009) and have a narrow view of proof as verification (Knuth, 2002). As such, this delineation is a useful starting point to consider what and how proof understandings are pursued in professional development.

Pedagogical Content Knowledge for Teaching Proof

Knowledge of Content and Students (KCS) refers to knowledge of students' typical conceptions or misconceptions of proof as well as an understanding of developmental sequences. Researchers have used a number of frameworks to categorize students' proof productions and describe the progression from inductive or empirical justifications toward deductive arguments (Balacheff, 1987; Harel & Sowder, 1998; Lannin, 2005). The proof scheme taxonomy developed by Harel and Sowder (1998; 2007) is one of the most comprehensive and frequently cited in mathematics education research. According to Harel and Sowder (2007) an individual's proof scheme consists of what constitutes ascertaining (convincing oneself) and persuading (convincing others) for that person. Proof schemes are broadly classified as externally based, empirical, or analytical. External proof schemes are those in which a person is convinced by an authority or the ritualistic, symbolic form of the argument presentation. Empirical proof schemes are based on perceptual observations or inductive reasoning in which statements are verified by testing a number of examples. Finally, in an analytic proof scheme, conjectures are validated by means of logical deductions and are thus accepted as proof within the mathematics community.

It is important for teachers to know that while neither external conviction nor empirical proof schemes are consistent with mathematical standards of proof, both are commonly observed in research with students (Harel & Sowder, 2007; Kuchemann & Hoyles, 2009; Martin & Harel, 1989). Explicit knowledge of this taxonomy, or a similar categorization grounded in research on student thinking, would support teachers' ability to interpret student thinking and decide on the next instructional move. Additional elements within KCS relate to teachers' ability to appraise student arguments in relation to mathematical standards of proof. This requires knowledge of the definitions, forms of representation, and argumentation that are developmentally appropriate and accessible at a particular grade level.

Finally, *Knowledge of Content and Teaching* (KCT) intertwines knowledge of proof from the other domains with pedagogical principles and strategies. KCT for proof includes methods of representing, explaining, or connecting proof ideas as well as methods of responding to student contributions. For example, to support students' ability to move beyond authoritarian or empirical justification toward more sophisticated notions of proof, teachers must first recognize how methods of answering questions and responding to students are related to the proof schemes students develop (Harel & Rabin, 2010). Moreover, teachers must know what examples to present or questions to pose in order to focus students on key proof ideas, elicit further justification, or extend student thinking toward the general case (Bieda, 2010; Harel & Rabin, 2010; Martino & Maher, 1999). Table 2 below summarizes pedagogical aspects of MKT for Teaching Proof.

In their conceptualization of MKT, Ball and colleagues (2008) included two additional domains, knowledge at the mathematical horizon and knowledge of curriculum. Given that both of these domains remain underconceptualized, they were not included in the MKT for Proof framework. My focus on CCK, SCK, KCS and KCT is consistent with recent work that seeks to develop measures to assess mathematical knowledge for teaching in other content areas (e.g. Herbst & Kosko, 2014; Hill, Schillings & Ball, 2008).

To date, researchers have used MKT constructs to infer mathematical knowledge of proof drawn upon during classroom teaching (e.g. Stylianides & Ball, 2008; Steele & Rogers, 2012) and, to a lesser extent, have considered how knowledge of proof might be developed in teacher preparation programs (e.g. Stylianides & Stylianides, 2009). In contrast, my study is empirically grounded in professional development (PD). My purpose is not to make claims about what individual mathematics teachers understand about proof, but rather to investigate how and when knowledge of proof or the teaching of proof is evidenced.

Table 2. Pedagogical content knowledge components of the MKT for proof framework

Pedagogical Content Knowledge for Teaching Proof	
Knowledge of Content and Students	Knowledge of Content and Teaching
<p><i>Explicit knowledge of student proof schemes</i></p> <ul style="list-style-type: none"> • Characteristics of external, empirical and deductive proof schemes • Students' tendency to rely on authority or empirical examples • Typical progression from inductive to deductive proof <p><i>Developmental aspects of proof</i></p> <ul style="list-style-type: none"> • Definitions & statements available to students • Representations within students' conceptual reach • Forms of argumentation appropriate for students' level • Relationship between mathematical and everyday use of terms 	<p><i>Relationship between instruction and proof schemes</i></p> <ul style="list-style-type: none"> • Methods of answering questions, responding to student ideas, using examples and lecturing that either promote or diminish authoritarian or empirical proof schemes <p><i>Questioning strategies</i></p> <ul style="list-style-type: none"> • To elicit justification beyond procedures • To encourage thinking about the general case <p><i>Use of pivotal examples or counter-examples</i></p> <ul style="list-style-type: none"> • To extend, bridge or scaffold thinking • To focus on key proof ideas <p><i>Knowledge of proof connections</i></p> <ul style="list-style-type: none"> • How to link visual, symbolic & verbal proofs • How argument structure depends on accepted definitions • How to produce a general argument from a numerical example or specific diagram

Research Questions

This study examined leaders and teachers' engagement in a proof-related task across a variety of settings with an overarching goal to detail the Mathematical Knowledge for Teaching Proof framework and assess its utility for designing and analyzing teacher experiences with proof in PD. With this goal in mind, analysis was guided by the following questions:

1. What proof ideas emerge in leader and teacher discussions while working on a proof-related task in professional development?
2. How are leader and teacher discussions around the task related to components within the MKT for Proof framework?

Method

Participants and Context

This study was conducted in the context of Leader Project, (project name removed for blind review) a 5-year research and development project intended to support the preparation of mathematics PD facilitators. In particular, the Leader Project was designed to enhance leaders' ability to facilitate mathematically rich learning opportunities for teachers (Elliott, Kazemi, Lesseig, Mumme, Carroll, & Kelley-Petersen, 2009). Data for this paper comes from phase II of the project, which involved thirty-five teacher-leaders, (i.e. mathematics coaches, teachers on special assignment and full-time classroom teachers with school or district leadership responsibilities) from three US school districts. Leaders came with a range of teaching backgrounds with 17 working at the elementary level (US grades K-5, or students aged 5-11), 10 working in middle school (US grades 6-8, or students aged 11-14) and 8 from the high school (US grades 9-12, students aged 14-18). While the majority of participants also had teaching responsibilities, this group is referred to as 'leaders' throughout the paper to distinguish them from the teachers who participated in PD sessions these leaders facilitated.

Leaders attended four 2-day seminars across the academic year. In each seminar, leaders engaged with mathematics tasks (doing math), analyzed PD videocases, and participated in other activities to support them in their role as PD facilitators. Between seminars, leaders facilitated PD sessions with practicing teachers at the building or district level. Researchers identified a subgroup of the participating leaders as case-study leaders and gathered data on the PD sessions that those leaders facilitated across the year. [Note that throughout this paper, 'seminars' refers to sessions led by the research team *for* all the leaders and 'PD sessions' refers to sessions led *by* selected case-study leaders for classroom teachers.]

To provide a detailed analysis of how the MKT for Proof framework can be operationalized, this paper focuses on leader and teacher discussion around the Halving and Doubling Task (see Figure 1). This task was introduced on Day 1 of the second Leader Project seminar. In addition, several case-study leaders used this task in PD sessions they facilitated.

Halving and Doubling Task
In a professional development session for elementary teachers, teachers were asked to use a variety of approaches to try to verify this conjecture:
<i>When you multiply two numbers, you can cut one of the numbers in half and double the other number, and the product will be the same.</i>
What methods do you think teachers would use to prove the conjecture?

Figure 1. Halving and doubling task presented in leader seminars (Carroll & Mumme, 2007).

Data and Analysis

Video from Leader Project seminar activities associated with the Halving and Doubling task together with video from three PD sessions that case-study leaders facilitated using the task, one each at the elementary, middle, and high school level, was the primary data source. During researcher-led seminars two cameras were used to capture leaders' mathematical work and interaction within two small groups and the whole-group discussions. Fieldnotes, seminar materials (e.g. facilitation notes, videocase transcripts and other handouts), and artifacts (e.g. small group posters) were also analyzed. Data from the case-study leader PD sessions included video of whole group interactions, artifacts from the session, and pre-post interviews with the case-study leaders who facilitated the session.

Systematic coding and analysis of the video data was facilitated by Studiocode© (Studiocode Business Group, 2012), a qualitative video analysis software. Video data was first chunked into idea units to denote a conversation thread around a particular topic and applicable codes were applied to each idea unit. Codes for empirical justification versus deductive reasoning (i.e. 'testing numbers' and 'generic example'), modes of representation (i.e. 'algebra' and 'visual model') and pedagogical references (i.e. 'students' and 'teaching') attended to specific features of the MKT for Proof framework. Additional codes captured how leaders and teachers used resources such as the levels of justification that were provided during seminars.

Matrix and databasing features within Studicode were used to calculate the percentage of idea units containing each code. This analysis supported the identification of themes within and across PD activities and provided direction for further qualitative analysis. For example, to explore how participants made connections across different representations, idea units coded as both 'generic example' and 'algebra' or 'visual model' could be collected and viewed in succession. As a final analysis step, proof ideas were connected to the framework to create a comprehensive description of MKT for Proof evidenced within each activity. These summaries were used to identify the primary foci of leader and teacher discussions and facilitate comparisons across settings. Throughout this process coding examples and emerging themes were vetted with other researchers engaged in the project (Merriam, 2009).

Findings

The goal of this study was to investigate MKT for Proof in professional development settings and to examine the utility of the MKT for Proof framework. Analysis revealed that during the seminars the four knowledge domains, CCK, SCK, KCS and KCT, were evenly represented in discussions when leaders worked on the Halving and Doubling task. Although each domain was also evident during the subsequent videocase discussion, these conversations were more focused on SCK. In contrast, during the PD sessions that case-study leaders facilitated, differences were found regarding the emphasis on different facets of MKT for Proof. Within the elementary PD session discussion was primarily aimed at building CCK, whereas the middle and high school PD sessions focused more on KCS and KCT.

MKT for Proof Evidenced in Leader Seminar

Doing math discussions encompassed all four knowledge domains: During Leader Project seminars, leaders first worked individually and then in small groups to produce a variety of arguments teachers might construct to prove the Halving and Doubling conjecture (Figure 1). This press to consider methods others may use to prove the conjecture moved leaders beyond constructing a single proof and led them to create a range of empirical and deductive arguments. For example, leaders in one small group began by sharing their individual work, which included numeric examples, an area model using a two-by-two rectangle, and a short algebraic proof. The group then went on to evaluate an example-based proof Flo had constructed using a grouping model for multiplication. Flo had drawn a diagram to illustrate that three sets of two boxes was equivalent to one set of six boxes. Leaders' quickly acknowledged that the model was problematic because it only showed one pair of numbers and was thus "not generalized." However, as indicated in the continued discussion, leaders still valued this method for its accessibility and explanatory power.

- Jack: Your example is neat because it shows... it makes sense a little bit more of multiplication. Like three groups of five... or three groups of two is actually what your picture showed.
- Flo: Yeah two three times, then one six times.
- Bob: I think that is a good visual that everyone can probably grasp, especially if you use blocks and show physical movement. That's a real easy proof.
- Flo: Right, and I think it's good to go back to the "what does multiplication mean."

Quantitative data indicate that leaders spent considerable time comparing argument representations with roughly 40% of idea units coded as testing specific numbers, visual models, or algebra (Figure 2). These discussions elicited both CCK and SCK as leaders considered whether their proofs demonstrated the conjecture was true for all cases or discussed what properties or meanings of multiplication each representation made visible. In addition to noting that proofs should be general and based on key ideas, several functions of proof were also implicit in leaders' evaluation. As illustrated in Jack and Flo's comments above, leaders recognized that the role of proof is not only to establish validity, but also to provide insight into why a statement must be true and to build mathematical understanding (e.g. understanding the meaning of operations). Indeed, as shown in Figure 2, 29% of the doing math idea units were coded as 'explain why.' This same emphasis on "proofs that explain" (Hanna, 1989) was not as evident in two of the three case-study PD sessions.

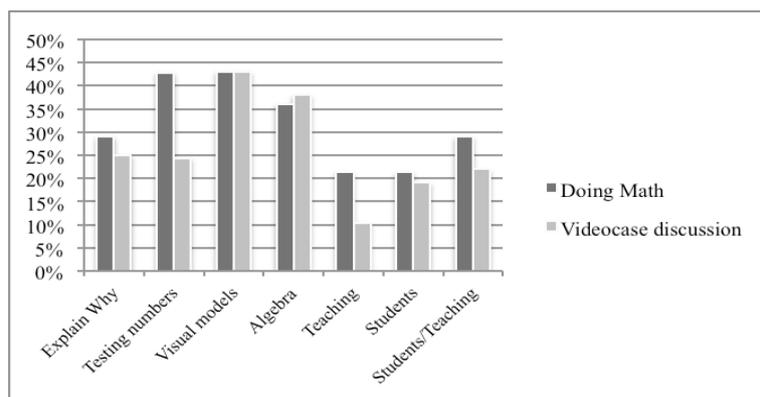


Figure 2. Percentage of idea units with select topic codes during leader seminar activity

Connections to pedagogical domains, evidenced in codes for classroom teaching or students, occurred in 29% of idea units. These opportunities arose naturally as leaders evaluated proofs their colleagues constructed. For example, Bob's judgment that Flo's visual model, or a similar physical representation, would be an easy one to grasp illustrates how KCS emerged. Leaders also asked questions about students' familiarity with array models or at what grade level variables were typically introduced. Evidence of KCT often took the form of leaders describing specific teaching strategies they may employ. This was illustrated in the continuation of the small group discussion about Flo's array model.

To further explore the usefulness of Flo's visual model, Jack constructed a similar example; this time showing that two groups of three was the same as four groups of one and one-half (Figure 3). After Jack completed his drawing, the group evaluated and revised his model to consider how one might present it to students.

- Jack: So, this is a proof?
- Flo: You could show them [students] it still works. That it's going to be the same, 2, 3, 4, 5, 6 [counting the boxes]
- Jack: It's not as transparent of a strategy as this model [referring to a rectangular area model presented previously].
- Bob: Well I think that if you were to take this and maybe dot it in up here, it would be easier to see that, okay I'm going to cut these in half [drawing a dotted line down the middle of each box] which will double my groups.
- Jack: Okay, yea draw some arrows.

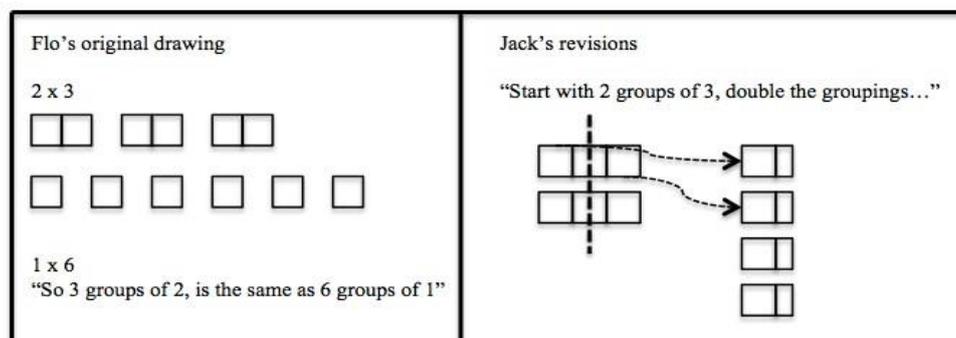


Figure 3. Flo's original 2 x 3 example together with the groups' revisions

Jack's question as to whether this example would suffice as a proof provided an opportunity for leaders to consider the affordances and constraints of this model from mathematical and pedagogical perspectives. For example, does this model demonstrate the conjecture is true for all numbers? And if so, how can we make the general process transparent so students understand why the conjecture must always be true? This small group exchange illustrates the tight link between subject matter and pedagogical content knowledge for proof seen throughout leaders' initial work on the Halving and Doubling task. Here, leaders were using an example to focus on key ideas in this proof (e.g. the commutative and reciprocal properties of multiplication) and exploring ways to make those ideas apparent to students.

Videocase Discussions Centered on SCK

After working on the task themselves, leaders viewed a PD videocase of the task being implemented with a group of elementary teachers (Carroll & Mumme, 2007). In the videocase, three types of arguments were presented in some detail: (a) Carrie's 3x3 array model and verbal argument, (b) Cheryl's rectangular area model and (c) a deductive argument using algebraic notation generated from Cheryl's diagram (see Appendix for a presentation of these arguments). A third teacher in the video, Judy, also mentions how testing a number of examples convinced her that halving and doubling works, but she does not propose this as proof. In the seminar discussion, leaders were prompted to compare the methods and representations used in these teacher arguments.

Prior to watching the videocase a second time, leaders were introduced to different levels of justification (Figure 4). This classification, which corresponds to student proof schemes (Harel & Sowder, 2007), was meant to raise awareness that not all arguments or justifications that students, or teachers, might produce constitute mathematical proof as they do not provide secure methods of verification. Justification types ranged from nonproof arguments including appeals to authority and empirical justifications to two forms of valid mathematical proofs, generic example proofs and deductive arguments. As described by Mason and Pimm (1984), a generic example refers to a specific example that is presented in such a way as to communicate generality across cases.

Levels of Justification

- No justification (no feeling of obligation to provide a justification for an answer)
- Appeal to an external authority (relying on textbook, leader, etc. for verification)
- Test specific examples (testing one or a few instances)
- Use generic examples (using an example to communicate generality across cases)
- Provide a deductive argument (creating a general argument that applies across all cases based on previously accepted "truths")

Figure 4. Levels of justification (adapted from Lannin, 2005) provided during leader seminars

In the subsequent small group discussion, leaders used these levels to categorize the arguments in the videocase and reconsider whether or not each argument constituted a valid proof of the original conjecture. The proof arguments presented in the videocase coupled with the levels of justification (Figure 4) grounded leaders' discussion in SCK for proof. Leaders were attuned to making connections across visual and symbolic representations and valid forms of argumentation, including generic examples. Although aspects of KCS and KCT were also present, they arose less frequently during videocase discussions and were not explored in depth. As indicated in Figure 2, leaders made explicit connections to teaching in only 10% of idea units (compared to 21% while doing the math) and to students 22% of the time. These references generally occurred as leaders considered whether students would perceive Carrie's argument as a specific versus a general case.

An excerpt from the other small group on camera illustrates how MKT for Proof components were evidenced in videocase discussions. Leaders in this group spent considerable time trying to categorize Carrie and Cheryl's methods within the levels of justification (Figure 4). In the process, they debated the definition of a generic example as well as if and how a visual model could be used to prove a general statement. Barb began the discussion by defending Carrie's argument.

- Barb: I think Carrie provided a valid justification because she used an example, but the example will work for any number. So we looked at this three by three, this idea of visually cutting in half one number and swinging or transferring that half. You could do that with anything and you still conserve the same number. It doesn't change the number, it just changes the arrangement and that would happen with any number, so I think she provided a valid justification.
- Maria: At first I thought it was just a test of specific examples... this does involve a fraction so it shows it is working in another area. But there is something about the explanation and the visual that pushes it a little more into the generalization.
- Sam: I don't know that Carrie's method lends itself to an algebraic proof or justification because it is only showing concrete, discrete numbers; it's not showing me how I would connect a variable standing for any number.

Barb suggests that Carrie's verbal description, cutting in half and rotating the boxes, makes this a valid general proof. Although not entirely convinced, Maria agrees that the visual somehow moves this beyond a specific case toward a generalizable method that would work for any number. Sam remains skeptical, indicating that to him, variables play a key role in representing a general proof. After further debate as to whether Carrie's method represented a specific case or a generic example, the group then proceeded to discuss Cheryl's approach (see Appendix). Cheryl had drawn a rectangular model with dimensions $a \times b$ that she proceeded to cut in half and rearrange. Her diagram was meant to demonstrate that the original area ($a*b$) was equivalent to the new area ($\frac{1}{2}a(2b)$).

- Maria: Cheryl's is definitely using an example to show generality across cases.
- Sam: And lending toward deductive argument I think, because now I can't count blocks anymore. You've taken and just said this is "some" blocks wide not three blocks wide, but this is some blocks wide and some blocks long as opposed to a specific example that can be generalized with different numbers.
- Maria: And I think sometimes it comes back to how explicit the understanding is. Because some of us can look at this [pointing back to Carrie's method] and we are generalizing it in our minds so it takes it to that next bullet [generic example]. But maybe it needs to be more explicit than that to actually be classified as generalizing across cases.
- Sam: Because I could see students saying here [pointing to Carrie's method] what is the product of three by three, well its nine blocks. What's the product of a and b ? Well it's just a big square or rectangle.

Leaders' work to distinguish between empirical arguments and valid deductive proofs, to identify characteristics of generic examples, and to compare the visual and symbolic proofs presented in the videocase, was clearly within SCK terrain. But as evidenced by Sam's reference to what students may or may not be able to generalize from Carrie's diagram, leaders were also connecting these ideas to teaching. To summarize, conversations in leader seminars were primarily focused on subject matter knowledge for proof. In the process of critically examining arguments created by colleagues or presented in the videocase, leaders had opportunities to develop or deepen their understanding of different modes of representation and argumentation. In the midst of evaluating whether particular arguments constituted a valid proof, leaders also made natural connections to

students and teaching. These references were slightly less frequent in videocase discussions and primarily consisted of general wonderings about what justifications were accessible to students.

MKT for Proof Evidenced in Case-study PD Sessions

Three different enactments of PD facilitated by case-study leaders were analyzed, one each at the elementary, middle and high school level. Table 3 summarizes the participants and the goals that leaders identified for each of these PD sessions. During the researcher-led seminars, leaders had been given time to plan a PD session using the Halving and Doubling task. Given this pre-planning and leaders' own experiences with the task, it is not surprising that all three case-study PD sessions followed a similar trajectory of activities. Teachers first worked in small groups to develop their own justifications for the Halving and Doubling conjecture and shared their work in a whole group discussion. Leaders then introduced the levels of justification from the Leader Project seminar (Figure 4) and asked teachers to use these to categorize the arguments presented and/or consider what might be expected of students at different grade levels. Despite similar activities, aspects of MKT for Proof were not always evidenced to the same degree or elicited in the same manner. Instead, the PD context, including session participants and goals, played a role in which facets of MKT for Proof were more or less central in each session.

Table 3. Participants and goals for case study leader PD sessions

	Leaders facilitating	Participants	Stated session goals
Elementary case	Maria – elementary math coach	24 K-6 teachers	Consider the importance of proof and link these ideas to “how teachers press students to explain their thinking or demonstrate understanding”
Middle school case	Cori - 7 th grade math/science teacher Sam- 7 th - 8 th grade math teacher	10 middle school math & science teachers	Consider justifications to expect at each grade level and “how we can support each other in holding students to the standard at each grade level and build an actual progression”
High school case	Flo – high school math teacher Ken – high school math teacher & department chair	6 high school math teachers	Consider “what constitutes a proof” and understand there are different levels

Middle and high school PD sessions focused on pedagogical content knowledge: Figure 5 shows the percentage of idea units for select topics across the three PD sessions. The opening task, to verify the conjecture, grounded teachers' initial work in constructing valid proofs, and leaders worked to make sure a variety of representations and types of argument were shared in the whole group discussion. Thus, there was a fairly even distribution of coded as testing numbers, visual models and algebra in both the middle school and high school sessions, though the frequencies were all slightly less in the high school session. In the middle school session the leaders, Sam and Cori, specifically sequenced teacher presentations to begin with examplebased arguments and progress to a final deductive proof. In the high school session, teachers had to be prodded to develop something other than a standard algebraic proof. It was not until Flo asked for a different approach that “showed the same thing, for those students who might not grasp the algebraic methods presented so far,” that teachers generated array models to demonstrate the conjecture.

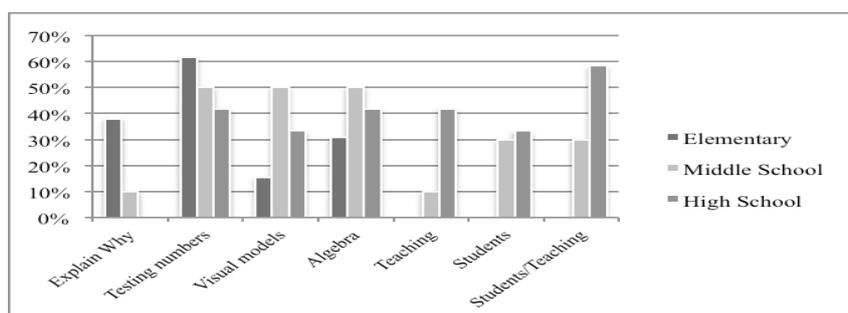


Figure 5. Percentage of idea units with select topic codes across the case leader PD sessions

The range of arguments that were eventually produced opened opportunities for teachers to make comparisons across proof representations and modes of argumentation, key aspects of SCK. However, unlike in leader seminars, teachers did not spend time discussing whether the examples or visual models constituted a proof or provided insights into why the conjecture was true. Instead arguments were presented in rapid succession with little discussion or were evaluated in terms of how to present them to students. In short, the initial proof construction activities remained at the level of CCK or KCS. Discussions moved more deeply into KCS and KCT domains as teachers were then prompted to use the levels of justification (Figure 4) to consider what might be expected at different grade levels. This was also in contrast to seminar discussions in which leaders used these descriptions to dissect the structure of generic example arguments and critically analyze how visual models might support a deductive argument.

As summarized in Table 3, the goal of the middle school PD session was to have teachers think about how they could support students along a progression of proof. In line with this goal, Cori started with the first argument presented, a picture demonstrating the conjecture for both even and odd factors, and asked teachers, “what level of proof might this have been and at what grade levels would you expect to see this?” After considering this question for each of the arguments presented, teachers concluded that while they would expect more empirical justifications at the pre-algebra level they would like to see pictures combined with algebraic notation by 7th or 8th grade and might expect two-column proofs in 10th grade geometry. Similarly, in the high school session, Flo stated that teachers needed to understand the different levels so both they and their students would know what was expected. She then provided examples of specific grade-level expectations:

Flo: I know that if I were teaching geometry, I would probably go to a more formal proof just because that might be what it is about. And in algebra, I might start verification at a lower level and then say, further on we need to go to a generalization.

Flo’s comment about the progression of proof demonstrates the focus on pedagogical knowledge evident across the middle and high school sessions. This focus is also highlighted in statistics showing that nearly one third of the idea units in the middle school session, and twice that percentage in the high school session, were coded as either ‘students’ or ‘classroom teaching’ (Figure 5). Taken together, more than 45% of the conversations in the middle and high school sessions were focused on pedagogical knowledge of proof. Specifically, teachers considered what representations and forms of argumentation were available and appropriate at different grade levels, KCS, and how to link various verbal, visual and symbolic arguments through instruction, KCT.

The Elementary PD session centered on subject matter knowledge: The conversations in the elementary session were quite different. This session centered on CCK and SCK with no explicit references to either students or classroom teaching. A second significant difference was the focus on explaining why, which was coded in 38% of the idea units as compared to 10% in the middle school case and 0% in the high school case (Figure 5). Instead, elementary teachers were building their own understandings about how to construct general proofs and considering the explanatory role of proof. The case leader, Maria, facilitated this work by continually asking teachers whether the arguments presented proved the conjecture for all possible cases.

For example, Maria began the whole group discussion by eliciting numeric examples but pressed teachers to consider the limitations of empirical justification. The first teacher shared the example $15 \times 4 = 60$ and $7.5 \times 8 = 60$, and explained how if you cut the fifteen in half and doubled the four you get the same result. Maria then asked another teacher to build on this example,

Maria: Because, you guys were experimenting... and our question is, does that prove the conjecture?
 Teacher: Well, we talked about evens and odds. And then after we realized it worked for an even number and an odd number, we went to fractions. So for $\frac{3}{4} \times \frac{1}{4}$, I doubled the first one, so six-fourths, and then cut one-fourth in halve to get one-eighth so when you simplify, you get three-sixteenths again. So it works with fractions too.
 Maria: Do you agree with that? Does that prove that our conjecture is true for all numbers?
 Teachers: No.
 Maria: So, this group started with some whole numbers, did some decimals, did some negatives, it worked for zero. So they tried lots of different examples. It worked for everything. They haven’t found one non-example. Because the next question too is *why* does this work?

In Maria's summary, she emphasized that testing a range of examples was not a secure method of mathematical proof. Moreover, this empirical work did not help explain why Halving and Doubling worked. Maria continued to press on both of these points as teachers shared visual examples and eventually an algebraic proof based on multiplication properties. In these conversations, teachers' work was focused on CCK and SCK, building essential proof understandings and reinforcing that two key functions of proof are to establish truth for all cases and provide insight into why the statement is true.

To summarize, the facilitator's goals and actions during each PD session led to the foregrounding of different aspects of MKT for Proof. In the middle school PD session, teachers' arguments were specifically sequenced to match a progression from numeric examples, to a visual model, to an algebraic proof. The levels of justification were then used to ground discussion around expectations of students at particular grade levels. In the high school session, Flo and Will needed to invoke students or teaching scenarios to elicit visual arguments and orchestrate discussions across representations to achieve similar KCS related goals. In the elementary session a very different profile of MKT for Proof was evidenced. Unlike in the middle and high school sessions, teachers did not tie the levels of justification to their own students. Instead, the elementary teachers spent time exploring number properties and delving more deeply into whether the numeric, algebraic and visual arguments they produced proved the conjecture worked for all numbers; opening opportunities to develop or draw upon CCK and SCK.

Discussion

The purpose of this study was to examine the nature of teachers' proof activity in professional development and explore the utility of the MKT for Proof framework. The intent was not to assess teachers or leaders' proof knowledge or make judgments about the effectiveness of the PD these participants experienced, but rather to demonstrate how the MKT for Proof framework can serve as a tool for both research and practice. As illustrated in this study, the framework enables researchers (or PD leaders) to recognize when teacher discussions are focused deeply on one component or whether multiple knowledge domains are being integrated. By detailing both mathematical and pedagogical knowledge of proof, the framework can assist leaders in identifying proof specific learning goals for teachers and designing experiences to meet those goals.

Utility of the MKT for Proof Framework for Research

The MKT for Proof framework provided an analytic tool to interpret leader and teachers' mathematical work on the Halving and Doubling task and draw conclusions about mathematical knowledge for teaching proof evidenced across PD settings. As leaders and teachers evaluated arguments constructed by peers or presented in the videocase, a range of proof ideas were elicited. These ideas were tightly connected to the work of teaching and crossed all four domains of MKT for Proof. Interjections about what representations students have access to (KCS for proof) were intermingled with judgments about whether an argument constituted a valid proof of the general statement (CCK).

By investigating how proof ideas were taken up by teachers and leaders across PD settings, comparisons could be made between MKT for Proof evident in leader seminars versus those evident in case leader PD sessions. For example, the levels of justification afforded opportunities for teachers in the elementary PD session to consider characteristics of empirical justification, generic examples, and deductive proof, important facets of SCK for proof. In the middle and high school PD sessions, teachers used these levels of justification to consider what representations and forms of argumentation to expect at different grade levels, addressing aspects of KCS and KCT for proof.

In sum, the MKT for Proof framework contains sufficient detail to answer questions regarding teachers' proof experiences in PD. Research questions that might be investigated using this tool include: To what extent do teachers integrate their knowledge of proof with knowledge of typical student thinking around proof? To what extent do specific PD activities (e.g. constructing proofs vs. evaluating proofs presented by others) promote knowledge integration and/or allow for deep conversations within a particular MKT for Proof domain? To what extent are teachers afforded opportunities to develop proof knowledge within any one domain? And how do teachers take up these opportunities (or not)?

Utility of the MKT for Proof Framework in Practice

Smith (2014) discusses how tools used to gather data might also be re-envisioned as tools to communicate a standard or shared understanding of practice. I argue similarly that the MKT for Proof framework can support the practice of those charged with facilitating teacher learning. The MKT for Proof framework draws attention to mathematical and pedagogical knowledge that would support teachers' efforts to promote proof in the classroom and can thus guide facilitators as they design learning opportunities to meet specific proof-related goals. The MKT for Proof framework can also help facilitators recognize when teacher conversations are straying or staying focused on those goals so they can respond accordingly. For example, results from this study indicated that opportunities to unpack visual representations and explore characteristics of generic examples did not occur with the same frequency or depth across PD sessions.

Unpacking Visual Representations

Given research documenting difficulties teachers have evaluating or valuing non-symbolic arguments (Barkai et al., 2009; Knuth, 2002) it is important for teachers to have a sophisticated understanding of how and when visual models constitute, or can support, a valid mathematical proof. During seminars, leaders considered in detail how visual models might demonstrate a general process rather than a specific case. Leaders also noted how mathematical concepts (e.g. inverse operations or the meaning of multiplication) were more or less transparent in particular representations.

This unpacking of the specific characteristics of visual diagrams, a key facet of the SCK discussions in the leader seminars, did not always occur during the case-study PD sessions. While visual models were elicited in all of these PD sessions, the conversations around them remained at a more general level. Teachers commented that the visual model "showed the same thing" as an algebraic proof or noted how adding variables to the diagrams could support students in making the transition to proof, but they failed to explore how the diagram, with its own explanation, could suffice.

Exploring Generic Examples

Coupled with a limited understanding of the role of visual representations, teachers also lack experience with generic example proofs (Karunakaran, Freeburn, Konuk & Arbaugh, 2014). Given classroom research indicating how generic examples support students' progression from inductive to deductive reasoning (Lannin, Barker & Townsend, 2006; Pedemonte & Buchbinder, 2011), opportunities for teachers to develop this component of SCK seem particularly salient. As demonstrated in the second small group seminar discussion, leaders critically evaluated the array and area models presented in the Halving and Doubling videocase. Specifically, leaders debated whether Carrie's diagram would be considered a generic example, or if some aspects needed to be more specific so that everyone could "see" the generality.

Again, this same level of detailing or unpacking mathematical ideas was not always evident in the case-study PD sessions. For example, teachers in the middle and high school sessions acknowledged the utility of visual models to support a deductive proof but did not necessarily see these as valid general proofs. In the elementary PD session, teachers developed some tentative understandings but had difficulty actually constructing a generic example to prove the Halving and Doubling conjecture. Simply sharing solutions did not guarantee teacher discussions reached a level of mathematical detailing to support SCK for teaching proof. Teachers were exposed to a range of proof and non-proof arguments. But, depending on the PD session goals, teachers were not always pressed to consider specific characteristics of generic examples or to fully unpack the mathematical properties underlying non-symbolic arguments. These findings suggest that structuring teachers' proof activity to make a variety of solutions public and to compare or evaluate proofs was certainly productive. However, more focused facilitation prompts might be necessary to elicit SCK for proof.

Limitations

Given that MKT is conceptualized as knowledge useful for the work of teaching, and the fact that PD settings are often removed from classroom teaching interactions in which such knowledge would be deployed, there are obvious limitations to the approach taken in this study. For example, some interpretation is needed to categorize discussions and infer how or if teachers are making connections to their work. Also, as noted earlier, the distinctions between categories are not always clear, nor are they necessarily productive in practice. This limitation is not unique to this study. For example, when using classroom scenarios to measure MKT, or

analyzing teachers in-the-moment decision making, it is difficult to discern whether teachers are relying on purely mathematical knowledge (SCK), or knowledge they have garnered from years of experience interacting with students (KCS) (Herbst & Kosko, 2014). However, by detailing how proof knowledge surfaces in PD settings, this work has the potential to inform those designing learning experiences for teachers.

Finally, it is important to reiterate that the MKT for Proof framework is not meant to be a comprehensive list of proof understandings, nor does it provide a specific curriculum through which teachers may develop such knowledge. However, by categorizing the multi-faceted knowledge components that would support the teaching of proof, the framework presents a useful set of learning objectives to guide teacher preparation or PD. For example, in their work with preservice teachers, Stylianides & Stylianides (2009) have developed an instructional sequence to specifically address the misconception that empirical justification is a valid method for proving a general statement. Their work thus targets essential proof understandings (CCK) and explicit understandings of modes of argumentation (SCK). Similarly, others might use the MKT for Proof framework, or other common tools, to explore how specific components of MKT for Proof might be developed.

Conclusion

Recent attention has been drawn to the importance of providing all students consistent opportunities to engage in proof (Stylianou et al., 2009; Stylianides, 2007). In addition, researchers have called for common analytic tools to both investigate and draw conclusions about what teachers learn from professional development (Desimone, 2009; Stylianides & Silver, 2009). The MKT for Proof framework, developed through a synthesis of proof literature and refined through an empirical study, is positioned to support both of these efforts.

As a research tool, the MKT for Proof framework provided language to categorize teachers' proof discussions and enabled comparisons across PD settings. As a practitioner tool, the framework can be used to define learning goals for teachers, design professional development to meet those goals, and assess the results. The underlying premise of this study is that teachers will need continued support to promote proof as an integral part of school mathematics. Detailing mathematical knowledge for teaching proof and demonstrating how professional development can support the integration of proof knowledge is a first step toward this important goal.

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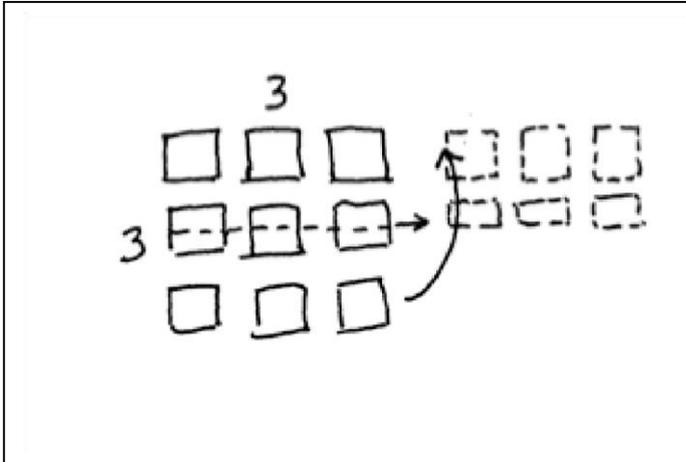
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Appendix

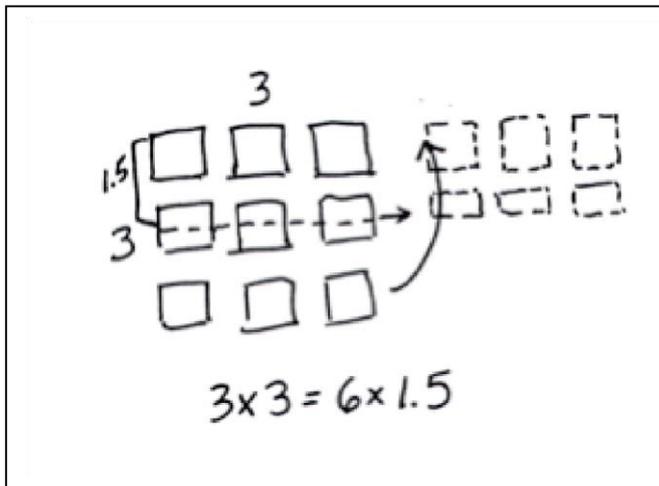
Halving and Doubling Videocase Solutions (Carroll & Mumme, 2007)

Carrie's Method



“We were wondering about the odd ones, so I just drew a picture of blocks.”

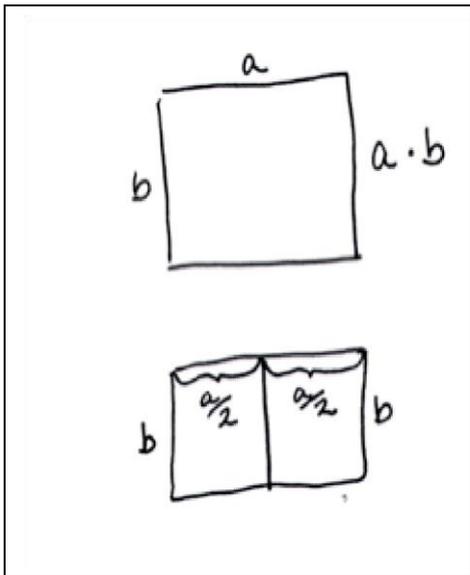
“We looked at 3 x 3, and all you do is just visually cut it in half, put a little hinge here and swing this guy up there, and what you get is 1, 2, 3, and then these halves.”



“You can see you’ve cut this one in half and doubled this one but you’ve conserved the number of squares that you’ve used in your example.”

“This is half here, so 1, 2, 3, 4, 5, 6 halves and 1, 2, 3, 4, 5, 6 wholes. So this then becomes a six by one and a half. Okay?”

Cheryl's Method



1) "We used an array model that was a by b , we looked and decided this gives us ab ... this distance is a over 2 and this distance is a over 2"

2) "... Then we do what Carrie did, which is shift this over here... so you've got a over two times $2b$ so it's the same."

3) "If you wanted to connect it you can make it an a over two times two b and the twos cancel out"

4) After more discussion... another teacher in the group suggests this alternative way to record